Metric Spaces Worksheet 4

Sequences III

The final aspect of sequences we'll be interested in – and the one you're most likely to meet in your continued education – is the notion of a subsequence. To discuss these objects we need the following notion.

Definition 1 (strictly increasing function)**.** A function $f : \mathbb{N} \to \mathbb{N}$ is *strictly increasing* if for all $n, m \in \mathbb{N}$, $n < m \to f(n) < f(m)$. ⌟

Definition 2 (subsequence)**.** If (a_n) is a sequence in (X, d) then a *subsequence* of (a_n) is a strictly increasing function $k : \mathbb{N} \to \mathbb{N}$, which we think of as generating a new sequence $a \circ k : \mathbb{N} \to X$ and which we write as (a_{k_n}). ⌟

It is useful to think of a subsequence of (a_n) as an infinite *list* or *specification* of terms of (a_n) we wish to keep, such that these terms maintain their relative locations form (a_n).

Example 3 (*some examples of subsequences in* $\mathbb{R}$)

Examples of subsequences are plentiful indeed. In the Euclidean space $\mathbb{R}$, the following are a list of sequences and some associated subsequences.

1. The sequence (a_n) where $a_n :\equiv n$, whose terms are $0, 1, 2, 3, \ldots$, has subsequences such as
   i. (a_{k_n}) where $k(n) :\equiv$ "the $(n+1)^{\text{th}}$ prime", the terms of the subsequence here are $2, 3, 5, 7, 11 \ldots$
   ii. (a_{j_n}) where $j(n) :\equiv 2n$, the terms of the subsequence here are $0, 2, 4, 6, \ldots$
   iii. (a_{i_n}) where $i(n) :\equiv n + 100$, the terms of the subsequence here are $100, 101, 102, 103, \ldots$
2. The sequence (b_n) where $b_n :\equiv n/(n+1)$, whose terms are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$, has subsequences such as
   i. (b_{k_n}) where $k(n) :\equiv$ "the $(n+1)^{\text{th}}$ prime", the terms of the subsequence here are $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12} \ldots$
   ii. (b_{j_n}) where $j(n) :\equiv 2n$, the terms of the subsequence here are $0, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \ldots$
   iii. (b_{i_n}) where $i(n) :\equiv n + 100$, the terms of the subsequence here are $\frac{100}{101}, \frac{101}{102}, \frac{102}{103}, \frac{103}{104}, \ldots$
3. The sequence (c_n) where $c_n :\equiv (-1)^n$, whose terms are $1, -1, 1, -1, \ldots$, has subsequences such as
   i. (c_{k_n}) where $k(n) :\equiv$ "the $(n+1)^{\text{th}}$ prime", the terms of the subsequence here are $1, -1, -1, -1, \ldots$
   ii. (c_{j_n}) where $j(n) :\equiv 2n$, the terms of the subsequence here are $1, 1, 1, 1, \ldots$
   iii. (c_{i_n}) where $i(n) :\equiv n+100$, the terms of the subsequence here are $1, -1, 1, -1, \ldots$

Question 4. Last time we looked at the sequences (a_n), (b_n), and (c_n) we decided whether they were convergent, divergent, constant, eventually constant, or none of these. *Without proving anything*, for each sequence in this list guess what type of sequence it is.

Let's warm up by relating constant-ness to subsequences as follows.

Lemma 5 (subsequence of eventually constant is eventually constant)**.** *If (a_n) is an eventually constant sequence in a metric space (X, d), and (a_{k_n}) is a subsequence of (a_n) defined by a strictly increasing function $k\colon \mathbb{N} \to \mathbb{N}$, then (a_{k_n}) is eventually constant too.*

Complete the proof here

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Question 6. Why are all subsequences of a constant sequence themselves constant? Can you find a non-eventually-constant sequence with a constant subsequence?

Complete the proof here

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Now we're ready to establish some results connecting convergence and subsequences. Try to prove the following lemma and its corollary.

Lemma 7 (subsequence of convergent means convergent)**.** *If (a_n) is a convergent sequence in a metric space (X, d) and (a_{k_n}) is a subsequence then (a_{k_n}) is convergent.*

Complete the proof here

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Corollary 8 (divergent subsequence means divergent)**.** *If (a_n) is a sequence in a metric space (X, d) and (a_{k_n}) is a divergent subsequence then (a_n) is divergent.*

Complete the proof here

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At this point we have worked out some of the relationships between convergence and subsequences. Try to complete the tables below.

If (a_n) is	Convergent	Divergent
Then (a_{k_n}) can be	only convergent (lemma 7)	?

If (a_{k_n}) is	Convergent	Divergent
Then (a_n) can be	?	only divergent (corollary 8)

Although somewhat counter-intuitive, there are sometimes metric spaces wherein every sequence has a convergent subsequence.

Lemma 9 (finite sets force convergent subsequences)**.** *If (X, d) is a metric space and $F \subseteq X$ is a finite subset of X then any sequence (a_n) in F must have a convergent subsequence.*

Hint 10. Reduce this problem to proving that every sequence must have a constant subsequence. Why is this enough?

Complete the proof here

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