MATH 301: HOMEWORK 8

Problem 1. Let $n, k \in \mathbb{N}$ with $k \leq n$. Find a bijection between $\binom{[n]}{k}$ and $\binom{[n]}{n-k}$ to deduce that $\binom{n}{k} = \binom{n}{n-k}$.

Problem 2. Use double counting to prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

for all $n \in \mathbb{N}$.

Problem 3. Use the pigeonhole principle to show that given 100 integers, one can choose 15 of them such that the difference of any two is divisible by 7.

Problem 4. For X a finite set and $m_1, m_2, \ldots, m_k \in \mathbb{N}$, denote by $\binom{X}{m_1\ m_2\ \ldots\ m_k}$ the set of k-tuples $(U_1, \ldots, U_k)$ of pair-wise disjoint subsets of X with $|U_i| = m_i$. Let $n, k, m_1, \ldots, m_k \in \mathbb{N}$ be integers. Show that

$$\left|\binom{[n]}{m_1\ m_2\ \ldots\ m_k}\right| = \begin{cases} 0 & m_1 + \cdots + m_k > n \\ \frac{n!}{m_1! \cdot m_2! \cdot \cdots \cdot m_k! \cdot (n - m_1 - m_2 - \cdots - m_k)!} & \text{otherwise} \end{cases}$$

Problem 5.

a) Find the number of surjective maps from [5] to [4].
b) Find the number of injective maps from [3] to [4].

Problem 6. How many natural numbers less than 100 are multiples of 2, 3 or 5?

Problem 7.

a) Let $n \in \mathbb{N}$. Show that the set of subsets of $\mathbb{N}$ of size n is countable.
b) Show that the set of finite subsets of $\mathbb{N}$ is countable.

Problem 8. Consider the set X of all subset $U \subset \mathbb{N}$ such that neither U nor $\mathbb{N} \setminus U$ is finite. Show that X is uncountable.