MATH 301: HOMEWORK 7

Problem 1.

a) Let $A = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} | b \neq 0\}$ be the set of pairs of integers (a,b) where $b \neq 0$. Show that the relation $\sim$ on A given by

$$(a,b) \sim (a',b') \quad \text{if} \quad ab' = a'b$$

is an equivalence relation.

b) Find a bijection between the set of rational numbers and the quotient set $A/\sim$

Problem 2. Let $f : X \to Y$ be a function. Define a relation $\sim_f$ on X by

$$x \sim_f y \quad \Leftrightarrow f(x) = f(y).$$

a) Show that $\sim_f$ is an equivalence relation.

b) Construct a function $g : X/\sim_f \to Y$ such that

$$f = g \circ h$$

where h is the quotient function $h : X \to X/\sim_f$ given by $h(x) = [x]_{\sim_f}$.

c) Show that g is an injection.

Problem 3. Let n be a natural number and define an equivalence relation $\sim_n$ on $\mathbb{Z}$ by

$$a \sim_n b \text{ if } n|(a-b).$$

We showed in class that $\sim_n$ is an equivalence relation. Denote the quotient set $\mathbb{Z}/\sim_n$ by C_n.

a) Show that the assignment

$$[a]_{\sim_n} + [b]_{\sim_n} = [a+b]_{\sim_n}$$

defines an addition function $+ : C_n \times C_n \to C_n$. In other words, show that the equivalence class $[a+b]$ only depends on the equivalence classes $[a], [b]$ and not on the representatives a, b.

b) Similarly, show that the assignment

$$[a]_{\sim_n} \cdot [b]_{\sim_n} = [a \cdot b]_{\sim_n}$$

defines a multiplication function $\cdot : C_n \times C_n \to C_n$

c) Fill in the multiplication table for C_4.

d) Prove that for any integer m, the remainder of m^2 when divided by 4 is not 3.

TABLE 1. C_4 Multiplication table

	[0]	[1]	[2]	[3]
[0]				
[1]				
[2]				
[3]				

Problem 4. Let $\mathcal{U} = \{U_i \subset X | i \in I\}$ be a partition of a set X. Show that the relation $\sim_{\mathcal{U}}$ given by

$$x \sim_{\mathcal{U}} y \quad \Leftrightarrow \quad \exists i \in I, x \in U_i \wedge y \in U_i$$

is an equivalence relation.

Problem 5. Let X be a finite set with $|X| = n > 1$. Let $a \in X$ and $b \notin X$. Prove that

a) $X \setminus \{a\}$ is finite and $|X \setminus \{a\}| = n - 1$,
b) $X \cup \{b\}$ is finite and $|X \cup \{b\}| = n + 1$.

Problem 6. Let X be a finite set and $U \subset X$. Prove that

$$|X \setminus U| = |X| - |U|.$$

Problem 7. Let X, Y be disjoint finite sets of size n and m respectively. Let $f : [n] \to X$ and $g : [m] \to Y$ be enumeration functions of X and Y. Prove that the function h defined by

$$h : [n+m] \to X \cup Y$$
$$a \mapsto \begin{cases} f(a) & a \leq n \\ g(a-n) & a > n \end{cases}$$

is an enumeration of $X \cup Y$.

Problem 8. Let X, Y be finite sets of size n and m respectively. Find, with a proof, the number of relations from X to Y?