MATH 301: HOMEWORK 5

Problem 1. Let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) Show that if f and g are surjective, then $g \circ f$ is surjective.

Solution. Suppose that f, g are surjective and let $z \in Z$. We will show that there exists $x \in X$ with $g \circ f(x) = z$. Since g is surjective, there exists an element $y \in Y$ such that $g(y) = z$. Since f is surjective, there exists an element $x \in X$ such that $f(x) = y$. Then $g \circ f(x) = g(f(x)) = g(y) = z$, as desired.

(b) Show that if $g \circ f$ is surjective, then g is surjective.

Solution. Suppose that $g \circ f$ is surjective and let $z \in Z$. We will show that there exists $y \in Y$ such that $g(y) = z$. Since $g \circ f$ is surjective, there exists an element $x \in X$ such that $g \circ f(x) = z$. Letting $y = f(x)$, we see that $g(y) = g(f(x)) = g \circ f(x) = z$, as desired.

Problem 2. Let $f : X \to Y$ be a function and $V \subset Y$. Prove or find a counterexample for each of the following assertions.

(a) $g(V) \subset f^{-1}(V)$ for every left inverse g of f.

Solution. This statement is false. As a counterexample, let $X = \{1\}$ and $Y = \{1, 2\}$. Define functions f, g by $f(x) = 1$ and $g(y) = 1$ for all $x \in X$ and $y \in Y$. It's easy to verify that g is a left inverse of f. Letting $V = \{2\}$, we have

$$g(\{2\}) = \{1\} \not\subset \emptyset = f^{-1}(\{2\})$$

.

(b) $f^{-1}(V) \subset g(V)$ for every left inverse g of f.

Solution. This statement is true. Let $x \in f^{-1}(V)$; we will prove that $x \in g(V)$. By the definition of the preimage, we have $f(x) \in V$. By the definition of left inverse $x = g(f(x))$. Since $f(x) \in V$, we have $x \in g(V)$, as desired.

(c) $f^{-1}(V) \subset g(V)$ for every right inverse g of f.

Solution. This statement is false. As a counterexample, let $X = \{1, 2\}$ and $Y = \{1\}$. Define functions f, g by $f(x) = 1$ and $g(y) = 1$ for all $x \in X$ and $y \in Y$. It's easy to verify that g is a right inverse of f. Letting $V = \{1\}$, we have

$$f^{-1}(\{1\}) = \{1, 2\} \not\subset \{1\} = g(\{1\})$$

.

(d) $g(V) \subset f^{-1}(V)$ for every right inverse g of f.

Solution. This statement is true. Let $x \in g(V)$; we will prove that $x \in f^{-1}(V)$. By definition of the set image, there exists $y \in V$ such that $g(y) = x$. By the definition of right inverse $y = f(g(y))$. Hence $f(x) = y$ and therefore $x \in f^{-1}(V)$.

Problem 3.

(a) Let $f : X \to Y$ be a function. Suppose that f has a left inverse g_L and a right inverse g_R. Show that $g_L = g_R$.

Solution. Let $y \in Y$. We have

$$g_L(y) = g_L(f(g_R(y))) = g_R(y)$$

where the first equality follows from the fact that $y = f(g_R(y))$, since g_R is a right inverse of f, and the second equality follows from the fact that g_L is a left inverse of f. Since $g_L(y) = g_R(y)$ for all $y \in Y$, the two functions are equal.

(b) Let $f : X \to Y$ be a bijection. Show that a function $g : Y \to X$ is a right inverse of f if and only if it is a left inverse of f. Moreover, show that such function is unique. In this case, we denote this function $f^{-1} : Y \to X$.

Solution. Since f is a bijection, it is both an injection and a surjection. By propositions 3.2.28 and 3.2.37, f has a left inverse g_L and a right inverse g_R. If g is a right inverse to f, then $g = g_L$ by part a) and hence also a left inverse. If g is a left inverse of f, then by part a), $g = g_R$ and therefore also a right inverse. Therefore g is a right inverse of f if and only if it is a left inverse of f.

There exists a left inverse of f, since g_L is a left inverse. If g, g' are two left inverses of f, then by part a), $g = g_R$ and $g' = g_R$. In particular, $g = g'$. It follows that there is a unique left inverse of f.

(c) Let $f : X \to Y$ be a function. Show that if f has a two-sided inverse, then it is bijective.

Solution. If g is a two-sided inverse of f, then f is an injection since it has a left inverse and a surjection since it has a right inverse, hence it is a bijection.

Problem 4. Let X, Y, Z be sets. Find a bijection (with proof) between $X \times (Y \times Z)$ and $X \times Y \times Z$.

Solution. Define functions

$$f : X \times (Y \times Z) \to X \times Y \times Z$$

and

$$g : X \times Y \times Z \to X \times (Y \times Z)$$

by the following formulas,

$$f((x,(y,z))) = (x,y,z);$$
$$g((x,y,z)) = ((x,(y,z))).$$

It is straightforward to verify that $f \circ g = id$ and $g \circ f = id$. Hence g is a two-sided inverse of f and by part c) of question 3, f is a bijection.

Problem 5. For each of the following pairs of sets X, Y, construct a bijective function (with proof) $f : X \to Y$.

(a) $X = \mathbb{Z}, Y = \mathbb{N}$

Solution. We will construct a function $f : \mathbb{Z} \to \mathbb{N}$ and prove that it is a bijection by exhibiting a two-sided inverse g of f. Define

$$f(a) = \begin{cases} 2a & a \geq 0 \\ -2a-1 & a < 0 \end{cases}$$

$$g(n) = \begin{cases} n/2 & n \text{ is even} \\ -\frac{n+1}{2} & n \text{ is odd} \end{cases}$$

We first show that $g \circ f(a)$ for all $a \in \mathbb{Z}$ and hence g is a left inverse of f. Let $a \in \mathbb{Z}$. If $a \geq 0$, then

$$g(f(a)) = g(2a) = a.$$

If $a < 0$, then $f(a) = -2a - 1$ is odd. Therefore

$$g(f(a)) = g(-2a-1) = -\frac{-2a-1+1}{2} = a.$$

In both cases, we have $g(f(a)) = a$.

We now show that $f \circ g(n) = n$ for all $n \in \mathbb{N}$ and hence g is a right inverse of f. Let $n \in \mathbb{N}$. If n is even, then

$$f(g(n)) = f(n/2) = n.$$

If n is odd, then

$$f(g(n)) = f(-\frac{n+1}{2}) = -2 \cdot -\frac{n+1}{2} - 1 = n.$$

In both cases, we have $f(g(n)) = n$.

(b) $X = \mathbb{R}, Y = (-1, 1)$

Solution. We again construct a function $f : \mathbb{R} \to (-1, 1)$ and then find its two-sided inverse g. Define functions f and g by $f(x) = \frac{x}{1+|x|}$ and $g(x) = \frac{y}{1-|y|}$. Note that $1 + |x| > |x|$ and therefore $|f(x)| < 1$ for all $x \in \mathbb{R}$ and hence defines a function with codomain $(-1, 1)$. It is straight forward to verify that g is a two-sided inverse of f.

Problem 6. Let $f : X \to \mathcal{P}(X)$ be a function where $\mathcal{P}(X)$ is the power set of X. By considering the set $B = \{x \in X | x \notin f(x)\}$, prove that f is not surjective.

Solution. We claim that there does not exist an element $x \in X$ such that $f(x) = B$. Assume towards a contradiction that such element x exists. Then there are two cases, $x \in B$ or $x \notin B$.

- Case 1: Assume that $x \in B$. By definition of B, $x \notin f(x)$. Since $f(x) = B$, we have $x \notin B$. This is a contradiction.
- Case 2: Assume that $x \notin B$. By definition of B, $x \in f(x)$. Since $f(x) = B$, we have $x \in B$. This is again a contradiction.

Problem 7. Given $m \in \mathbb{N}$, define m^n for all $n \in \mathbb{N}$ by recursion on n, and prove that $2^2 = 4$.

Solution. We define

$$m^0 = 1; \quad m^{s(n)} = m^n \cdot m$$

where we use the fact that we already defined the multiplication function in class. Recall that $2 = s(s(0))$ and $4 = s(s(s(s(0))))$. We therefore have

$$\begin{aligned} 2^2 = s(s(0))^{s(s(0))} &= s(s(0))^{s(0)} \cdot s(s(0)) \\ &= (s(s(0))^0 \cdot s(s(0))) \cdot (s(s(0))) \\ &= (1 \cdot s(s(0))) \cdot (s(s(0))) \\ &= s(s(0)) \cdot s(s(0)) = s(s(s(s(0)))) \end{aligned}$$

where in the last line we used that we already know that $1 \cdot n = n$ for all n and that $2 \cdot 2 = 4$. Equivalently, we can write the same computation as

$$2^2 = 2^1 \cdot 2 = (2^0 \cdot 2) \cdot 2 = (1 \cdot 2) \cdot 2 = (2) \cdot 2 = 4$$