MATH 301: HOMEWORK 5

Problem 1. Let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) Show that if f and g are surjective, then $g \circ f$ is surjective.
(b) Show that if $g \circ f$ is surjective, then g is surjective.

Problem 2. Let $f : X \to Y$ be a function and $V \subset Y$. Prove or find a counterexample for each of the following assertions.

(a) $g(V) \subset f^{-1}(V)$ for every left inverse g of f.
(b) $f^{-1}(V) \subset g(V)$ for every left inverse g of f.
(c) $f^{-1}(V) \subset g(V)$ for every right inverse g of f.
(d) $g(V) \subset f^{-1}(V)$ for every right inverse g of f.

Problem 3.

(a) Let $f : X \to Y$ be a function. Suppose that f has a left inverse g_L and a right inverse g_R. Show that $g_L = g_R$.
(b) Let $f : X \to Y$ be a bijection. Show that a function $g : Y \to X$ is a right inverse of f if and only if it is a left inverse of f. Moreover, show that such function is unique. In this case, we denote this function $f^{-1} : Y \to X$.
(c) Let $f : X \to Y$ be a function. Show that if f has a two-sided inverse, then it is bijective.

Problem 4. Let X, Y, Z be sets. Find a bijection (with proof) between $X \times (Y \times Z)$ and $X \times Y \times Z$.

Problem 5. For each of the following pairs of sets X, Y, construct a bijective function (with proof) $f : X \to Y$.

(a) $X = \mathbb{Z}, Y = \mathbb{N}$
(b) $X = \mathbb{R}, Y = (-1, 1)$

Problem 6. Let $f : X \to \mathcal{P}(X)$ be a function where $\mathcal{P}(X)$ is the power set of X. By considering the set $B = \{x \in X | x \notin f(x)\}$, prove that f is not surjective.

Problem 7. Given $m \in \mathbb{N}$, define m^n for all $n \in \mathbb{N}$ by recursion on n, and prove that $2^2 = 4$.