MATH 301: HOMEWORK 4

Problem 1. Let X and Y be sets. Prove that $X \subset Y$ if and only if $X \cup Y = Y$.

Problem 2. Let X, Y be sets. Prove that $X \setminus (X \setminus Y) = X \cap Y$.

Problem 3. Let U, V, W be sets. Prove
(a) $U \cap (V \cup W) = (U \cap V) \cup (U \cap W)$
(b) $U \cup (V \cap W) = (U \cup V) \cap (U \cup W)$

(Hint: Write out the logical formulas expressing that an element x belongs to each of the sets above)

Problem 4. Prove the De Morgan's laws for sets A, X, Y and a family of sets $\{X_i | i \in I\}$
(a) $A \setminus (X \cup Y) = (A \setminus X) \cap (A \setminus Y)$
(b) $A \setminus (X \cap Y) = (A \setminus X) \cup (A \setminus Y)$
(c) $A \setminus \bigcup_{i \in I} X_i = \bigcap_{i \in I} (A \setminus X_i)$
(d) $A \setminus \bigcap_{i \in I} X_i = \bigcup_{i \in I} (A \setminus X_i)$

Problem 5. Find a family of sets $\{X_n | n \in \mathbb{N}\}$ such that the following properties are satisfied
(a) $\bigcup_{n \in \mathbb{N}} X_n = \mathbb{N}$
(b) $\bigcap_{n \in \mathbb{N}} X_n = \emptyset$
(c) For all $i \neq j \in \mathbb{N}$, $X_i \cap X_j = \emptyset$.

Problem 6. For which of the following specifications of sets $X, Y, G \subset X \times Y$ is G a graph of a function?
(a) $X = \mathbb{R}, Y = \mathbb{R}, G = \{(a, a^2) | a \in \mathbb{R}\}$
(b) $X = \mathbb{R}, Y = \mathbb{R}, G = \{(a^2, a) | a \in \mathbb{R}\}$
(c) $X = \mathbb{R}_{\geq 0}, Y = \mathbb{R}_{\geq 0}, G = \{(a^2, a) | a \in \mathbb{R}_{\geq 0}\}$
(d) $X = \mathbb{Q}_{\geq 0}, Y = \mathbb{Q}_{\geq 0}, G = \{(a^2, a) | a \in \mathbb{Q}_{\geq 0}\}$
(e) $X = \mathbb{N}, Y = \mathbb{N}, G = \{(x, y) \in \mathbb{N} \times \mathbb{N} | x \text{ divides } y\}$
(f) $X = \mathbb{N}_{\geq 1}, Y = \mathbb{N}, G = \{(x, y) \in \mathbb{N}_{\geq 1} \times \mathbb{N} | y \text{ is the greatest power of 2 dividing } x\}$

Problem 7. Let $f : X \to Y$ be a function. Prove or find a counterexample for the following assertions
(a) $U \subset f^{-1}(f(U))$ for all $U \subset X$
(b) $f^{-1}(f(U)) \subset U$ for all $U \subset X$
(c) $V \subset f(f^{-1}(V))$ for all $V \subset Y$
(d) $f(f^{-1}(V)) \subset V$ for all $V \subset Y$