MATH 301: HOMEWORK 3

Problem 1. Prove that

$$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (\neg p \Rightarrow \neg q)$$

What proof strategy does this equivalence suggest for proving a statement of the form $p \Leftrightarrow q$.

Problem 2. Prove using a truth table that $p \Rightarrow q \not\equiv q \Rightarrow p$. Give an example of propositions p, q such that $p \Rightarrow q$ is true but $q \Rightarrow p$ is false.

Problem 3. Maximally negate the following logical formula and then prove that it is true or prove that it is false

$$\exists x \in \mathbb{R}, [x > 1 \wedge (\forall y \in \mathbb{R}, [x < y \Rightarrow x^2 \leq y])].$$

Problem 4. Let X be $\mathbb{Z}$ or $\mathbb{Q}$ and define a logical formula p by

$$\forall x \in X, \exists y \in X, (x < y \wedge [\forall z \in X, \neg(x < z \wedge z < y)]).$$

Describe what p asserts about the set X. Find the maximally negated logical formula equivalent to $\neg p$. Prove that p is true when $X = \mathbb{Z}$ and false when $X = \mathbb{Q}$.

Problem 5. Which of the following formulas are equivalent?

a) $p \Rightarrow (q \Rightarrow r)$
b) $q \Rightarrow (p \Rightarrow r)$
c) $(p \Rightarrow q) \wedge (p \Rightarrow r)$
d) $(p \wedge q) \Rightarrow r$
e) $p \Rightarrow (q \wedge r)$

Problem 6. The set $\{x \in \mathbb{R} | x^2 < x\}$ is equal to an interval. Find this interval and prove the equality of the two sets.

Problem 7. Identify the set

$$\{r \in \mathbb{N} |\ 0 \leq r < 3 \wedge [\exists n \in \mathbb{Z}, \exists q \in \mathbb{Z}, n^2 = 3q + r]\}.$$

Problem 8. Prove that a natural number is divisible by 3 if and only if the sum of its base 10 digits is divisible by 3. (Hint: You might need the fact that for a natural number n, $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$.)