MATH 301: HOMEWORK 2

Problem 1. Find logical formulae that represent each of the following statements.

(1) There is no greatest odd integer.
(2) If an integer has a rational square root, then that root is an integer.
(3) If the product of two integers is negative, then one of the integers is negative, but not both.
(4) There is an integer that is divisible by every integer.

Problem 2. Find the statements in plain English, involving as few variables as possible, that are represented by each of the following logical formulae.

(1) $\forall n \in \mathbb{N}, [(\exists q \in \mathbb{Z}, 4q = n) \Rightarrow (\exists q \in \mathbb{Z}, 2q = n)]$.
(2) $\forall d \in \mathbb{N}, [(\exists q \in \mathbb{Z}, n = qd) \Rightarrow (d = 1 \vee d = n)]$.
(3) $\forall a \in \mathbb{R}, [a > 0 \Rightarrow (\exists b \in \mathbb{R}, (b > 0 \wedge a < b))]$.
(4) $\exists a \in \mathbb{Z}, [(\exists q \in \mathbb{Z}, 2q = a) \wedge (\exists q \in \mathbb{Z}, 2q + 1 = a)]$.

Problem 3. Prove

$$p \vee (q \wedge r) \Rightarrow (p \vee q) \wedge (p \vee r)$$

by constructing a proof tree whose premise is $p \vee (q \wedge r)$ and whose conclusion is $(p \vee q) \wedge (p \vee r)$.

Problem 4. Let $x \in \mathbb{R}$. Prove by contradiction that if x is irrational, then $-x$ and $1/x$, if $x \neq 0$, are irrational.

Problem 5. Prove that, for all real numbers x and y, if x is irrational, then $x + y$ and $x - y$ are not both rational.

Problem 6. Write the logical formula corresponding to the following statement and then prove it.

"There is no smallest positive real number."

Can you identify the proof techniques and rules of inference that you used in your proof?

Problem 7. Let X be a set and let $p(x)$ be a predicate. Find a logical formula representing the assertion that there are exactly two elements in X for which $p(x)$ is true. Use the structure of this logical formula to prove that there are exactly two real numbers x such that $x^2 = 1$.