M405 - HOMEWORK SET #9- SOLUTIONS

6.1.8 The derivative of g, if it exists, equals

$$g'(x) = \lim_{h\to 0} \frac{1}{h}(g(x+h) - g(x)) \tag{1}$$

$$= \lim_{h\to 0} \frac{1}{h} \int_0^1 \left(f((x+h)y) - f(xy)\right) y^2 dy \tag{2}$$

Since f is C^1, for any x, we have

$$f(x+z) = f(x) + f'(x)z + R_x(z)$$

for some error function R_x. Therefore

$$f((x+h)y) - f(xy) = f'(xy)hy + R_{xy}(hy)$$

and by (2)

$$g'(x) = \lim_{h\to 0} \frac{1}{h} \int_0^1 \left(f'(xy)hy + R_{xy}(hy)\right) y^2 dy$$
$$= \int_0^1 f'(xy)y^3 dy + \lim_{h\to 0} \frac{1}{h} \int_0^1 R_{xy}(hy)y^2 dy$$

We claim that the second term vanishes, which we now prove.

Let x be fixed and assume, for notational simplicity, that $x > 0$. We show that

$$\lim_{h\to 0} \frac{1}{h} \int_0^1 R_{xy}(hy)y^2 dy = 0$$

Since f' is continuous, it is uniformly continuous on $[0, x+1]$. Given $\frac{1}{m}$, let $\frac{1}{n}$ be such that for $|z_0 - z_1| < \frac{1}{n}$ $|f'(z_0) - f'(z_1)| < \frac{1}{m}$. For any y, h we have

$$f((x+h)y) - f(xy) = f'(z^*)(hy)$$

for some z^* between xy and $xy + hy$. Therefore

$$R_{xy}(hy) = f((x+h)y) - f(xy) - f'(xy)hy$$
$$= \left(f'(z^*) - f'(xy)\right) hy$$

If $|h| < \frac{1}{n}$, since $|y| \leq 1$, we have $|z^* - xy| < h < \frac{1}{n}$ and hence

$$|R_{xy}(hy)| \leq \frac{1}{m}|h|y$$

It follows that if $|h| < \frac{1}{n}$,

$$\left|\frac{1}{h} \int_0^1 R_{xy}(hy)y^2 dy\right| \leq \frac{1}{m} \int_0^1 y^3 dy.$$

Since $\int_0^1 y^3 dy$ is a constant, it follows that

$$\lim_{h\to 0} \frac{1}{h} \int_0^1 R_{xy}(hy)y^2 dy = 0.$$

We showed that

$$g'(x) = \int_0^1 f'(xy)y^3 dy.$$

It remains to show that g' is continuous. Fix $M > 0$ and restrict the functions f, f', g, g' to $[0, M]$. We show that g' is uniformly continuous on $[0, M]$. Since f' is continuous on $[0, M]$, it is uniformly continuous. Given $\frac{1}{m}$, let $\frac{1}{n}$ be such that for all $x_1, x_2 \in [0, M]$, $|f'(x_1) - f'(x_2)| < \frac{1}{m}$. Then for all $x_1, x_2 \in [0, M]$ with $|x_1 - x_2| < \frac{1}{n}$ we have

$$\begin{aligned} |g'(x_1) - g'(x_2)| &= |\int_0^1 f'(x_1 y)y^3 dy - \int_0^1 f'(x_2 y)y^3 dy| \\ &= |\int_0^1 (f'(x_1 y) - f(x_2 y))y^3 dy| \\ &\leq \int_0^1 |(f'(x_1 y) - f(x_2 y))|y^3 dy \\ &= \int_0^1 \frac{1}{m} y^3 dy = \frac{1}{4m} < \frac{1}{m} \end{aligned}$$

where we used the fact that $0 \leq y \leq 1$ to deduce that $|x_1 y - x_2 y| \leq |x_1 - x_2| < \frac{1}{n}$.

Or, you could have used Theorem 6.1.7 although we have not proved it.

6.1.14 a) Fix $x \in (a, b)$ and assume that $f(x) \neq 0$. (If $f(x) = 0$ the statement follows trivially). By the mean value theorem, there exist $y_1 < x < y_2$ such that

$$\begin{aligned} f(x) &= f'(y_1)(x - a) \\ f(x) &= f'(y_2)(x - b) \end{aligned}$$

In particular, $f'(y_1)$ and $f'(y_2)$ have different signs and therefore by the intermediate value theorem, there exists z with $y_1 < z < y_2$ such that $f'(z) = 0$. Since

$$|y_1 - z| + |y_2 - z| = |y_1 - y_2| < b - a$$

either $|y_2 - z| \leq |x - a|$ or $|y_1 - z| \leq |x - b|$. There are now two cases to check

Case 1 If $|y_2 - z| \leq |x - a|$, then by the intermediate value theorem, there exists w such that $f'(y_2) = f''(w)(y_2 - z)$ and hence

$$\begin{aligned} |f(x)| &= |f'(y_2)(x - b)| = |f''(w)(y_2 - z)(x - b)| \\ &\leq M_2 |y_2 - z||x - b| \leq M_2 |x - a||x - b| \end{aligned}$$

Case 2 If $|y_1 - z| \leq |x - b|$, then by the intermediate value theorem, there exists w such that $f'(y_1) = f''(w)(y_1 - z)$ and hence

$$\begin{aligned} |f(x)| &= |f'(y_1)(x - a)| = |f''(w)(y_1 - z)(x - a)| \\ &\leq M_2 |y_1 - z||x - a| \leq M_2 |x - b||x - a| \end{aligned}$$

b) Let f be a function with $|f''(x) \leq M_2$ for all x. We would like to compare the value of the integral $\int_a^b f(x)dx$ with the trapezoidal rule approximation of the integral with respect to some partition $P = x_i$. Consider a particular interval in the partition $[x_{i-1}, x_i]$. The difference between the integral and the trapezoid gule over this partition is

$$\int_{x_{i-1}}^{x_i} f(x)dx - \frac{1}{2}(f(x_i) + f(x_{i-1}))(x_i - x_{i-1}). \tag{3}$$

We can rewrite the second term as the integral of the linear function $f(x_{i-1}) + \frac{f(x_i)-f(x_{i-1}))}{x_i - x_{i-1}}x$ and hence (3) becomes

$$\int_{x_{i-1}}^{x_i} f(x) - \left(f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1}))}{x_i - x_{i-1}}x\right) dx \tag{4}$$

In this equation, the integrand vanishes on the boundary points and its second derivative coincides with the second derivative of f. By part a) of this quesion, the integrand is bounded by $M_2(x_i - x)(x - x_{i-1})$ and (4) is bounded by

$$\int_{x-i}^{x_i} M_2(x_i - x)(x - x_{i-1})dx = M_2\frac{(x_i - x_{i-1})^3}{6}$$

To evaluate the integral, note that the result only depends on $x_i - x_{i-1}$, so assuming $x_{i-1} = 0$, the integral simplifies to

$$\int_0^{x_i} (x_i x - x^2)dx = \frac{x_i^3}{2} - \frac{x_i^3}{3} = \frac{x_i^3}{6}.$$

Hence if P is a partition such that $(x_i - x_{i-1}) < \delta$, we have

$$\begin{aligned}
&\int_a^b f(x)dx - \sum_{k=1}^{n} \frac{1}{2}\left(f(x_i) + f(x_{i-1})\right)(x_i - x_{i-1}) \\
&\le \sum_{k=1}^{n} M_2\frac{(x_i - x_{i-1})^3}{6} \le \sum_{k=1}^{n} M_2\delta^2\frac{(x_i - x_{i-1})}{6} \\
&= \frac{M_2\delta^2}{6}(b-a)
\end{aligned}$$

6.2.1 Let f^* be the restriction of f to $[a, b]$. We show that $Osc(f^*, P) \to 0$ as $|P| \to 0$. Since f is integrable, $Osc(f, P') \to 0$ as $|P'| \to 0$. Given $\frac{1}{m}$, let $\frac{1}{n}$ be such that $Osc(f, P') < \frac{1}{m}$ for all partitions P' of $[a, c]$ such that $|P'| < \frac{1}{n}$. For any partition P of $[a, b]$ such that $|P| < \frac{1}{n}$ let P' be an extension of P to $[a, c]$ such that $|P'| < \frac{1}{n}$ as well. We have

$$Osc(f^*, P) \le Osc(f, P') < \frac{1}{n}$$

6.2.4 Since f, g are Riemann integrable on $[a, b]$, they are in particular bounded. Let $|f(x)| \le M_1$ and $|g(x)| < M_2$ for all x. The oscillation $Osc(f \cdot g, P)$ is the sum of the terms

$$\sup_{x\in[x_{i-1},x_i]} f(x)g(x) - \inf_{x\in[x_{i-1},x_i]} f(x)g(x)$$

multiplied by $(x_i - x_{i-1})$. These terms can equivalently be expressed as

$$\sup_{x,y\in[x_{i-1},x_i]} |f(x)g(x) - f(y)g(y)|$$

By triangle inequality,

$$\begin{aligned}
&\sup_{x,y\in[x_{i-1},x_i]} |f(x)g(x) - f(y)g(y)| \\
&= \sup_{x,y\in[x_{i-1},x_i]} |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&= \sup_{x,y\in[x_{i-1},x_i]} |f(x)||g(x) - g(y)| + |g(y)|f(x) - f(y)| \\
&\leq M_1 \sup_{x,y\in[x_{i-1},x_i]} (g(x) - g(y)) + M_2 \sup_{x,y\in[x_{i-1},x_i]} (f(x) - f(y))
\end{aligned}$$

It follows that $Osc(f \cdot g, P) \leq M_1 Osc(g, P) + M_2 Osc(f, P)$ from which it follows that $Osc(f \cdot g, P) \to 0$ when $|P| \to 0$ if f and g are Riemann integrable.

6.2.6 Consider $h(x) = f(x) - g(x)$. We have $h(x) = 0$ for all but finitely many x. Hence h is Riemann integrable. Then $g = f(x) - h(x)$ is Riemann integrable being the sum of two Riemann integrable functions.

6.2.7 We will construct a partition P with an arbitrarily small value of $Osc(g \circ f, P)$. For simplicity, we will only consider equidistant partitions, i.e., for $N \in \mathbb{N}$, we take $x_i = a + (b - a) \cdot \frac{i}{N}$. Letting M_i, m_i be the supremum and infinum of f over the ith interval we have

$$Osc(f, P) = \frac{a-b}{N} \sum_{i=1}^{N} M_i - m_i$$

and this quantity goes to 0 as N goes to infinity since f is Riemann integrable. If all of the differences $M_i - m_i$ got small as N increased then using uniform continuity of g we could easily argue that $Osc(g \circ f, P)$ also goes to 0 when N goes to infinity. In general though, some of the terms $M_i - m_i$ might remain large, for example if f is discontinuous. Only few of the terms $M_i - m_i$ can remain large though and we would need to bound contributions from those intervals separately. We will need a quantitative measure of how many of the terms in $M_i - m_i$ might remain large.

Let M_i' and m_i' be the supremum and infinum of $g \circ f$ over the ith interval. Let $2|g(x)| \leq M$ for all x. Then $|M_i' - m_i'| \leq M$ for all i. Given $\epsilon > 0$, let $\delta > 0$ be such that if $|y_1 - y_0| < \delta$ then $|g(y_1) - g(y_2)| < \frac{\epsilon}{2}$. Hence if for some i we have $M_i - m_i < \delta$, then $M_i' - m_i' < \frac{\epsilon}{2}$.

Let $K \in \mathbb{N}$ be such that $K > \frac{2M}{\epsilon}$. Let P be an equidistant partition of $[a, b]$ with NK subintervals such that

$$Osc(f, P) = \frac{a-b}{NK} \sum_{i=1}^{NK} M_i - m_i < \frac{(b-a)\delta}{K}$$

This inequality implies that at most N of the terms $M_i - m_i$ can be bigger than δ since otherwise the left hand side would add up to a number bigger than $\frac{(b-a)\delta}{K}$. The corresponding terms in $Osc(g \circ f, P)$ will contribute at most

$$N \cdot \frac{a-b}{NK} \cdot M \leq \frac{(b-a)\epsilon}{2}$$

since there will be at most N terms each one of which will be of the form $\frac{a-b}{NK}(M_i' - m_i')$ and $M_i' - m_i' < M$. The remaining terms in $Osc(g \circ f, P)$ will have $(M_i' - m_i') < \frac{\epsilon}{2}$

since for those i, $M_i - m_i < \delta$. Those will contribute at most

$$NK \cdot \frac{a-b}{NK} \cdot \frac{\epsilon}{2} = \frac{(b-a)\epsilon}{2}$$

Therefore

$$Osc(g \circ f, P) \leq (b-a)\epsilon$$

Since ϵ can be chosen arbitrarily small, the result follows.

6.2.10 Let $\lim_{x \to x_0^+} f(x) = f^+(x_0)$ which we assume exists. Then

$$\lim_{x \to x_0^+} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \to x_0^+} \frac{1}{x - x_0} \int_{x_0}^{x} f(t)dt$$

Since $\lim_{x \to x_0^+} f(x) = f^+(x_0)$ given any $\epsilon > 0$ if $x > x_0$ is sufficiently close to x_0 then $|f(x) - f^+(x_0)| < \epsilon$. For all such x we have

$$\begin{aligned} |\frac{1}{x-x_0} \int_{x_0}^{x} f(t)dt - f^+(x_0)| &= |\frac{1}{x-x_0} \int_{x_0}^{x} f(t) - f^+(x_0)dt| \\ &\leq \frac{1}{x-x_0}(x-x_0) \cdot \epsilon = \epsilon \end{aligned}$$

In other words, for $x > x_0$ close enough to x_0, $\frac{1}{x-x_0} \int_{x_0}^{x} f(t)dt$ is within ϵ of $f^+(x_0)$. We thus get

$$\lim_{x \to x_0^+} \frac{F(x) - F(x_0)}{x - x_0} = f^+(x_0)$$

Completely analogously, for $f^-(x_0) = \lim_{x \to x_0^-}$, we have

$$\lim_{x \to x_0^-} \frac{F(x) - F(x_0)}{x - x_0} = f^-(x_0)$$

This shows that if f is continuous at x_0, i.e. $f^-(x_0) = f^+(x_0) = f(x_0)$, then $F'(x_0) = f(x_0)$ and if $f^-(x_0) \neq f^+(x_0)$, then the limit defining the derivative of F at x_0 does not exist.