M405 - HOMEWORK SET #8 - SOLUTIONS

5.3.1 An easy computation shows that

$$f'(x) = \begin{cases} kx^{k-1} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which is a continuous function.

5.3.3 We first construct a C^1 function f whose zero set is the complement of $(0,1)$. The idea is to integrate a piece-wise linear function. We will define f with the desired property such that

$$f'(x) = \begin{cases} 2x & 0 < x \leq \frac{1}{4} \\ 1 - 2x & \frac{1}{4} < x \leq \frac{3}{4} \\ -2 + 2x & \frac{3}{4} < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

which is continuous. To be more explicit, define f by

$$f(x) = \begin{cases} x^2 & 0 < x \leq \frac{1}{4} \\ -\frac{1}{8} + x - x^2 & \frac{1}{4} < x \leq \frac{3}{4} \\ 1 + x^2 - 2x & \frac{3}{4} < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Given an open interval $I = (a,b)$, let f_I be the C^1 function defined by $f_I(x) = f(\frac{x-a}{b-a})$. The zero set of f_I is the complement of I.

Given a closed set A, the complement of A is a countable union of disjoint open intervals $\{I_k\}_{k \in K}$. The function

$$g(x) = \sum_{k \in K} f_{I_k}(x)$$

is a C^1 function whose zero set is A. Notice that for each x, at most one $f_{I_k}(x)$ is non-zero, so the above sum does indeed define a function on $\mathbb{R}$.

5.3.7 The derivative of $g(x) = x^k$ is $g'(x) = k \cdot x^{k-1}$ and hence for all $x > 0$, $g'(x) > 0$. By the inverse function theorem, for any $0 < a < b$, the restriction of g to (a,b) is invertible. Since $\lim_{x \to 0} x^k = 0$ and $\lim_{x \to \infty} x^k = 0\infty$, for any $y > 0$, there exist $a < b$ such that $a^k < y$ and $b^k > y$. Hence y is in the image of the restriction of g to (a,b) and there exists a unique $g^{-1}(y) \in (a,b)$ such that $g(g^{-1}(y)) = y$. The value $g^{-1}(y)$ clearly does not depend on the choice of (a,b). This defines g^{-1} on $(0,\infty)$. Since $x = 0$ is the unique value such that $x^k = 0$, defining $g^{-1}(0) = 0$ defines the desired function $f(x) = x^{1/k}$ on $[0,\infty]$. By the inverse function theorem, the derivative of g^{-1} is given by

$$(g^{-1})'(y) = \frac{1}{g'(g^{-1}(y))} = \frac{1}{k \cdot (y^{1/k})^{k-1}} = \frac{1}{k} y^{-\frac{k-1}{k}}$$

5.4.2 Assume that $f(x_0) = 0$. If that is not the case, we can subtract $f(x_0)$ from f without affecting its derivatives or whether or not x_0 is a local maximum or a local minimum.

Since f is C^n, $f^{(n)}$ is continuous. Since $f^{(n)}(x_0) > 0$, there exists an open interval $(x_0 - \epsilon, x_0 + \epsilon)$ for some $\epsilon > 0$ on which $f^{(n)} > 0$. We first show that for all $x \in (x_0, x_0 + \epsilon)$, $f^{(k)}(x) > 0$ for $k = 0, 1, \ldots, n$. We induct on k as k decreases from n to 0. We already know the result for $k = n$. To prove the inductive step, assume that $f^{(k+1)}(x) > 0$ for all $x \in (x_0, x_0 + \epsilon)$. We also have $f^{(k)}(x_0) = 0$. By the Mean Value Theorem, for any $x \in (x_0, x_0 + \epsilon)$, there exists $x' \in (x_0, x)$ such that

$$f^{(k)}(x) = f^{(k+1)}(x')(x - x_0) > 0.$$

This completes the inductive step and we have thus showed that $f(x) > 0$ for $x \in (x_0, x_0 + \epsilon)$ We now prove an analogous statement for $x \in (x_0 - \epsilon, x_0)$. In particular, we show that for $k = 0, 1, \ldots, n$ and $x \in (x_0 - \epsilon, x_0)$ we have $f^{(k)}(x) > 0$ if $n - k$ is even and $f^{(k)}(x) < 0$ if $n - k$ is odd. We again induct on k as k decreases from n to 0. The case $k = n$ is the base case and we prove the inductive step. Assume that for all $x \in (x_0 - \epsilon, x_0)$, $f^{(k+1)}(x) > 0$ if $n - k - 1$ is even and $f^{(k+1)} < 0$ if $n - k - 1$ is odd. By the Mean Value Theorem, for every $x \in (x_0 - \epsilon, x_0)$ there exists $x' \in (x, x_0)$ such that

$$f^{(k)}(x) = f^{(k+1)}(x')(x - x_0)$$

The sign of $x - x_0$ is negative and therefore the sign of $f^{(k)}(x)$ is opposite of the sign of $f^{(k+1)}(x')$. In particular, $f^{(k)}(x) > 0$ if $n-k$ is even and $f^{(k)}(x) < 0$ if $n-k$ is odd. This completes the inductive step. We have showed that if n is even, then $f(x) > 0$ for all $x \in (x_0 - \epsilon, x_0)$ and if n is odd, then $f(x) < 0$ for all $x \in (x_0 - \epsilon, x_0)$. This shows that if n is even, then x_0 is a local minimum and if n is odd, x_0 is neither a local maximum nor a local minimum.

5.4.3 Let

$$g_h(x) = \frac{f(x+h) - f(x)}{h}.$$

Note that

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \frac{g_h(x) - g_h(x-h)}{h}$$

By the Mean Value Theorem, there exists x_1 between x and $x - h$ such that

$$\frac{g_h(x) - g_h(x-h)}{h} = g_h'(x_1).$$

Now

$$g_h'(x_1) = \frac{f'(x_1 + h) - f'(x_1)}{h}$$

and by the Mean Value Theorem applied to f'

$$g_h'(x_1) = f''(x_2)$$

for some x_2 between x_1 and $x_1 + h$. Since f'' is continuous and $x_2 \xrightarrow{x}$ and $h \to 0$, we conclude that

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$$

5.4.4 To simplify notation, assume that $x_0 = 0$. We will not need to full strength of the statement that f, g are C^3; we will only use that f, g are thrice differentiable and f''', g''' are bounded in some neighborhood of 0. Let $h(x) = \frac{f(x)}{g(x)}$. Since $g'(0) \neq 0$, there is some neighborhood U of 0 such that g does not vanish on $U \backslash \{0\}$. In particular, h is well defined on $U \backslash \{0\}$. We first show that $h(0)$ can be defined as the limit $\lim_{x\to 0} h(x)$ which in fact equals $\frac{f'(0)}{g'(0)}$. We will use the mean value theorem many times in this argument, and in order to not introduce new symbols every time we apply the mean value theorem, we will denote a "some point" between 0 and x by $\bullet$. For example, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(\bullet)$$

and the only thing we will use about $f'''(\bullet)$ is that it is bounded by some $M > 0$ which is independent of x.

Having introduced this notation, the rest is fairly straight forward. For $x \neq 0$, we have

$$h(x) = \frac{f'(0)x + \frac{f''(\bullet)}{2}x^2}{g'(0)x + \frac{g''(\bullet)}{2}x^2}$$

where we have used that $f(0) = g(0) = 0$. Dividing by x, noting that $g''(\bullet)$ and $f''(\bullet)$ are bounded, and taking the limit as $x \to 0$, we get

$$\lim_{x\to 0} h(x) = \frac{f'(0)}{g'(0)}$$

which we define to be the value of $h(0)$.

We now evaluate $h'(0)$. We have

$$\begin{aligned} h'(0) &= \lim_{x\to 0} \frac{1}{x} \cdot \left(\frac{f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(\bullet)}{3!}x^3}{g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(\bullet)}{3!}x^3} - \frac{f'(0)}{g'(0)} \right) \\ &= \lim_{x\to 0} \frac{1}{x} \cdot \left(\frac{g'(0)(f'(0)x + \frac{(f''(0)}{2}x^2 + \frac{f'''(\bullet)}{3!}x^3)) - f'(0)(g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(\bullet)}{3!}x^3)}{g'(0)(g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(\bullet)}{3!}x^3)} \right) \end{aligned}$$

Noting that $g'(0)f'(0)x$ appears with both, a positive sign and a negative sign in the numerator and that that is the only term with x raised to power 1, we get

$$h'(0) = \lim_{x\to 0} \frac{g'(0)\frac{f''(0)}{2} - f'(0)\frac{g''(0)}{2} + \dots}{g'(0)^2 + \dots} = \frac{1}{2} \cdot \frac{g'(0)f''(0) - f'(0)g''(0)}{g'(0)^2}$$

where by $\dots$ we denote the terms with non-zero power of x.

It remains to show that $\lim_{x\to 0} h'(x) = h'(0)$. By the quotient rule for derivatives, we have

$$h'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g(x)^2}$$

We apply the mean value theorem to all 4 functions: $f(x), g(x), f'(x), g'(x)$ to get

$$h'(x) = \frac{(g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(\bullet)}{3!}x^3)(f'(0) + f''(0)x + \frac{f'''(\bullet)}{2}x^2)}{(g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(\bullet)}{3!}x^3)^2} - \frac{(g'(0) + g''(0)x + \frac{g'''(\bullet)}{2}x^2)(f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(\bullet)}{3!}x^3)}{(g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(\bullet)}{3!}x^3)^2}$$

This expression is pretty horrid, but recall that we are interested in the limit as $x \to 0$ and so we are only interested in the lowest powers of x, which in this case is x^2 since $g'(0)f'(0)x$ cancels out in the numerator. We thus get

$$\begin{aligned}\lim_{x\to 0} h'(x) &= \lim_{h\to 0} \frac{x^2(g'(0)f''(0) + \frac{g''(0)}{2}f'(0) - g'(0)\frac{f''(0)}{2} - g''(0)f'(0))) + \dots}{g'(0)^2x^2 + \dots} \\ &= \frac{1}{2}\frac{g'(0)f''(0) + g''(0)f'()}{g'(0)^2}\end{aligned}$$

5.4.13 Let $f(x) = \sum_{k=0}^{n} a_k(x-x_0)^k + R_f(x-x_0)$ and $g(x) = \sum_{j=0}^{n} b_j(y-y_0)^j + R_g(y-y_0)$ where both R_f and R_g are $o(|x|^n)$. We then have

$$\begin{aligned} g\circ f(x) &= \sum_{j=0}^{n} b_j(f(x)-y_0)^j + R_g(f(x)-y_0) \\ &= \sum_{j=0}^{n} b_j(\sum_{k=1}^{n} a_k(x-x_0)^k + R_f(x-x_0))^j + R_g(f(x)-y_0)\end{aligned}$$

This expression has all the terms we want, but also many that we don't. It thus suffices to show that the terms we don't want are $o(|x-x_0|^n)$. It's clear for terms that contain $(x-x_0)$ raised to a power greater than n and the terms which have $R_f(x-x_0)$ as a factor. The only remaining term is $R_g(f(x)-y_0)$.

Since $f(x)-y_0 = a_1(x-x_0)+o(x-x_0)$, for any a_1' such that $a_1' > |a_1|$, there exists $\frac{1}{n}$ such that $|f(x)-y_0| < a_1'|x-x_0|$ for all x such that $|x-x_0| < \frac{1}{n}$. We then have that for all such x,

$$\begin{aligned}\frac{R_g(f(x)-y_0)}{|x-x_0|^n} &= \frac{R_g(f(x)-y_0)}{|f(x)-y_0|^n}\cdot\frac{|f(x)-y_0|^n}{|x-x_0|^n} \\ &\le \frac{R_g(f(x)-y_0)}{|f(x)-y_0|^n}\cdot(a_1')^n\end{aligned}$$

Since $(f(x)-y_0) \to 0$ as $x \to 0$, and $R_g = o(|x|^n)$, it follows that

$$\lim_{x\to x_0}\frac{R_g(f(x)-y_0)}{|x-x_0|^n} = 0$$

and hence $R_g(f(x)-y_0) = o(|x-x_0|^n)$.

5.4.20 Let $f(x) = \sqrt{x}$ and let us find the second order Taylor polynomial of f at 100. We have

$$f(x) = x^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$$
$$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$

and so

$$f(100) = 10$$
$$f'(100) = \frac{1}{20}$$
$$f''(100) = -\frac{1}{4000}$$
$$f'''(100) = \frac{3}{800000}$$

So

$$f(100+1) = 10 + \frac{1}{20} - \frac{1}{2 \cdot 4000} + \frac{1}{3!}f'''(x')$$

for some x' between 100 and 101. The function f''' is decreasing, so the maximal value $f'''(x')$ can attain is $f'''(100) = \frac{3}{800000}$. Therefore

$$10 + \frac{1}{20} - \frac{1}{2 \cdot 4000} < \sqrt{101} < 10 + \frac{1}{20} - \frac{1}{2 \cdot 4000} + \frac{3}{800000}$$

Pretty good estimate if you ask me.

6.1.2 The absolute value is a continuous function so it commutes with limits. Let P_i be a sequence of partitions such that $|P_k| \to 0$. Then

$$\begin{aligned} |\int_a^b f(x)dx| &= |\lim_{k\to\infty} S(f, P_k, \{x^*_{k,i}\})| \\ &= \lim_{k\to\infty} |S(f, P_k, \{x^*_{k,i}\})| \\ &= \lim_{k\to\infty} |\sum_{i=1}^n f(x^*_i)(x_{k,i} - x_{k,i-1})| \\ &\le \lim_{k\to\infty} \sum_{i=1}^n |f(x^*_i)|(x_{k,i} - x_{k,i-1}) \\ &= \lim_{k\to\infty} S(|f|, P_k, \{x^*_{k,i}\}) \\ &= \int_a^b |f(x)|dx \end{aligned}$$

6.1.4 Since f is continuous on a compact domain $[a, b]$, it attains its maximum M and minimum m. Moreover, by the intermediate value theorem, $f([a, b]) = [m, M]$, i.e., every value between m and M is in the image of f. Since $f(x) - m \geq 0$ and $f(x) - M \leq 0$ we get

$$0 \leq \int_a^b f(x) - m dx = \int_a^b f(x)dx - m(b-a)$$
$$m \leq \frac{\int_a^b f(x)dx}{b-a}$$

Similarly,

$$0 \geq \int_a^b f(x) - M dx = \int_a^b f(x)dx - M(b-a)$$
$$M \geq \frac{\int_a^b f(x)dx}{b-a}$$

Hence by the intermediate value theorem, there exists $y \in [a,b]$ such that

$$f(y) = \frac{\int_a^b f(x)dx}{b-a}$$

6.1.5 We use Theorem 6.1.7 to compute

$$f'(x) = (x-x)g(x) + \int_a^x g(t)dt = \int_a^x g(t)dt$$
$$f''(x) = g(x).$$

Also

$$f(a) = \int_a^a (a-t)g(g)dt = 0$$
$$f'(a) = \int_a^a g(t)dt = 0.$$