M405 - HOMEWORK SET #7- SOLUTIONS

5.1.1 Assume $f(x) = O(|x-x_0|^2)$ as $x \to x_0$ and let $\frac{1}{m}$ be given. Let $c > 0$ and $\frac{1}{n}$ be such that $|f(x)| < c \cdot |x-x_0|^2$ whenever $|x - x_0| < \frac{1}{n}$. Fix an integer $n' > \max(n, mc)$. Then for all x such that $|x-x_0| < \frac{1}{n'}$ we have

$$|f(x)| < c \cdot |x-x_0|^2 < c \cdot \frac{1}{mc} \cdot |x-x_0| = \frac{|x-x_0|}{m}$$

and hence $f(x) = o(|x-x_0|)$.

To see that the converse is not true, let $f(x) = |x-x_0|^{3/2}$. Then $f(x) = o(|x-x_0|)$ but $f(x) \neq O(|x-x_0|^2)$.

5.1.3 Assume that $f(x) = O(|x-x_0|^k)$ and $g(x) = o(|x-x_0|^j)$ and let $\frac{1}{m}$ be given. Let $c > 0$ and $\frac{1}{n}$ be such that $|f(x)| < c \cdot |x-x_0|^k$ whenever $|x-x_0| < \frac{1}{n}$. Let n' be such that $|g(x)| < \frac{|x-x_0|^j}{cm}$ whenever $|x-x_0| < \frac{1}{n'}$. Let $n'' = \max(n, n')$. Then for all x such that $|x-x_0| < \frac{1}{n''}$ we have

$$|f(x)g(x)| < c \cdot |x-x_0|^k \cdot \frac{|x-x_0|^j}{cm} = \frac{|x-x_0|^{k+j}}{m}$$

In particular, $f(x)g(x) = o(|x-x_0|^{k+j})$.

5.1.9 By the definition of the derivative of f at x_0, there exists a function R such that

(1) $$f(x) = f(x_0) + f'(x_0)(x-x_0) + R(x-x_0)$$

and $R(x) = o(|x|)$ as $x \to 0$. Let g_M be the function whose graph is the zoom with the magnification factor of M of f at x_0. Implicitly, it is defined by $f(x_0+x/M) = y_0 + \frac{y}{M}$. Solving for y in the above equation we obtain the explicit formula for g_M:

$$g_M(x) = Mf(x_0 + \frac{x}{M}) - Mf(x_0).$$

Plugging in equation 1, we get

$$\begin{aligned} g_M(x) &= M\left(f(x_0) + f'(x_0)(\frac{x}{M}) + R(\frac{x}{M})\right) - Mf(x_0) \\ &= f'(x_0)x + M \cdot R(\frac{x}{M}) \end{aligned}$$

The first term gives us precisely the equation of the straight line with slope $f'(x_0)$. We thus need to show $M \cdot R(\frac{x}{M})$ converges to 0 as $M \to \infty$. We will discuss the notion of the convergence of sequences of functions later in the semester in more detail. For the purpose of this exercise, let's prove the point-wise convergence, i.e. for a fixed $x_0 \in \mathbb{R}$, we will show that $\lim_{M\to\infty} M \cdot R(\frac{x_0}{M}) = 0$. To that end, let $\frac{1}{m}$ be given. Let $\frac{1}{n}$ be such that for all x such that $|x| < \frac{1}{n}$, we have $|R(x)| < \frac{|x|}{|x_0|m}$. Let M' be such that $\frac{x_0}{M'} < \frac{1}{n}$. Then for all $M > M'$ we have $\frac{x_0}{M'} < \frac{1}{n}$ and hence

$$|M \cdot R(\frac{x_0}{M})| < M \cdot \left|\frac{x_0/M}{m|x_0|}\right| = \frac{1}{m}$$

Hence $\lim_{M\to\infty} M \cdot R(\frac{x_0}{M}) = 0$.

5.2.2 For any x_0, we have

$$\left|\frac{f(x+h)-f(x)}{h}\right| \leq \frac{M|h|^\alpha}{|h|} = M|h|^{\alpha-1}$$

Since $\alpha > 1$, $\lim_{h\to 0}|h|^{\alpha-1} = 0$ and hence $f'(x_0) = 0$ for all x_0. In particular, f is constant.

5.2.3 The converse of the mean value theorem is not true. A counterexample is the function $f(x) = x^3$ and the point $x_0 = 0$. We have $f'(x_0) = 0$ but since f is strictly increasing, there are no distinct points x_1, x_2 such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$.

5.2.6 We will prove the contrapositive of the statement. Assume $f'(x) \geq 0$ on (a,b) but f is not strictly increasing. We will show that f' vanishes on an interval in (a,b). Since f is not strictly increasing, there exist points $x_1 < x_2$ in the domain of f such that $f(x_1) \geq x_2$. Since $f'(x) \geq 0$ for all $x \in (a,b)$, by Theorem 5.2.2 a, f is monotone increasing, and therefore $f(x_1) \leq f(x_2$. We conclude that $f(x_1) = f(x_2)$. Moreover, since f is monotone increasing, for every $x \in (x_1, x_2)$, we have

$$f(x_1) \leq f(x) \leq f(x_2)$$

and since $f(x_1) = f(x_2)$, we also have $f(x) = f(x_1)$. In particular, f is constant on (x_1, x_2) and hence f' vanishes on (x_1, x_2).

5.2.8 Since f' is continuous on the closed interval $[c,d]$, it is uniformly continuous. In particular, given $\frac{1}{m}$, there exists $\frac{1}{n}$ such that if $|x_1 - x_2| < \frac{1}{n}$, we have $|f'(x_1) - f'(x_2)| < \frac{1}{m}$. Let x, x_0 be such that $|x - x_0| < \frac{1}{n}$. By the mean value theorem, there exists x' between x and x_0 such that $f(x) - f(x_0) = f'(x')(x - x_0)$. Since x' lies between x and x_0, we also have $|x_0 - x'| < \frac{1}{n}$. Putting all this together, we get

$$\begin{aligned}
|f(x) - f(x_0) - f'(x_0)(x - x_0)| &= |f'(x')(x - x_0) - f'(x_0)(x - x_0)| \\
&= |f'(x') - f'(x_0)| \cdot |x - x_0| \\
&< \frac{1}{m} \cdot |x - x_0|.
\end{aligned}$$

Which shows that f is uniformly differentiable on $[c,d]$.