M405 - HOMEWORK SET #6- SOLUTIONS

4.1.1 Let $f : D \to \mathbb{R}$ be a function defined on a closed domain D. We show that f is continuous if and only if the inverse image of every closed set is closed.

Assume first that f is continuous and let $B \subset \mathbb{R}$ be closed. We would like to show that $f^{-1}(B)$ is closed. We will show that $f^{-1}(B)$ contains all its limit points. Let x be a limit point of $f^{-1}(B)$. Then there exists a sequence $\{x_i\}$ in $f^{-1}(B)$ converging to x. Since $\{x_i\}$ is also a sequence in D, which is closed, $x \in D$. Since f is continuous, the sequence $\{f(x_i)\}$ converges to $f(x)$. Since $f(x_i) \in B$ for all i and B is closed, $f(x) = \lim_{i\to\infty} f(x_i)$ is in B. Hence $x \in f^{-1}(B)$.

Assume that the preimage $f^{-1}(B)$ is closed for every closed $B \subset \mathbb{R}$. We show that f is continuous at every $x \in D$. Given $\frac{1}{n}$, let $B = (f(x) - \frac{1}{n}, f(x) + \frac{1}{n})^c$ be the complement of the open interval centered at $f(x)$ of width $\frac{2}{n}$. The preimage $f^{-1}(B)$ is closed and hence the complement $(f^{-1}(B))^c$ is open. Since $f(x) \notin B$, x is contained in $(f^{-1}(B))^c$. Therefore there exists an open interval $(x - \frac{1}{m}, x + \frac{1}{m}) \subset (f^{-1}(B))^c$. This implies that for all $x \in (x - \frac{1}{m}, x + \frac{1}{m})$, either $x \notin D$ or $f(x) \in (f(x) - \frac{1}{n}, f(x) + \frac{1}{n})$. Hence for any $\frac{1}{n}$ there exists $\frac{1}{m}$ such that for all $y \in D$ with $|y - x| < \frac{1}{m}$, $|f(x) - f(y)| < \frac{1}{n}$. Hence f is continuous at x.

4.1.3 Define sets $A_i = f_i^{-1}([0, \infty])$. Since $[0, \infty]$ is closed, each set A_i is closed by exercise 1. The set A, of points x satisfying $f_i(x) \geq 0$ for all i, is the intersection of the sets A_i. Since the intersection of closed sets is closed, A is closed.

The sets $A_i' = f_i^{-1}((0, \infty))$ on the other hand are open, being the preimages of open sets by a continuous function with open domain. The intersection of the sets A_i' is open, being a finite intersection of open sets, and consists of points satisfying $f_i(x) > 0$ for all i.

4.1.14 We show that an element x is contained in $f^{-1}(A \cup B)$ if and only if it is contained in $f^{-1}(A) \cup f^{-1}(B)$ by a series of equivalences:

$$
\begin{aligned}
x \in f^{-1}(A \cup B) &\iff f(x) \in A \cup B \\
&\iff f(x) \in A \text{ or } f(x) \in B \\
&\iff x \in f^{-1}(A) \text{ or } x \in f^{-1}(B) \\
&\iff x \in f^{-1}(A) \cup f^{-1}(B)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
x \in f^{-1}(A \cap B) &\iff f(x) \in A \cap B \\
&\iff f(x) \in A \text{ and } f(x) \in B \\
&\iff x \in f^{-1}(A) \text{ and } x \in f^{-1}(B) \\
&\iff x \in f^{-1}(A) \cap f^{-1}(B)
\end{aligned}
$$

The analogous statements for images of sets instead of preimages of sets is true for the unions but not true for the intersection. Let $A, B \subset D$ be two subsets of the domain of a function f. Then $f(A\cup B)=f(A)\cup f(B)$. Indeed,

$$\begin{aligned} y\in f(A\cup B) &\iff \exists x\in A\cup B, f(x)=y\\ &\iff (\exists x\in A, f(x)=y) \text{ or } (\exists x\in B, f(x)=y)\\ &\iff y\in f(A) \text{ or } y\in f(B)\\ &\iff y\in f(A)\cup f(B)\end{aligned}$$

For the intersection of images of sets, consider the following counterexample. Let f be the function $f(x)=x^2$ and let A, B be the sets

$$A=(-2,-1);\quad B=(1,2).$$

Then $A\cap B=\emptyset$ while

$$f(A)=f(B)=f(A)\cap f(B)=(1,4).$$

4.1.15 Let $f:(a,b)\to\mathbb{R}$ be a uniformly continuous function. We would like to extend f to the closed interval $[a,b]$ by defining $f(a)=\lim_{x\to a}f(x)$ and $f(b)=\lim_{x\to b}f(x)$. If these limits exist, then the resulting function is continuous since a function is continuous if and only if $f(x_0)=\lim_{x\to x_0}f(x)$ for all x_0 limit points of the domain. By Theorem 4.1.1, the limit $\lim_{x\to a}f(x)$ exists if and only if for every sequence $\{x_i\}$ in (a,b) converging to a, the sequence $\{f(x_i)\}$ is convergent. We now show that in this case $\{f(x_i)\}$ is Cauchy, and hence convergent. Given $\frac{1}{n}$, since f is uniformly continuous, there exists $\frac{1}{m}$ such that for all $x,y\in(a,b)$ with $|x-y|<\frac{1}{m}$, we have $|f(x)-f(y)|<\frac{1}{n}$. Since $\{x_i\}$ is convergent, it is Cauchy. Therefore there exists $l\in\mathbb{N}$ such that for all $i,j>l$, $|x_i-x_j|<\frac{1}{m}$. We then have $|f(x_i)-f(x_j)|<\frac{1}{n}$ for all $i.j>l$ and hence $\{f(x_i)\}$ is Cauchy. The limit $\lim_{x\to b}f(x)$ exists by the same argument.

It remains to show that the extended function is uniformly continuous on the closed interval $[a,b]$. Given $\frac{1}{n}$, let $\frac{1}{m}$ be such that for all $x,y\in(a,b)$ with $|x-y|<\frac{1}{m}$ we have $|f(x)-f(y)|<\frac{1}{2n}$. We claim that for all $x,y\in[a,b]$ such that $|x-y|<\frac{1}{m}$ we have $|f(x)-f(y)|<\frac{1}{n}$. The statement is clear if neither of the x,y lie on the boundary. Suppose $x=a$ and $y\in(a,b)$ and let $\{x_i\}$ be a sequence in (a,y) converging to a. Since $|a-y|<\frac{1}{m}$, also $|x_i-y|<\frac{1}{m}$ for all i. Therefore

$$|f(x_i)-f(y)|<\frac{1}{2n}.$$

Since limits preserve non-strict inequality, and the absolute value function is continuous, we have

$$|f(a)-f(y)|=|\lim_{i\to\infty}f(x_i)-f(y)|=\lim_{i\to\infty}|f(x_i)-f(y)|\le\frac{1}{2n}<\frac{1}{n}.$$

We only case we have not considered yet is when both x,y are the end-points. We can avoid discussing this case by increasing the value of m if necessary so that $\frac{1}{m}<b-a$. In this case, if $|x-y|<\frac{1}{m}$, then x,y cannot be the the opposite endpoints of $[a,b]$.

4.2.3 For $a,b\in\mathbb{R}$, let $I_{a,b}$ be $[a,b]$ or $[b,a]$ according to whether $a\le b$ or $b\le a$. Intervals are sets A characterized by the property that if $a,b\in A$, then $I_{a,b}\subset A$. Suppose $f:D\to\mathbb{R}$ is a continuous function whose domain D is an interval. Let $a,b\in f(D)$ and let $x,y\in D$ be such that $f(x)=a, f(y)=b$. Since D is an interval, the closed interval $I_{x,y} \subset D$ is contained in D. Applying the intermediate value theorem to f restricted to $I_{x,y}$, we obtain that $I_{a,b} \subset f(I_{x,y}) \subset f(D)$. Hence $f(D)$ is an interval.

If $D = (0,1)$ and $f(x) = x$, then clearly $f(D)$ is an open interval.

4.2.8 We will show that for some large value x_0, the values $f(x_0)$ and $f(-x_0)$ are non-zero and have opposite signs. In that case, applying the Intermediate value theorem to the restriction of f to $[-x_0, x_0]$ implies that there exists $x \in (-x_0, x_0)$ such that $f(x) = 0$. To show the existence of x_0, we will use the fact that an odd degree polynomial $p(x)$ satisfies $\lim_{x\to\pm\infty} |p(x)| = \infty$ with the sign of $p(x)$ and $p(-x)$ being opposite for x large enough. Adding a bounded function to p does not change this property. We now present a more careful proof of that.

Without loss of generality, we may assume that the leading coefficient of p is 1. If it is not, we can divide f by the leading coefficient. Let k odd be the degree of p. We have

$$p(x) = x^k + c_{k-1}x^{k-1} + \dots c_1 x + c_0$$

for some constants c_i. We have

$$\lim_{x\to\pm\infty} \frac{|c_{k-1}x^{k-1} + \dots c_1 x + c_0|}{|x^k|} = \lim_{x\to\pm\infty} |c_{k-1}x^{-1} + \dots c_1 x^{-k+1} + c_0 x^{-k}| = 0.$$

Let N be such that for all x such that $|x| \geq N$ we have

$$\frac{|c_{k-1}x^{k-1} + \dots c_1 x + c_0|}{|x^k|} < \frac{1}{2}$$

or equivalently

$$|c_{k-1}x^{k-1} + \dots c_1 x + c_0| < \frac{1}{2}|x^k|.$$

By the reverse triangle inequality, it follows that for all such x we have

$$\begin{aligned}|p(x)| &\geq |x^k| - |c_{k-1}x^{k-1} + \dots c_1 x + c_0| \\ &> |x^k| - \frac{1}{2}|x^k| = \frac{1}{2}|x^k|\end{aligned}$$

Moreover, the sign of $p(x)$ is the same as the sign of x^k. Since g is bounded, there exists $M > 0$ such that for all $x \in \mathbb{R}$, $|g(x)| < M$. Let $N' = \max(N, \sqrt[k]{4M})$. Then for for all x with $|x| \geq N'$ we have

$$|p(x)| > \frac{1}{2}|x^k| \geq \frac{1}{2}4M = 2M.$$

Letting $f(x) = p(x) + g(x)$, we have that for $|x| \geq N'$,

$$|f(x)| \geq |p(x)| - |g(x)| > 2M - M = M$$

and moreover, the sign of $f(x)$ is the same as the sign of $p(x)$ which is the same as the sign of x^k. Since the sign of x^k is different for $x = N'$ and $x = -N'$, we are done.

4.2.13 Let f, g be continuous functions on $[a,b]$ and $[b,c]$ respectively. Define $h(x)$ on $[a,c]$ by

$$h(x) = \begin{cases} f(x) & a \leq x \leq b \\ g(x) & b < x \leq c \end{cases}$$

Suppose $h(x)$ is continuous. Then $\lim_{x\to b} h(x)$ exists and equals to both $\lim_{x\to b^-} h(x) = \lim_{x\to b} f(x) = f(b)$ and $\lim_{x\to b^+} h(x) = \lim_{x\to b} g(x) = g(b)$. Hence $f(b) = g(b)$.

For the other direction, assume that $f(b) = g(b)$. It is clear that $h(x)$ is continuous at all $x \neq b$. To show continuity of h, it thus suffices to show that h is continuous at b. Given $\frac{1}{n}$ let $\frac{1}{m}$ be such that for all $x \in [a, b]$ with $|x-b| < \frac{1}{m}$ we have $|f(x)-f(b)| < \frac{1}{n}$ and such that for all $x \in [b, c]$ with $|x - b| < \frac{1}{m}$ we have $|g(x) - g(b)| < \frac{1}{n}$. Then for any $x \in [a, c]$ such that $|x - b| < \frac{1}{m}$ we have that either $x \in [a, b]$, in which case

$$|(h(x) - h(b)| = |f(x) - f(b)| < \frac{1}{n}$$

or $x \in [b, c]$, in which case

$$|h(x) - h(b)| = |g(x) - g(b)| < \frac{1}{n}.$$

It follows that h is continuous at b.

4.2.17 The function

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

satisfies the intermediate value property but is not continuous at 0.

A function having the intermediate value property can not have jump discontinuities. Suppose f has a jump discontinuity at x_0. Let $\lim_{x \to x_0^-} f(x) = a$ and $\lim_{x \to x_0^+} f(x) = b$ Assume for concreteness that $a < b$ and let $\delta = b - a$. Let $\frac{1}{n}$ be such that for all $x < x_0$ with $|x_0 - x| \leq \frac{1}{n}$ we have

$$|f(x) - a| < \delta/3$$

and such that for all $x > x_0$ with $|x - x_0| \leq \frac{1}{n}$ we have

$$|f(x) - b| < \delta/3.$$

Then the restriction of f to $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$ does not satisfy the intermediate value property. Indeed,

$$f(x_0 - \frac{1}{n}) \in [a - \delta/3, a + \delta/3],$$
$$f(x_0 + \frac{1}{n}) \in [b - \delta/3, b + \delta/3],$$

but $[a + \delta/3, b - \delta/3]$ is not in the image of f restricted to $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$ except possibly for $f(x_0)$. In other words, for any $y \in [a + \delta/3, b - \delta/3]$ with $y \neq f(x_0)$, there does not exist $x \in [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$ such that $f(x) = y$.