M405 - HOMEWORK SET #5- SOLUTIONS

3.2.1 Let $\{x_1, \ldots, x_n\} \subset A$ be a finite collection of points. Define $A_n = A - \{x_n\} = A \cap (\mathbb{R} - \{x_n\})$, which are open being the intersection of two open sets. Then

$$A - \{x_1, \ldots, x_n\} = \bigcap_{i=1}^{n} A_n$$

is open, being a finite intersection of open sets. It should now also be clear what might go wrong if we remove countable number of points from an open set: the the intersection above might not be open. For a concrete example, let $A = \mathbb{R}$ and $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ be a countable collection of point of A. We have that $A - \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is not open since it contains 0, but no open interval containing 0.

3.2.2 Let x be a limit point of A. We define a subsequence of $\{x_i\}$ whose limit is x. Define an increasing function $k : \mathbb{N} \to \mathbb{N}$ as follows. Let $k(1)$ be such that $|x_{k(1)} - x| < 1$. Such value exists since x is a limit point of A. Assuming we defined $k(i)$ for all $i < n$, we choose $k(n) > k(n-1)$ such that $|x_{k(n)} - x| < \frac{1}{n}$. There are infinitely many points of A satisfying this property which in particular means there are infinitely many terms of $\{x_i\}$ which satisfy this property, and therefore such $k(n)$ exists. It should be clear that $\{x_{k(i)}\}$ is a convergent subsequence of $\{x_i\}$ with limit x.

In the other direction, assume that no point of A occurs more than a finite number of times in the sequence and that x is a limit point of the sequence $\{x_i\}$. Let $\{x_{k(i)}\}$ be a subsequence of $\{x_i\}$ which converges to x. Assume by contradiction that x is not a limit point of A. Then there exists $\frac{1}{n}$ such that $A \cap (x - \frac{1}{n}, x + \frac{1}{n})$ is either empty or equals $\{x\}$. Since $\lim_{i\to\infty} x_{k(i)} = x$, there exists $m \in \mathbb{N}$ such that for all $i > m$, $|x_{k(i)} - x| < \frac{1}{n}$ and hence $x_{k(i)} = x$ for all $i > m$. This contradicts that no point of A occurs more than a finite number of times in the sequence.

3.2.6 Let $A \subset \mathbb{R}$. Define a subset

$$B = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} | \exists a \in A \text{ s.t. } x < a < y\}.$$

The set B is countable since it is a subset of a countable set $\mathbb{Q} \times \mathbb{Q}$. Choose an element $a_{x,y} \in (x, y)$ for every $(x, y) \in B$ and let $A' \subset A$ be the union of all $a_{x,y}$. The set A' is countable since there is an onto map $B \to A'$ sending (x, y) to $a_{x,y}$.

We claim that A' is dense in A, i.e., that $A \subset closure(A')$. The closure of A' consists of A' and all the limit points of A'. We therefore have to show that for every $a \in A$, either $a \in A'$ or it is a limit point of A'. Let $a \in A - A'$. We show that a is a limit point of A'. Given $\frac{1}{n}$, choose $x \in (a - \frac{1}{n}, a) \cap \mathbb{Q}$ and $y \in (a, a + \frac{1}{n}) \cap \mathbb{Q}$. We have that $(x, y) \in B$ since $x < a < y$. Therefore there is $a_{x,y} \in A'$ satisfying $|a_{x,y} - a| < \frac{1}{n}$. Moreover $a_{x,y} \neq a$ since $a \notin A'$. It follows that a is a limit point of A'.

An example of a set A such that the intersection of A with the rational numbers is not dense in A is the set of irrational numbers $\mathbb{R} - \mathbb{Q}$. The intersection is in fact empty.

3.2.8 Let $\{x_i\}$ be a sequence and A be the set of limit points of $\{x_i\}$. We want to show that A is closed. We show that A contains all of its limit points. Let a be a limit point of A. Then for any $\frac{1}{n}$, there exists $x \in A$ with $|x-a| < \frac{1}{2n}$. Since $x \in A$, there are infinitely many terms x_i such that $|x_i - x| < \frac{1}{2n}$. For every such term, by the triangle inequality, we have

$$|x_i - a| \leq |x_i - x| + |x - a| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

Having infinitely many terms x_i satisfying this property for any $\frac{1}{n}$ means a is a limit point of $\{x_i\}$. In particular, $a \in A$ and hence A is closed.

3.2.13 Let $A \subset \mathbb{R}$ and let the derived set A' be the set of limit points of A. We would like to show that A' is closed. Let a be a limit point of A'. We would like to show that $a \in A'$, i.e., a is a limit point of A. We show that given any $\frac{1}{n}$, there is a point $x \in A$, $x \neq a$ such that $|x - a| < \frac{1}{n}$. Since a is a limit point of A', there exists $y \in A'$, not equal to a such that $|y - a| < \frac{1}{2n}$. Since y is a limit point of A, there exists a point $x \in A$ such that $|x - y| < |y - a|/2$. Then

$$|x - a| \leq |x - y| + |y - a| < \frac{|y-a|}{2} + \frac{1}{2n} < \frac{1}{4n} + \frac{1}{2n} < \frac{1}{n}$$

We additionally need to verify that $x \neq a$. For that, we apply the reverse triangle inequality

$$|a - x| \geq |a - y| - |x - y| \geq |a - y| - |y - a|/2 = |y - a|/2 > 0.$$

As per examples:

– $A = \{0\}$ is a closed set whose derived set is empty.

– Let $A = \{\frac{1}{n} | n \in \mathbb{N}\}$. Then $A' = \{0\}$ and $A'' = \emptyset$.

3.3.2 Let us give a name to the property which we would like to show is equivalent to compactness.

Definition 1. We say that a set A satisfies property B if for any collection $\mathcal{B}$ of closed sets such that the intersection of any finite number of them contains a point of A, the intersection of all of them contains a point of A.

We now show that a set A is compact if and only if it satisfies property B.

Assume A satisfies property B. Let $\mathcal{U}$ be a collection of open sets which covers A. Consider the collection $\mathcal{B} = \{U^c | U \in \mathcal{U}\}$ of the complements of the sets in $\mathcal{U}$. It is a collection of closed sets. We have

$$\bigcap_{U \in \mathcal{U}} U^c = (\bigcup_{U \in \mathcal{U}} U)^c \subset A^c$$

where the last inclusion follows from the fact that $\mathcal{U}$ is a cover of A. In particular, the intersection of all elements of $\mathcal{B}$ does NOT contain points of A. Since A satisfies property B, there must be a finite sub-collection $\{U_1^c, U_2^c, \ldots, U_k^c\}$ of $\mathcal{B}$ such that $\bigcap_{i=1}^k U_k^c$ does not contain points of A. In other words

$$\bigcap_{i=1}^{k} U_i^c = (\bigcap_{i=1}^{k} U_i)^c \subset A^c$$

and hence the finite subcollection $\{U_1, U_2, \ldots, U_k\}$ of $\mathcal{U}$ covers A.

Assume A is compact and let $\mathcal{B}$ be a collection of closed sets such that the intersection of any finite subcollection of $\mathcal{B}$ contains elements of A. Let $\mathcal{U} = \{B^c | B \in \mathcal{B}\}$.

The condition on $\mathcal{B}$ implies that for any finite subcollection $\{B_1^c, B_2^c, \ldots, B_3^c\}$ of $\mathcal{U}$, we have

$$\bigcup_{i=1}^{k} B_i^c = (\bigcap_{i=1}^{k} B_i)^c \not\supset A$$

since for $a \in A$, $a \in \bigcap_{i=1}^k B_i$ implies $a \notin (\bigcap_{i=1}^k B_i)^c$. Therefore $(\bigcap_{i=1}^k B_i)^c \not\supset A$. In particular, no finite subcollection of $\mathcal{U}$ covers A. Since A is compact, we have that the entire collection $\mathcal{U}$ also does not cover A, i.e.,

$$\bigcup_{B\in\mathcal{B}} B^c = (\bigcap_{B\in\mathcal{B}} B)^c \not\supset A$$

which then implies that $\bigcap_{B\in\mathcal{B}} B$ contains points of A. Therefore A satisfies property B.

3.3.4 If B_1 and B_2 are disjoint and $A \subset B_1 \cup B_2$, then $A \cap B_1 = A \cap B_2^c$. Assume A is compact and hence closed and bounded. Since B_2^c is closed, so is $A \cap B_2^c$. Since A is bounded, so is $A \cap B_2^c$. Therefore, $A \cap B_2^c$ is closed and bounded and hence compact.

3.3.6 Assume A is open and $a+b \in A+B$ with $a \in A, b \in B$. Since A is open, there exists an open interval $U \subset A$ containing a. Then the set $U + \{b\}$ is contained in $A + B$ and is an open interval, being a translation of U by b. Hence $A + B$ is open.

Assume A, B are compact. Let $\{x_i\}$ be a sequence in $A + B$. For each x_i, choose $a_i \in A$ and $b_i \in B$ such that $a_i+b_i = x_i$. Since A is compact, there exists a convergent subsequence $\{a_{k(i)}\}$ of $\{a_i\}$ with the limit in A. Since B is compact, there exists a convergent subsequence $\{b_{l(i)}\}$ of $\{b_{k(i)}\}$ with the limit in B. Then, since $\{a_{l(i)}\}$ is a subsequence of a convergent sequence $\{a_{k(i)}\}$, it is also convergent. We therefore have that $\{x_{l(i)}\}$ is a convergent sequence of $\{x_i\}$. Its limit is $\lim_{i\to\infty} a_{l(i)} + \lim_{i\to\infty} b_{l(i)} \in A+B$. Since an arbitrary sequence in $A + B$ has a convergent subsequence with the limit in $A + B$, the set $A + B$ is compact.

Let $A = \{n + \frac{1}{n+1} | n \in \mathbb{N}\}$ and $B = \{-n | n \in \mathbb{N}\}$. Then both A and B are closed. On the other hand, the set $A + B$ contains elements $\frac{1}{n+1}$ for all $n \in \mathbb{N}$ but not 0 and therefore is not closed.

- Let $\mathcal{U}$ be a collection of open sets which covers A. Then $\mathcal{U} \cup \{A^c\}$ covers B and therefore has a finite subcover $\{U_1, U_2, \ldots, U_k, A^c\}$. Since $A \subset B$, the collection $\{U_1, U_2, \ldots, U_k, A^c\}$ covers A. Moreover, since $A \cap A^c = \emptyset$, removing A^c from the collection, does not change the fact that it covers A. Therefore $\{U_1, \ldots, U_k\}$ is a finite subcover of $\mathcal{U}$. Hence A is compact.