M405 - HOMEWORK SET #4- SOLUTIONS

3.1.1 .

x_n	sup	inf	lim sup	lim inf	limit points
$1/n + (-1)^n$	3/2	-1	1	-1	± 1
$1 + (-1)^n/n$	3/2	0	1	1	1
$(-1)^n + 1/n + 2\sin n\pi/2$	2	-3	1	-3	1, -3

3.1.2 No! For example $x_n = (-1)^n$ is a bounded non-convergent sequence but is the sum $x_n = y_n + z_n$ where $y_n = 3n$ is monotone increasing and $z_n = -3n + (-1)^n$ is monotone decreasing.

If on the other hand $\{y_n\}, \{z_n\}$ are themselves bounded, then they are convergent and therefore so is their sum.

3.1.4 Since $\sup(A)$ is the least upper bound of A, to show that $\sup(A \cup B) \geq \sup(A)$, it suffices to show that $x = \sup(A \cup B)$ is an upper bound of A. We have that for all $y \in A \cup B$, $x \geq y$, hence in particular it is true for all $y \in A$ since $A \subset A \cup B$.

For the same reason, to show that $\sup(A \cap B) \leq \sup(A)$ it suffices to show that $x = \sup(A)$ is an upper bound of $A \cap B$. We have that for all $y \in A$, $x \geq y$. Since $A \cap B \subset A$, it in particular holds for all y in $A \cap B$ and hence x is an upper bound of $A \cap B$.

3.1.5 Recall the definition

$$\limsup x_i = \lim_{m \to \infty} \sup\{x_i | i \geq m\}.$$

Hence for every $\frac{1}{n}$, there exists $m \in \mathbb{N}$ such that $\sup\{x_i | i \geq m\} < \limsup\{x_n\} + \frac{1}{2n}$. By the same argument applied to the sequence $\{y_i\}$, we can pick m so that also $\sup\{y_i | i \geq m\} < \limsup\{y_n\} + \frac{1}{2n}$. Hence for all $i \geq m$,

$$\begin{aligned} x_i + y_i &< \limsup\{x_n\} + \frac{1}{2n} + \limsup\{y_n\} + \frac{1}{2n} \\ &= \limsup\{x_n\} + \limsup\{y_n\} + \frac{1}{n} \end{aligned}$$

Therefore

$$\limsup\{x_i + y_i\} \leq \limsup\{x_n\} + \limsup\{y_n\} + \frac{1}{n}$$

for all $\frac{1}{n}$. By the Archimedean property of real numbers, we have

$$\limsup\{x_i + y_i\} \leq \limsup\{x_n\} + \limsup\{y_n\}$$

An example where the inequality is strict is $x_i = (-1)^i$ and $y_i = (-1)^{i+1}$. In this case, $\limsup\{x_i + y_i\} = 0$ while $\limsup\{x_n\} + \limsup\{y_n\} = 2$.

3.1.8 Let us recall the definition of ∞ being a limit point of a sequence.

Definition 1. Let $\{x_i\}$ be a sequence of real numbers. Then ∞ is a limit point of $\{x_i\}$ if for every $N \in \mathbb{N}$, there are infinitely many terms x_i satisfying $x_i > N$.

Let $\{x_i\}$ be a sequence of real numbers. Suppose that ∞ is a limit point of $\{x_i\}$. We construct a subsequence whose limit is ∞. We define an increasing function $k : \mathbb{N} \to \mathbb{N}$ inductively. Let $k(1) \in \mathbb{N}$ be such that $x_{k(1)} > 1$. It exists since ∞ is a limit point of $\{x_i\}$. Suppose we have defined $k(i)$ for all $i \leq n$. Since ∞ is a limit point of $\{x_i\}$, there are infinitely many j so that $|x_j > n + 1|$. In particular, there exists such j with $j > k(n)$. We let $k(n + 1) = j$. This defines an increasing function, so $\{x_{k(i)}\}$ is a subsequence of $\{x_i\}$. Moreover, since $x_{k(i)} > i$, we have $\lim_{i\to\infty} x_{k(i)} = \infty$.

In the other direction, let $\{x_i\}$ be a sequence of real numbers, $k : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\lim_{i\to\infty} x_{k(i)} = \infty$. We show that ∞ is a limit point of $\{x_i\}$. For $N \in \mathbb{N}$, let $m \in \mathbb{N}$ be such that for all $i > m$, $x_{k(i)} > N$, which exists since $\lim_{i\to\infty} x_{k(i)} = \infty$. In particular, $x_{k(i)}$ with $i > m$ constitute an infinite number of terms of $\{x_i\}$ with the property $x_i > N$.

3.1.9 No! If $1, 1/2, 1/3, \ldots$ are limit points of $\{x_i\}$, then 0 must also be a limit point of $\{x_i\}$, which we now demonstrate. Let $\{x_i\}$ be a sequence such that $1, 1/2, 1/3, \ldots$ are limit points. We construct a subsequence whose limit is 0. We define an increasing function $k : \mathbb{N} \to \mathbb{N}$ inductively. Let $k(1) = 1$. Suppose we have defined $k(i)$ for all $i < n$. Choose $k(n) > k(n-1)$ such that $|x_{k(n)} - \frac{1}{2n}| < \frac{1}{2n}$, which is possible since $\frac{1}{2n}$ is a limit point of $\{x_i\}$. This defines a subsequence $\{x_{k(i)}\}$. Moreover, by triangle inequality, this subsequence satisfies

$$|x_{k(i)}| \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

In particular, $\lim_{i\to\infty} x_{k(i)} = 0$.

3.1.11 If x is a limit point of a row or a column, then in particular, there is a subsequence consisting of the elements of that row or column converging to x. Since such subsequence would also be a subsequence of the sequence obtained by diagonalizing, x is also a limit point of that sequence.

On the other hand, you don't necessary get all the limit points of the total sequence in this manner. For example, consider the case

$$a_{ij} = \begin{cases} 0 & i = j \\ 1 & \text{otherwise} \end{cases}$$

The only limit point of every row and every column is 1, but 0 is a limit point of the total sequence.