# M405 - HOMEWORK SET #3- SOLUTIONS

2.2.2 Let $S_{0,1}$ be the set of sequences of 0s and 1s. We saw in class that $S_{0,1} \cong 2^{\mathbb{N}}$. We show that $\mathbb{R}$ has the same cardinality as $S_{0,1}$. We break up the proof into two smaller steps:

Claim 1: The set $[0,1) \subset \mathbb{R}$ has the same cardinality at $S_{0,1}$, i.e., there is a bijection

$$\phi : [0,1) \xrightarrow{\sim} S_{0,1}.$$

Claim 2: The set $\mathbb{Z} \times S_{0,1}$ has the same cardinality as $S_{0,1}$, i.e., there is a bijection

$$\chi : \mathbb{Z} \times S_{0,1} \xrightarrow{\sim} S_{0,1}.$$

Assuming the two claims above, we get the desired bijection

$$\mathbb{R} \to \mathbb{Z} \times [0,1) \xrightarrow{\mathrm{id}\times\phi} \mathbb{Z} \times S_{0,1} \xrightarrow{\chi} S_{0,1}$$

where the left-most map is $x \mapsto (\lfloor x \rfloor, x - \lfloor x \rfloor)$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

*Proof of Claim 1.* The basic idea is simple: we can express any real number $x \in [0,1)$ in binary expansion as "0." followed by a sequence of 0s and 1s. The detail in which the devil lies is that for some $x \in [0,1)$, there are more ways than one to express $x$ in binary expansion. We first identify such $x$.

Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two distinct sequences of 0s and 1s such that the corresponding real numbers $0.x_1x_2x_3\ldots$ and $0.y_1y_2y_3$ are equal. We claim that up to interchanging $\{x_i\}$ and $\{y_i\}$, there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} x_i &= y_i \qquad \text{for } i < n \\ x_n &= 0; \quad y_n = 1 \\ x_i &= 1; \quad y_i = 0 \qquad \text{for } i > m \end{aligned}$$

In other words, two distinct sequences correspond to the same real number only if they have trailing 0s and trailing 1s following some finite sequence of 0s and 1s. For example

$$0.10111111111\cdots = 0.11000000000\ldots$$

To see why that is true, note that the condition on the sequences translates to

$$\sum_{i=1}^{\infty} 2^{-i} x_i = \sum_{i=1}^{\infty} 2^{-i} y_i$$

Let $n$ be the first position where the sequences differ and assume that $x_n = 0$ and $y_n = 1$, which we can do by interchanging $\{x_i\}$ and $\{y_i\}$ if necessary. We then have

(1) $$\sum_{i=n+1}^{\infty} 2^{-i} x_i = 2^{-n} + \sum_{i=n+1}^{\infty} 2^{-i} y_i.$$

Every term in the above equation is positive. Moreover, we have

$$\sum_{i=n+1}^{\infty} 2^{-i} x_i \leq \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n}$$

with equality holding only if $x_i = 1$ for all $i \geq n+1$. The equality (1) can therefore only hold if $x_i = 1$ for $i \geq n+1$ and $\sum_{i=n+1}^{\infty} 2^{-i} y_i = 0$, i.e. $y_i = 0$ for $i \geq n+1$.

Define the following subsets $A, B, C \subset S_{0,1}$

$$A = \{(x_1, x_2, \dots) \in S_{0,1} | \text{ there exists } n \in \mathbb{N} \text{ s.t. for all } i > n, x_i = 0\}$$
$$B = \{(x_1, x_2, \dots) \in S_{0,1} | \text{ there exists } n \in \mathbb{N} \text{ s.t. for all } i > n, x_i = 1\}$$
$$C = S_{0,1} - (A \cup B)$$

In words, $A$ is the set of sequences with trailing 0s, $B$ is the set of sequences with trailing 1s, and $C$ is the subset of all other sequences. We write down a bijection between $[0,1)$ and $A \cup C$.

(2) $$[0,1) \to A \cup C$$

(3) $$x \mapsto \{x_i\} \quad \text{where } x_i = \lfloor 2^i \cdot x \rfloor \mod 2$$

This is precisely the procedure for obtaining a binary representation of a real number. Notice that it only results in sequences which do not have trailing 1s. Indeed, if $x \in [0,1)$ can be represented by a sequence with trailing 1s, then by the above analysis, $x$ can be written as a finite sum $x = \sum_{i=1}^{n} 2^{-i} x'_i$ and thus in particular $2^i \cdot x$ is an even integer for $i > n$. The inverse to the map $\rho$ is the map of exercise 6 from section 2.1.

We showed that $[0,1)$ has the cardinality of $A \cup C$. We still need to show that $A \cup C$ has the same cardinality as $S_{0,1}$. Note that the sets $A$ and $B$ are infinitely countable. Indeed, a sequence $\{x_i\}$ is specified by the finite sequence preceding the trailing 0s and the set of finite sequences of 0,1 is countable. Let $\phi_1 : \mathbb{N} \to A$ and $\phi_2 : \mathbb{N} \to B$ be the corresponding bijections. Also, let $\phi_3$ be a bijection $\mathbb{N} \times \{0,1\} \to \mathbb{N}$. We can then write the desired bijection

$$S_{0,1} = A \cup B \cup C \to A \cup C$$
$$x \mapsto \begin{cases} x & x \in C \\ \phi_1 \circ \phi_3((\phi_1^{-1}(x), 0)) & x \in A \\ \phi_1 \circ \phi_3((\phi_2^{-1}(x), 1)) & x \in B \end{cases}$$

To summarize the above construction, since $A$ and $B$ are countably infinite, $A \cup B$ has the same cardinality as $A$. Therefore $A \cup B \cup C$ has the same cardinality as $A \cup C$. This completes the proof of Claim 1. □

*Proof of Claim 2.* Let $\rho : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ be a bijection. The desired bijection is

$$\chi : \mathbb{Z} \times S_{0,1} \to S_{0,1}$$
$$(n, (x_1, x_2, x_3, \dots)) \mapsto (\underbrace{0, 0, \dots, 0}_{\rho(n) \text{ 0s}}, 1, x_1, x_2, \dots)$$

□

2.2.3 Let $x$ be a real number. Assume first that $x$ can be represented by a Cauchy sequence $\{x_n\}$ of rational numbers so that $x_i < x_{i+1}$ for all $i$. Then for any $n$, $x - x_n$ is represented by the sequence $\{x_i - x_n\}_{i\in\mathbb{N}}$ and for all $i > n$, $x_i - x_n > x_{n+1} - x_n > 0$. In particular $x_n < x$.

It remains to show that any $x \in \mathbb{R}$ can be represented by a strictly increasing Cauchy sequence of rational numbers. Let $\{y_i\}$ be any sequence representing $x$. For $k \in \mathbb{N}$, let $r_k \in \mathbb{N}$ be such that $|y_i - y_j| < 2^{-k}$ for all $i, j \geq r_k$. For simplicity, we take the constants $r_k$ satisfying $r_k \geq k$ and $r_{k+1} > r_k$. Define a new sequence $\{x_i\}$ by $x_k = y_{r_k} - 2^{-k+2}$. We claim that $\{x_i\}$ is a strictly increasing sequence which is equivalent to $\{y_i\}$.

We first show that $\{x_i\}$ is equivalent to $\{y_i\}$. Given $\frac{1}{n}$, let $m \in N$ be such that $2^{-m+2} < \frac{1}{2n}$ and for all $i, j \geq m$ we have $|y_i - y_j| < \frac{1}{2n}$. Then for all $i > m$ we have

$$|x_i - y_i| = |y_{r_i} - 2^{-i+2} - y_i| \leq |y_{n_i} - y_i| + 2^{-i+2} < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$$

This proves that $\{x_i\}$ is equivalent to $\{y_i\}$.

We now show that $\{x_i\}$ is strictly increasing. We have

$$\begin{aligned} x_{i+1} - x_i &= (y_{r_{i+1}} - 2^{-(i+1)+2}) - (y_{r_i} - 2^{-i+2}) = (y_{r_{i+1}} - y_{r_i}) + (2^{-i+2} - 2^{-i+1}) \\ &= (y_{r_{i+1}} - y_{r_i}) + 2^{-i+1}. \end{aligned}$$

By the choice of the constant $r_i$ and the fact that $r_{i+1} \geq r_i$ we have that $|(y_{r_{i+1}} - y_{r_i}) < 2^{-i}$. It follows that

$$x_{i+1} - x_i = 2^{-i+1} + (y_{r_{i+1}} - y_{r_i}) \geq 2^{-i+1} - |y_{r_{i+1}} - y_{r_i}| > 2^{-i+1} - 2^{-i} = 2^{-i}.$$

In particular, $x_{i+1} - x_i$ is strictly positive.

2.2.6 We first show that given any positive rational number $y$ and $\frac{1}{n}$, there exist $p, q \in \mathbb{N}$ such that

$$|y - \frac{p^2}{q^2}| < \frac{1}{n} \tag{4}$$

Fix $q \in \mathbb{N}$. We claim that if $y < \frac{p^2}{q^2}$ then there exists $p'$ such that $|y - \frac{p'^2}{q^2}| \leq \frac{2p-1}{q^2}$. Indeed, in such case there exists $p' \leq p$ such that

$$\frac{(p'-1)^2}{q^2} \leq y \leq \frac{(p')^2}{q^2} \tag{5}$$

The difference between the outer terms of the above inequality is

$$\frac{(p')^2}{q^2} - \frac{(p'-1)^2}{q^2} = \frac{2p'-1}{q^2}$$

and therefore

$$|y - \frac{p'^2}{q^2}| \leq \frac{2p'-1}{q^2} \leq \frac{2p-1}{q^2}$$

Fix $p, q$ such that $y < \frac{p^2}{q^2}$. For any $m \in \mathbb{N}$, be have

$$y < \frac{p^2}{q^2} = \frac{(np)^2}{(nq)^2}$$

and therefore by the argument of the previous paragraph, there exists $p'$ with

$$|y - \frac{p'^2}{(mq)^2}| \leq \frac{mp-1}{(mq)^2} = \frac{1}{m}\frac{p-\frac{1}{m}}{q^2}$$

The right hand side of this inequality can be made arbitrarily small by increasing $m$. In particular, given $\frac{1}{n}$, let $m$ be such that $\frac{1}{m}\frac{p-\frac{1}{m}}{q^2} < \frac{1}{n}$. Then by the above argument, there exists $p'$ such that

$$|y - \frac{p'^2}{(mq)^2}| \leq \frac{1}{m}\frac{p-\frac{1}{m}}{q^2} < \frac{1}{n}.$$

Now, given any positive real number $x$, there is a Cauchy sequence $\{x_i\}$ of positive rational numbers representing $x$. Let $p_i, q_i \in \mathbb{N}$ be such that $|x_i - \frac{p_i^2}{q_i^2}| < \frac{1}{i}$ which we just proved to exist. Then the sequence $\{\frac{p_i}{q_i}\}$ is the desired sequence.

2.2.9 Let us recall the definition of an ordered field.

**Definition 6.** An order field is a field $F$ with a strict total order $<$ such that for all $x, y, z \in F$

(a) If $x < y$, then $x + z < y + z$.
(b) If $x, y > 0$, then $xy > 0$.

Lets us first derive few simple consequences.

Claim 1 For all $y \neq 0$, $y > 0$ if and only if $-y < 0$.

*Proof.* By property (a) of definition 6, $y > 0$ is equivalent to $y + (-y) > 0 + (-y)$ which simplifies to $0 > -y$. □

Claim 2 For all $y \in F$, $y^2 \geq 0$

*Proof.* If $y > 0$, then $y^2 = y \cdot y > 0$ by property (b) of definition 6. If $y < 0$, then $-y > 0$ by Claim 1 and again we have $y^2 = (-y) \cdot (-y) > 0$ by property (b) of definition 6. The result clearly holds if $y = 0$. □

Claim 3: Let $x \neq 0$. Then $x > 0$ if and only if $x^{-1} > 0$.

*Proof.* We first note that $0 < 1$ by Claim 2, since $1 = 1^2$. Therefore by Claim 1, $-1 < 0$. Assume by contradiction that there exists $x \in F$ such that $x > 0$ and $x^{-1} < 0$. Then

$$-1 = x \cdot (-x^{-1}) > 0$$

where the inequality follows from property (b) of definition 6. □

We now go back to the original problem.

a. If $x \neq 0$, then $\frac{x^2+y^2}{x^2} \geq 1$.

*Proof.* By Claim 2, we have $y^2 \geq 0$. Therefore

$$y^2 \geq 0 \xRightarrow{\text{Def 6(a)}} y^2 + x^2 \geq x^2 \xRightarrow[\text{Claims 2\&3}]{\text{Def 6(b)}} \frac{y^2 + x^2}{x^2} \geq 1$$

where the last implication is obtained by multiplying by $(x^2)^{-1}$ which is positive by claims 2 and 3. □

b. $2xy \leq x^2 + y^2$.

*Proof.* We have $(x-y)^2 = x^2-2xy+y^2$ by the virtue of $F$ being a field. Therefore by Claim 2,

$$(x-y)^2 = x^2 - 2xy + y^2 \geq 0.$$

Adding $2xy$ to both sides gives the desired result. □

c. If $x > 0$ and $0 < y < 1$, then $x/y > x$.

*Proof.* Let $x, y \in F$ with $x > 0$ and $0 < y < 1$. We then have

$$1 > y \Longrightarrow x > xy \Longrightarrow \frac{x}{y} > x$$

where both implications follows from Definition 6(b). In the first we use $x > 0$ and in the second, $y^{-1} > 0$ which in turn follows from Claim 3 and $y > 0$. □

2.3.2 Given $x \in \mathbb{R}$, we construct two sequences $\{a_i\}, \{b_i\}$ of real numbers such that $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_0$, $a_i^3 \leq x \leq b_i^3$ for all $i$. Setting $y = \lim_{i\to\infty} a_i$, we then have $y^3 \leq x \leq y^3$ since limits preserve non-strict inequality, from which we can conclude $y^3 = x$.

We first pick $a_1, b_1$ such that $a_1^3 \leq x \leq b_1^3$ in the following manner: if $-1 \leq x \leq 1$, we set $a_1 = -1; b_1 = 1$. If $x > 1$, we set $a_1 = 1; b_1 = x^3$ and if $x < 1$, we set $a_1 = x^3; b_1 = -1$. We define $a_i, b_i$ inductively by the following rule. Suppose we defined $a_i, b_i$ for $i \leq n$. Let $m = \frac{a_n+b_n}{2}$, be the midpoint between $a_n$ and $b_n$. If $m^3 \leq x$, we set $a_{n+1} = m; b_{n+1} = b_n$. If $m^3 > x$, we set $a_{n+1} = a_n; b_{n+1} = m$. Note that $\{a_i\}$ is a monotone increasing sequence bounded by $b_1$ and therefore converges. Moreover, we have $|a_i - b_i| = \frac{|b_1-a_1|}{2^{i-1}}$ and therefore $\{b_i\}$ converges to the same limit $y$ as $\{a_i\}$. By the argument in the previous paragraph, we have $y^3 = x$.

To show uniqueness, suppose $y' \in \mathbb{R}$ is another real number such that $y'^3 = x$. Then $y^3 - y'^3 = 0$. We factor the left hand side

(7) $$y^3 - y'^3 = (y - y')(y^2 + yy' + y'^2) = 0.$$

We claim that $(y^2 + yy' + y'^2) = 0$ if and only if $y' = y = 0$. Assume that $(y^2 + yy' + y'^2) = 0$, then adding $yy'$ to both sides we obtain

$$(y' + y)^2 = (y^2 + 2yy' + y'^2) = yy'.$$

On the other hand, subtracting $yy'$ from both sides we obtain

$$y^2 + y'^2 = -yy'.$$

Together, since the left hand sides of those equations are necessarily non-negative, these imply that

$$y^2 + y'^2 = -yy' = 0.$$

In particular, equation 7 implies $y' = y$.

2.3.3 We first investigate partial sums of geometric series, i.e., we would like a closed formula for $\sum_{i=n}^{m} 2^{-i}$. Denote this quantity by $A$. We have

$$2A = \sum_{i=n}^{m} 2^{-(i-1)} = \sum_{i=n-1}^{m-1} 2^{-i} = 2^{-(n-1)} + \sum_{i=n}^{m} 2^{-i} - 2^{-m} = 2^{-(n-1)} + A - 2^{-m}$$

Subtracting $A$ from both sides, we obtain

$$A = \sum_{i=n}^{m} 2^{-i} = 2^{-(n-1)} - 2^{-m}$$

Going back to the problem, we would like to show that the $y_n = \sum_{i=1}^{n} 2^{-i}$ is a Cauchy sequence. Given $\frac{1}{n}$, let $m$ be such that $2^{-m} < \frac{1}{n}$. Then for all $i \geq j > m$ we have

$$\begin{aligned} |y_i - y_j| &= |\sum_{n=j+1}^{i} x_n| \leq \sum_{n=j+1}^{i} |x_n| \leq \sum_{n=j+1}^{i} 2^{-n} \\ &= 2^{-j} - 2^{-i} < 2^{-j} < 2^{-m} < \frac{1}{n} \end{aligned}$$

2.3.7 We prove the contrapositive, i.e., if $\sqrt{b} > \sqrt{a}$, then $b > a$. The condition $\sqrt{b} > \sqrt{a}$ is equivalent to $\sqrt{b} - \sqrt{a} > 0$. Since $\mathbb{R}$ is an ordered field, and $\sqrt{b} + \sqrt{a}$ is positive, we get

$$(\sqrt{b} + \sqrt{a})(\sqrt{b} - \sqrt{a}) = b - a > 0$$

which is equivalent to $b > a$.

2.3.10 We first show that there are irrational numbers arbitrarily close to 0. For example, given $\frac{1}{n}$, $\frac{\sqrt{2}}{2n}$ is irrational, since the product of a rational and an irrational number is irrational, and less than $\frac{1}{n}$ since $\frac{\sqrt{2}}{2} < 1$. Given $x \in \mathbb{R}$ and $\frac{1}{n}$, let $y \in \mathbb{Q}$ be such that $|x - y| < \frac{1}{2n}$ and let $z \in \mathbb{R}$ be irrational such that $|z| < \frac{1}{2n}$. Then $y + z$ is irrational, being a sum of an irrational number and a rational number, and moreover

$$|x - (y + z)| \leq |x - y| + |z| < \frac{1}{n}.$$

- Let $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} y_i = y$. Let $M \in \mathbb{N}$ be such that $|x_i| < M, |y_i| < M$ for all $i$ which exists since $\{x_i\}$ and $\{y_i\}$ are convergent. Since limits preserve non-strict inequalities, we also have $|x| \leq M$ and $|y| \leq M$. We prove that $\lim_{i\to\infty} x_i y_i = xy$. Given $\frac{1}{n}$, let $m$ be such that for all $i > m$,

$$|x - x_i| < \frac{1}{2Mn}$$

and

$$|y - y_i| < \frac{1}{2Mn}.$$

Then for all $i > m$ we have

$$\begin{aligned} |xy - x_i y_i| &= |x(y - y_i) + y_i(x - x_i)| \leq |x||y - y_i| + |y_i||x - x_i| \\ &< |x|\frac{1}{2Mn} + |y_i|\frac{1}{2Mn} \leq \frac{M}{2Mn} + \frac{M}{2Mn} = \frac{1}{n} \end{aligned}$$