M405 - HOMEWORK SET #2- SOLUTIONS

2.1.2 Let $\{x_i\}$ be a Cauchy sequence of rational numbers representing a real number x. Define a new sequence $\{x_i'\}$ by the following rule:

$$x_i' = \begin{cases} x_i & \text{if } x_i \notin \mathbb{Z} \\ x_i + \frac{1}{i+1} & \text{otherwise} \end{cases}$$

None of x_i' are integers. Moreover, $\{x_i'\} \sim \{x_i\}$ since for every $n \in \mathbb{N}$, for all $i > n$, we have

$$|x_i - x_i'| \leq \frac{1}{i+1} < \frac{1}{i}.$$

Therefore $x = [\{x_i'\}]$.

2.1.4 Denote the shuffled sequence $\{z_i\}$, i.e.,

$$z_i = \begin{cases} x_{(i+1)/2} & i \text{ odd} \\ y_{(i/2)} & i \text{ even} \end{cases}$$

We first prove that if the shuffled sequence $\{z_i\}$ is Cauchy, then $\{x_i\} \sim \{y_i\}$. Assume $\{z_i\}$ is Cauchy. Then for any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that for all $i, j > m$, $|z_i - z_j| < \frac{1}{n}$. Therefore for all $i > m$,

$$|x_i - y_i| = |z_{2i-1} - z_{2i}| < \frac{1}{n}$$

since $2i - 1, 2i > m$ if $i > m$. Therefore $\{x_i\} \sim \{y_i\}$.

Assume now that $\{x_i\} \sim \{y_i\}$. Given n, let $m_1, m_2, m_3 \in \mathbb{N}$ be such that for $i, j > m_1$, $i', j' > m_2$, $k > m_3$,

$$|x_i - x_j| < \frac{1}{2n},$$

$$|y_{i'} - y_{j'}| < \frac{1}{2n},$$

$$|x_k - y_k| < \frac{1}{2n}.$$

Constants m_1, m_2, m_3 exist since $\{x_i\}$ is Cauchy, $\{y_i\}$ is Cauchy and $\{x_i\} \sim \{y_i\}$ respectively. Let $m = 2 \cdot \max(m_1, m_2, m_3)$. Then for all $i, j > m$, we claim that

$$|z_i - z_j| < \frac{1}{n}.$$

Indeed, if both i, j are even, then

$$|z_i - z_j| = |y_{i/2} - y_{j/2}| < \frac{1}{2n} < \frac{1}{n}$$

since $i/2 > m_2$. The argument for when both i, j are odd is analogous. If i is even and j is odd, then

$$|z_i - z_j| = |y_{i/2} - x_{(j+1)/2}| \leq |y_{i/2} - x_{i/2}| + |x_{i/2} - x_{(j+1)/2}| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$$

where we used the triangle inequality and the fact that $\frac{i}{2} > m_3$ and $\frac{i}{2}, \frac{j+1}{2} > m_1$. The argument for when j is even and i is odd is analogous. It follows that $\{z_i\}$ is a Cauchy sequence.

2.1.6 Consider the decimal expansion $N.a_1a_2a_3, \dots$ where $N \in \mathbb{Z}$ and $a_i \in \{0, 1, 2, \dots, 9\}$. The corresponding sequence is

$$\begin{aligned} x_1 &= N \\ x_2 &= N.a_1 = \frac{10N + a_1}{10} \\ x_3 &= N.a_1a_2 = \frac{10^2N + 10a_1 + a_2}{10^2} \\ x_4 &= N.a_1a_2a_3 = \frac{10^3N + 10^2a_1 + 10a_2 + a_3}{10^3} \\ &\vdots \end{aligned}$$

For $i, j \in \mathbb{N}$ with $i < j$, we have

$$x_j - x_i = \pm 0.000\dots 00a_ia_{i+1}\dots a_{j-1} = \pm\frac{10^{j-i-1}a_i + 10^{j-i}a_{i+1} + \cdots + 10a_{j-2} + a_{j-1}}{10^{j-1}} \tag{1}$$

where the sign depends on the sign of N. For all k, we have $a_k \leq 9$. Therefore substituting 9 for each a_k in 1 we get the inequality

$$|x_j - x_i| \leq \frac{10^{j-i-1}\cdot 9 + 10^{j-i}\cdot 9 + \cdots + 10\cdot 9 + 9}{10^{j-1}} = \frac{10^{j-i} - 1}{10^{j-1}} = 10^{1-i} - 10^{1-j} < 10^{1-i}$$

Therefore, given $n \in \mathbb{N}$ letting $m \in \mathbb{N}$ be such that $10^{1-m} < \frac{1}{n}$ implies that for all $i > m$,

$$|x_j - x_i| < 10^{1-\min(i,j)} < 10^{1-m} < \frac{1}{n}.$$

2.1.7 Let $\{x_i\}, \{y_i\}$ be sequences where $y_i = 1$ for all i and $x_i = .99999\dots 99$, decimal point followed by i 9s. A better representation of the sequence x_i is $x_i = 1 - 10^{-i}$. Therefore

$$|y_i - x_i| = 10^{-i}.$$

Given $n \in \mathbb{N}$, let $m \in \mathbb{N}$ be the least integer such that $10^m > n$. Then for all $i > m$, we have

$$|y_i - x_i| = 10^{-i} < 10^{-m} < \frac{1}{n}$$

. It follows that $\{x_i\} \sim \{y_i\}$.

2.1.8 Yes! The sequences $\{x_i = \frac{1}{i}\}$ and $\{y_i = -\frac{1}{i}\}$ are equivalent.

- We prove the statement it two steps. We first prove that $(ad+bc, bd) \sim (a'd+b'c, b'd)$. Since $(a, b) \sim (a', b')$, we have

$$ab' = a'b$$

and therefore

$$adb'd = a'dbd$$
$$(ad + bc)b'd = (a'd + b'c)bd$$

implying $(ad+bc, bd) \sim (a'd+b'c, b'd)$. The same argument shows that $(a'd+b'c, b'd) \sim (a'd' + b'c', b'd')$. The result then follows from transitivity of $\sim$.