M405 - HOMEWORK SET #1- SOLUTIONS

1.2.2 The set $A \subset 2^{\mathbb{N}}$ of all finite subsets of $\mathbb{N}$ is countable. Let $A_n \subset 2^{\mathbb{N}}$ be the set of subsets of $\mathbb{N}$ of cardinality n. We have shown in class that a countable union of countable sets is countable. Therefore, since $A = \bigcup_{i=0}^{\infty} A_n$, it suffices to show that each A_n is countable. Define $B_n \subset \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} = \mathbb{N}^{\times n}$ by

$$B_n = \{(b_1, \ldots, b_n) \in \mathbb{N}^{\times n} | b_1 < b_2 < \cdots < b_n\}$$

There is a bijection between B_n and A_n which sends a sequence $(b_1, \ldots, b_n)$ to the set $\{b_1, \ldots, b_n\} \subset \mathbb{N}$. Since a subset of a countable set is countable, it suffices to show that $\mathbb{N}^{\times n}$ is countable to conclude that B_n, and hence A_n, is countable.

We showed in class that $\mathbb{N}^{\times 2}$ is countable, i.e., that there is a map $a : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ which is onto. We argue by induction that $\mathbb{N}^{\times n}$ is countable for all n. The base case is $n = 2$ which is what we proved in class. To prove the inductive step, fix $n \geq 2$ and assume that $\mathbb{N}^{\times n}$ is countable, i.e., that there is a map $b : \mathbb{N} \to \mathbb{N}^{\times n}$ which is onto. Then the composite map

$$\mathbb{N} \xrightarrow{a} \mathbb{N} \times \mathbb{N} \xrightarrow{b \times \mathrm{id}} \mathbb{N}^{\times n} \times \mathbb{N} \cong \mathbb{N}^{\times n+1}$$

is onto and therefore $\mathbb{N}^{n+1}$ is countable.

1.2.4 We can rephrase the statement we need to prove as follows:

- Let A be a set and $B \subset A$ be a countable subset. Then if A is uncountable then $A - B$ is also uncountable.

The contrapositive of the above statement is

- Let A be a set and $B \subset A$ be a countable subset. If $A - B$ is countable then A is countable.

Since $A = (A - B) \bigcup B$, the above statement follows from the fact that the union of countable sets is countable.

1.2.5 Assume that each set A_i has at least 2 elements. For each $i \in \mathbb{N}$, choose a subset $A'_i \subset A_i$ of cardinality 2. Up to relabeling the elements of A'_i, we may assume that $A'_i = \{0, 1\}$. Since subsets of countable sets are countable, to show that $A_1 \times A_2 \times ...$ is uncountable, it suffices to show that $A'_1 \times A'_2 \times \cdots \subset A_1 \times A_2 \times ...$ is uncountable. On the other hand, the set $A'_1 \times A'_2 \times \ldots$ is the set of sequences of 0s and 1s. We showed that this set is uncountable in class using the Cantor's diagonal argument.

1.2.6 Define the sets

$$A_k = \{a \in A | f(a) = k\}.$$

By the assumption, each set A_k is finite. Since a countable union of countable sets is countable, and $A = \bigcup_{k \in \mathbb{N}} A_k$, we conclude that A is countable. [1]

[1] Here, by countable we mean finite or infinitely countable.

1.2.7 We show that there is no function $f : A \to 2^A$ which is onto. Let $f : A \to 2^A$ be any function. We show that f is not onto. Define a subset $B \subset A$ by

$$B = \{a \in A | a \notin f(A) \subset A\}.$$

For any $a \in A$ we have that $a \in B$ if and only if $a \notin f(A)$. In particular $B \neq f(a)$. Since this holds for every $a \in A$, we conclude that B is not in the image of f and hence f is not onto.

(1) Let A, B, E be the predicates "Alice is a knight", "Bob is a knight", and "Eve is a knight" respectively. Alice saying that Bob is a knight corresponds to the predicate "A if and only if B". Bob saying that Alice is a knight but Eve is a knave corresponds to the predicate "B if and only if (A and not E)". Eve saying that both Alice and Bob are knights corresponds to the predicate "E if and only if (A and B)". In table 1, we draw the truth table to compute the conjunction of these predicates. In particular, there is only one consistent scenario which is where all three are knaves.

(2) (a) For every subset $U \subset \mathbb{Q}_+$ there exists $y \in U$ such that for all $x \in U$ we have $y \leq x$.

(b) There exists a subset $U \subset \mathbb{Q}_+$ such that for all $y \in U$ there exists $x \in U$ such that $y > x$.

(c) We prove the negation, i.e., the statement in (b). Let U be the entire set $\mathbb{Q}_+$. Then for any $y \in U$ let $x = y/2$. It is clear that $y > x$.

(3) (a) A sequence $x_1, x_2, \ldots$ of rational numbers does not converge to $x \in \mathbb{Q}$ if there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n > N$ such that $|x_n - x| > \epsilon$.

(b) Let $\epsilon = 1/2$. Then for any $N \in \mathbb{N}$ we have $|x_n - 0| = 1 > 1/2 = \epsilon$ where we can take $n = N + 1$.

(c) Let $x \in \mathbb{Q}$ and $\epsilon = 1/2$. Given $N \in \mathbb{N}$, choose $n > N$ such that n is even if $x < 0$ and n is odd if $x \geq 0$. We then have

$$|x_n - x| = |(-1)^n - x| = |x| + 1 > 1/2 = \epsilon.$$

TABLE 1. Truth Table

A	B	E	A and not E	A and B	A if and only if B	B if and only if (A and not E)	E if and only if (A and B)
T	F	F	T	F	F	F	T
T	F	T	F	F	F	T	F
T	T	F	T	T	T	T	F
T	T	T	F	T	T	F	T
F	F	F	F	F	T	T	T
F	F	T	F	F	T	T	F
F	T	F	F	F	F	F	T
F	T	T	F	F	F	F	F