M405 - PROBLEM SET #12- SOLUTIONS

7.5.1 For each k, we construct a polynomial $R_k(x)$ such that

$$R_k(x_i) = R_k'(x_i) = 0$$

if $i \neq k$ and

$$R_k(x_k) = a_k$$
$$R_k'(x_k) = b_k$$

The desired polynomial is then the sum $R_1 + \cdots + R_n$. Define an auxiliary polynomial

$$S_k(x) = \prod_{i \neq k} (x - x_i)^2.$$

It has the property that $S_k(x_i) = S_k'(x_i) = 0$ for $i \neq k$ and $S_k(x_k) \neq 0$. We then define

$$R_k(x) = S_k(x)(r + t(x - x_k))$$

where r, t are constant we will now determine. We want

$$a_k = R_k(x_k) = rS_k(x_k),$$

so we set

$$r = \frac{a_k}{S_k(x_k)}.$$

We also want

$$b_k = R_k'(x_k) = rS_k'(x_k) + tS_k(x_k),$$

so we set

$$t = \frac{b_k - rS_k'(x_k)}{S_k(x_k)}$$

7.5.3 Assume that f vanishes outside of $[a, b]$. Fix $x \in \mathbb{R}$ and let $|h| < 1$. We have

$$f * g(x + h) - f * g(x) = \int_a^b (g(x + h - y) - g(x - y))\, f(y)dy$$

The function g is uniformly continuous on $[x - b - 1, x - a + 1]$ and therefore given $\frac{1}{m}$ there exists $\frac{1}{n}$ such that

$$|g(x + h - y) - g(x - y)| \leq \frac{1}{mM(b - a)}$$

where $M = \sup_{x \in [a,b]} f(x)$ for all h with $|h| < \frac{1}{n}$ and all $y \in [a, b]$. We then have that for such h we have

$$|f * g(x + h) - f * g(x)| = |\int_a^b (g(x + h - y) - g(x - y))\, f(y)dy|$$
$$\leq \int_a^b \frac{1}{mM(b - a)} M = \frac{1}{m}$$

7.5.9 Given such f, the Weierstrass Approximation Theorem gives rise to a sequence of polynomials g_n such that $g_n \to f$ uniformly. Define the sequence of polynomials h_n by

$$h_n(x) = g_n(x) - g_n(c).$$

Since $g_n(c) \to f(c) = 0$, the sequence h_n also converges uniformly to f and now also satisfies $h_n(c) = 0$.

7.6.1 Given $\frac{1}{m}$ let $\frac{1}{n'}$ be such that for all n and all x, y with $|x - y| < \frac{1}{n}$ we have

$$|f_n(x) - f_n(y)| < \frac{1}{4m}$$

Let $y_1, \ldots, y_l \in [a, b]$ be such that for all $x \in [a, b]$ at least one of the y_i is within $\frac{1}{n'}$ of x. Let N be such that for all $n \geq N$,

$$|f_n(y_i) - f(y_i)| < \frac{1}{4m}$$

for all $y_1, \ldots, y_l$. This is possibly because there are only finitely many points $\{y_i\}$. Then for all $n, m \geq N$ and any x we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(y_i)| + |f_n(y_i) - f(y_i)| + |f(y_i) - f_m(y_i)| + |f_m(y_i) - f_m(x)| < \frac{1}{m}$$

Fixing n and letting m go to infinity, this implies that

$$|f_n(x) - f(x)| \leq \frac{1}{m}$$

for all $n \geq N$ and all x. Hence $f_n \to f$ uniformly.

7.6.2 For any $\frac{1}{m}$, let $\frac{1}{n'}$ be such that

$$M(\frac{1}{n'})^{\alpha} \leq \frac{1}{m}$$

Then for all x, y such that $|x - y| < \frac{1}{n'}$ we have

$$|f_n(x) - f_n(y)| \leq M|x - y|^{\alpha} \leq M(\frac{1}{n'})^{\alpha} \leq \frac{1}{m}.$$

7.6.4 Let $[a, b]$ be the compact domain on which we will show $sin(nx)$ is not uniformly equicontinuous. We show that given any $\frac{1}{m} < 2$ and $\frac{1}{n'}$, there exists $n \in \mathbb{N}$ and $x, y \in [a, b]$ such that $|x - y| < \frac{1}{n'}$ but

$$|\sin(nx) - \sin(ny)| = 2 > \frac{1}{m}.$$

Let n be large enough so that $\frac{2\pi}{n} < \frac{1}{n'}$ and so that $[a, b]$ contains points of the form

$$x = \frac{\pi + 4\pi k}{2n}; \quad y = \frac{3\pi + 4\pi k}{2n}$$

for some integer k. We have $\sin(nx) = 1$ and $\sin(ny) = -1$ and $|x - y| < \frac{1}{n'}$.

7.6.5 $f_n(x) = n$