M405 - HOMEWORK SET #11- SOLUTIONS

7.3.2 Assume that $f_n \to f$ point-wise and each function f is Lipschitz with the same constant M. Fix any two point $|x-y|$. We would like to show that $|f(x)-f(y)| \leq M|x-y|$. By continuity of the absolute value function, we have

$$|f(x)-f(y)| = \lim_{n\to\infty} |f_n(x)-f_n(y)|$$

Since all f_n are Lipschitz with the Lipschitz constant M, we have $|f_n(x)-f_n(y)| < M|x-y|$ for all n. Since limits preserve non-strict inequalities, we have $|f(x)-f(y)| \leq M|x-y|$.

7.3.5 Fix $x < y$ in the domain of the functions f. Since f_n are monotone increasing, $f_n(y)-f_n(x) \geq 0$ for all n. Since limits preserve non-strict inequalities, we have

$$f(y)-f(x) = \lim_{n\to\infty} f_n(y)-f_n(x) \geq 0.$$

Since limits do not preserve strict inequalities, the limit of strictly increasing functions need not be strictly increasing. For example, the functions $f_n(x) = x^n$ on the domain $(1/3, 2/3)$ are strictly increasing, while their limit, $f(x) = 0$ is not.

7.3.7 The series of functions converges point-wise since for any x, $\sum f_n(x)$ is an absolutely convergent series. Consider the absolute value of the following difference

$$|f - \sum_{n=1}^{N} f_n(x)| = |\sum_{n=N+1}^{\infty} f(x)| \leq \sum_{n=N+1}^{\infty} a_n$$

Since $\sum_{n=1}^{\infty} a_n$ converges absolutely, $\sum_{n=N+1}^{\infty} a_n$ goes to 0 as N goes to infinity. In particular, for any $\frac{1}{m}$ there exists $N' \in \mathbb{N}$ such that for all $N \geq N'$, we have

$$\sum_{n=N+1}^{\infty} a_n < \frac{1}{m}.$$

Then for all $N \geq N'$ and all x we have

$$|f - \sum_{n=1}^{N} f_n(x)| = |\sum_{n=N+1}^{\infty} f(x)| \leq \frac{1}{m}$$

and hence $\sum_{n=1}^{\infty} f_n(x)$ converges to f uniformly.

7.3.14 Let P_n be the equidistant partition of $[a,b]$ into n intervals. Let f_n be the linear spline defined by $f_n(x_i) = f(x_i)$ for every x_i in the partition P_n and f_n is linear on every interval of P_n. I claim that $f_n \to f$ uniformly.

Since f is continuous on the compact domain $[a,b]$, it is uniformly continuous. Given $\frac{1}{m}$, let $\frac{1}{n}$ be such that

$$|f(x)-f(y)| < \frac{1}{2m}$$

whenever $|x-y|<\frac{1}{n}$. Let N be such that for all $n \geq N$, the length of each interval in the partition P_n is shorter than $\frac{1}{n}$. For any $x \in [a,b]$, let x_i be the closest partition point in P_n to x with $x_i \leq x$. Then $|x-x_i|<\frac{1}{n}$. We thus have

$$|f(x)-f_n(x)| \leq |f(x)-f(x_i)| + |f(x_i)-f_n(x_i)| + |f_n(x_i)-f_n(x)|$$

The middle term in this sum is 0. The first term is bounded by $\frac{1}{2m}$. We still have to analyze the third term. We have

$$f_n(x) = f_n(x_i) + \frac{f(x_{i+1})-f(x_i)}{(x_{i+1}-x_i)} \cdot (x-x_i)$$

Therefore

$$|f_n(x_i)-f_n(x)| = |\frac{f(x_{i+1})-f(x_i)}{(x_{i+1}-x_i)} \cdot (x-x_i)| \leq |f(x_{i+1})-f(x_i)| \leq \frac{1}{2m}$$

where we used the fact that $x_{i+1}-x_i \geq x-x_i$. Hence

$$|f(x)-f_n(x)| < \frac{1}{m}$$

for all x and all $n \geq N$ and $f_n \to f$ uniformly.

7.4.2 Assume for simplicity that $x_0 = 0$. We have $f(x) = \sum_{n=0}^{\infty} a_n x^n$ in some neighborhood of 0, i.e., $\sum_{n=0}^{\infty} a_n x^n$ has a positive radius of convergence R. Since $f(0) = a_0 = 0$, $f(x)/x$, if well defined, is given by $\sum_{n=0}^{\infty} a_{n+1} x^n$. The radius of convergence of this power series is given by

$$\limsup_{n\to\infty} (a_{n+1})^{1/n} = \limsup_{n\to\infty} ((a_{n+1})^{1/n+1})^{(n+1)/n} = \lim_{n\to\infty} R^{(n+1)/n} = R$$

In particular, it has the same radius of convergence as $\sum_{n=0}^{\infty} a_n x^n$.

7.4.4 Fix $0<x<1$ and consider the series

$$1 + ax + \frac{(a)(a-1)}{2!}x^2 + \dots$$

The ratio of the n'th term and the $n-1$'st term is

$$\frac{a-n}{n}x$$

whose limit as $n \to \infty$ is $x < 1$. In particular, this series converges by the ratio test. Hence the radius of convergence of the power series satisfies $R \geq x$. Since this is true for all $0<x<1$, we have $R \geq 1$.

7.4.8 a) We have $\lim_{n\to\infty}(n!)^{\frac{1}{n}} = \infty$. To see it, note for example that all terms in $n!$ are ≥ 1 and half of them are greater than $n/2$. Therefore

$$n! \geq (\frac{n}{2})^{n/2}$$

and

$$(n!)^{1/n} \geq (\frac{n}{2})^{1/2} \to \infty$$

It follows that

$$\lim_{n\to\infty}(\frac{n^4}{n!})^{1/n} = 0$$

since $\lim_{n\to\infty}(n^4)^{1/n} = 1$, n^4 being a polynomial. Hence the radius of convergence is ∞.

b) We have $\sqrt{n}^{1/n} = (n^{1/n})^{1/2}$. Since $\lim_{n\to\infty}(n)^{1/n} = 1$, the radius of convergence of $\sum_{n=0}^{\infty}\sqrt{n}x^n$ is 1.

c) $R = 1/2$.