M405 - HOMEWORK SET 10 - SOLUTIONS

7.1.1 Let $F(x) = f(x) + ig(x)$ be a complex valued function where f and g are its real and imaginary parts respectively. For any partition P and a choice of sample points $\{x_i^*\}$ we have

$$S(F, P, \{x_i^*\}) = S(f, P, \{x_i^*\}) + iS(g, P, \{x_i^*\})$$

It then follows that if

$$\lim_{|P| \to 0} S(f, P, \{x_i^*\}) = A$$

and if

$$\lim_{|P| \to 0} S(g, P, \{x_i^*\}) = B$$

then

$$\lim_{|P| \to 0} S(F, P, \{x_i^*\}) = A + iB.$$

To spell out the details, given $\frac{1}{m}$ let $\frac{1}{n}$ be such that for all partitions P such that $|P| < \frac{1}{n}$ we have

$$|S(f, P, \{x_i^*\}) - A| < \frac{1}{2m}$$

and

$$|S(g, P, \{x_i^*\}) - B| < \frac{1}{2m}$$

. Then by the triangle inequality for complex numbers

$$|S(F, P, \{x_i^*\}) - (A + iB)| < \frac{1}{m}.$$

Hence if f, g are Riemann integrable, then so is F. For the other direction, assume

$$\lim_{|P| \to 0} S(F, P, \{x_i^*\}) = A + iB.$$

Given $\frac{1}{m}$ there exists $\frac{1}{n}$ such that for all partitions P such that $|P| < \frac{1}{n}$

$$|S(F, P, \{x_i^*\}) - (A + iB)| < \frac{1}{m} \tag{1}$$

For a complex number $a + bi$ we have $|a + bi| = \sqrt{a^2 + b^2} \geq |a|$. Hence equation (1) implies

$$|S(f, P, \{x_i^*\}) - A| < \frac{1}{m}$$

for all $|P| < \frac{1}{n}$. This shows that f is Riemann integrable. The argument for integrability of g is analogous.

7.1.10 a) Both sides of the equation are invariant under interchanging z and z_1. We will therefore assume without loss of generality that $|z| \geq |z_1$. In that case $|z| - |z_1| \geq 0$ and $||z| - |z_1|| = |z| - |z_1|$. We thus need to show that

$$|z| - |z_1| \leq |z - z_1|.$$

Adding $|z_1|$ to both sides, we get the usual triangle inequality

$$|z| \leq |z - z_1| + |z_1|.$$

b) We have $z - z_1 = x - x_1 + i(y - y_1)$. For any complex number $a + bi$ we have

$$|a + bi|^2 = a^2 + b^2 \leq a^2 + b^2 + 2|ab| = (|a| + |b|)^2$$

and therefore $|a + bi| \leq |a| + |b|$. We therefore have

$$||z| - |z_1|| \leq |z - z_1| \leq |x - x_1| + |y - y_1|$$

where the first inequality follows from part a).

7.2.1 Let $x_n = y_n = \frac{(-1)^n}{n^{1/2}}$. The series $\sum_{n=1}^{\infty} x_n$ converges by the alternating series test, but $\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

If $\sum_{n=1}^{\infty} x_n$ is absolutely convergent and $\sum_{n=1}^{\infty} y_n$ is convergent we show that $\sum_{n=1}^{\infty} x_n y_n$ is absolutely convergent. We have that $|y_n| < M$ for some constant M and hence $|x_n y_n| \leq |M||x_n|$. Since $\sum_{n=1}^{\infty} |x_n|$ is convergent, so is $\sum_{n=1}^{\infty} |M||x_n|$.

7.2.3 We have

$$|\sum_{k=1}^{n} r^k - \frac{r}{1-r}| = |\sum_{k=n+1}^{\infty} r^k| = \frac{|r|^{n+1}}{1 - |r|}.$$

For any fixed n, $\lim_{r \to 1} \frac{|r|^{n+1}}{1-|r|} = \infty$ and the result follows.

7.2.5 Let P be the partition of $[1, N]$ given by the integers, i.e. $x_i = 1 + i$. Over each interval $[x_{i-1}, x_i]$ the function $\frac{1}{x^a}$ attains its minimum value at x_i being a decreasing function. Hence the lower Riemann sum for $\int_1^N 1/x^a dx$ is

$$S^-(f, P) = \sum_{n=2}^{N} 1/n^a \leq \int_1^N \frac{1}{x^a} dx$$

For $a > 1$ we have

$$\int_1^N \frac{1}{x^a} dx = (a-1)(1 - \frac{1}{N^{a-1}}) \leq a - 1$$

And therefore

$$\sum_{n=2}^{N} 1/n^a \leq a - 1$$

Since all the terms of the series are positive, $\sum_{n=1}^{\infty} 1/n^a$ is convergent.

7.2.9 Define a_{mn} by

$$a_{mn} = \begin{cases} \frac{2^n-1}{2^{n-1}} & n = m \\ -\frac{2^n-1}{2^{n-1}} & m - n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Every row and every column has only finite number of non-zero entries. The sum of the rows vanishes while the sum of the mth column is $\frac{1}{2^m}$. We therefore have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = 2$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} = 0$$