Existence and boundedness of n-complements

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Abstract

Theory of n-complements with applications is presented.

1 Introduction

Recall two key concepts for us.

Definition 1. Let $(X/Z \ni o, D)$ be a pair with a [proper] local morphism $X/Z \ni o$ and with an $\mathbb{R}$-divisor D on X. Another pair $(X/Z \ni o, D^+)$ with the same local morphism and with an $\mathbb{R}$-divisor D^+ on X is called an $\mathbb{R}$-*complement* of $(X/Z \ni o, D)$ if

(1) $D^+ \geq D$;

(2) (X, D^+) is lc; and

(3) $K + D^+ \sim_{\mathbb{R}} 0/Z \ni o$.

In particular, $(X/Z \ni o, D^+)$ is a *[local relative] 0-pair*. [Note that the neighborhood of o for another pair can be different from the original one.]

The complement is *klt* if (X, D^+) is klt.

Definition 2 ([Sh92, Definition 5.1])**.** Let n be a positive integer, and $(X/Z \ni o, D)$ be a pair with a local morphism $X/Z \ni o$ and with an $\mathbb{R}$-divisor $D = \sum d_i D_i$ on X. A pair $(X/Z \ni o, D^+)$, with the same local morphism and with a $\mathbb{Q}$-divisor $D^+ = \sum d_i^+ D_i$ on X, is an n-*complement* of $(X/Z \ni o, D)$ if

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(1) for every prime divisor D_i on X,

$$d_i^+ \geq \begin{cases} 1, \text{ if} & d_i = 1; \\ \lfloor (n+1)d_i \rfloor / n & \text{otherwise;} \end{cases}$$

(2) (X, D^+) is lc; and

(3) $K + D^+ \sim_n 0/Z \ni o$.

The number n is called a *complementary index*. [Note that the neighborhood of o for another pair can be different from the original one.]

The n-complement is *monotonic* if $D^+ \geq D$.

Remark 1. (1) By (2) of Definitions 1 and 2, D^+ is a subboundary. So, by (1) of both definitions, D is a subboundary too. By (1) of both definitions, $d_i^+ = 1$ if $d_i = 1$.

Again by (1) of both definitions D^+ is a boundary if so does D.

(2) Immediate by definition an n-complement is monotonic if and only if it is an $\mathbb{R}$-complement. In general, an n-complement is not monotonic. However, the latter property can happen for certain n-complements or multiplicities, boundaries, e.g., for hyperstandard ones [PSh08, Theorem 1.4 and Lemma 3.5] [B, Theorem 1.7]. (Cf. [HLSh, Theorem 1.6] vs Example 5, (4) below.)

It is not surprising that the existence of an n-compliment does not imply the existence of an $\mathbb{R}$-compliment. But by (1) of Definition 2 and since $\lfloor (n+1)d_i \rfloor / n$ is very close to d_i for every $d_i \in [0, 1)$ and sufficiently large n, the multiplicity d_i^+ for an n-compliment become $\geq d_i$ or very close to d_i. This observation and the remark (1) will be used in a proof of the easy part of Theorem 1 below, one of our main results.

However, if $d_i \geq 1$ then $\lfloor (n+1)d_i \rfloor / n > 1$ and typically $> d_i$. On the other hand, if $d_i < 0$ then $\lfloor (n+1)d_i \rfloor / n < 0$ and typically $< d_i$.

(3) By (3) of Definition 2, the (Cartier and log canonical) index of $K + D^+$ divides n, in particular, nD^+ is integral. Thus D^+ is automatically $\mathbb{Q}$-divisor.

(4) If $X/Z \ni o$ is proper then $\sim_{\mathbb{R}}$ in (3) of Definition 1 can be replaced by the numerical equivalence $\equiv$ in the following two cases:

$X/Z \ni o$ has weak Fano type [ShCh, Corollary 4.5]; or

D^+ is a boundary [A05, Theorem 0.1,(1)] [G, Theorem 1.2].

(5) We can define also $\mathbb{R}$- and n-complements for pairs $(X/Z, D)$ with not necessarily local morphisms. But they have more complicated behaviour. In this situation it is better to use the relative version $\sim_Z, \sim_{n,Z}$ and $\sim_{\mathbb{R},Z}$ of linear equivalences instead of the usual one $\sim, \sim_n$ and $\sim_{\mathbb{R}}$ respectively (see 1.1 below). Then the local over Z existence of $\mathbb{R}$-complements usually implies the existence of an $\mathbb{R}$-complement over Z (see Addendum 8, [ShCh, Corollary 4.5] and cf. Proposition 9). However, for local over Z n-complements, the index n can depend on a point $o \in Z$. To find a universal n is a real challenge (cf. Addendum 1).

For both cases of (4), $\equiv$ over Z is equal to $\sim_{\mathbb{R},Z}$.

Example 1. (1) Every relative log Fano pair $(X/Z, B)$ over a quasiprojective variety Z with a boundary B has an $\mathbb{R}$-complement $(X/Z, B^+)$. However, $\sim_{\mathbb{R}}$ should be replaced by its relative version $\sim_{\mathbb{R},Z}$ (cf. Remark 1, (5)). In other words, in this situation (3) of Definition 1 has the following form:

$$K + B^+ \sim_{\mathbb{R}} \varphi^* H \text{ and } K + B^+ \equiv 0/Z,$$

where $\varphi\colon X \to Z$ and H an $\mathbb{R}$-ample divisor on Z. [Recall that] we suppose that Z is quasiprojective.

(2) By definition every pair $(X/Z, 0)$ with wFt X/Z has a klt $\mathbb{R}$-complement (see Fano and weak Fano types in 1.1 below and cf. [PSh08, Lemma-Definition 2.6, (ii)]).

(3) Every complete 0-pair (X, D) has an $\mathbb{R}$-complement and $D^+ = D$. Moreover, if $D^+ = D = B$ is a boundary then $K + D^+ = K + B \sim_{\mathbb{R}} 0 \Leftrightarrow \equiv 0$ [A05, Theorem 0.1,(1)] [G, Theorem 1.2].

Every relative proper 0-pair $(X/Z, D)$ also has an $\mathbb{R}$-complement, e.g., $D^+ = D$. However, $D^+ = D$ always if the pair is local and nonklt over o near every connected component of X_o, the central fiber. (Cf. with Maximal lc 0-pairs in Section 11.)

(4) Let (X, D) be a pair with a toric variety X and with torus invariant D. Then, for any morphism X/Z and any positive integer n, $(X/Z, D)$ has an n-compliment if D is a subboundary. If $K + D$ is $\mathbb{R}$-Cartier the last assumption is equivalent to the lc property of (X, D). For instance, we can take D^+ equal to the sum of invariant divisors. The complement is torus invariant too. This is a rear case where we can use $\sim$ instead of its relative version $\sim_Z$.

For complete toric X, D is a subboundary if and only if (X, D) has an $\mathbb{R}$-compliment. In this situation, the invariant complement is unique.

(5) Every n-complement $(\mathbb{P}^1, B^+)$ of $(\mathbb{P}^1, 0)$ corresponds to a polynomial $f \in k[x]$ of the degree $2n$:

$$B^+ = (f)_0/n,$$

such that every root of f has multiplicity $\leq n = (\deg f)/2$ (cf. the semistability of polynomials). Indeed, for an appropriate affine chart $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$ with a coordinate x, we can assume that B^+ is supported in $\mathbb{A}^1$ and nB^+ is given by the zeros of some polynomial f in x. Notice that B^+ is effective by (1) of Definition 2. By (3) of the definition $(\deg f)/n = -\deg K_{\mathbb{P}^1} = 2$ and f has the required degree. The multiplicities of roots $\leq n$ by (2) of Definition 2:

$$\operatorname{mult} B^+ = \frac{1}{n} \operatorname{mult}_a f \leq 1, a \in k.$$

If $\mathbb{P}^1$ is defined over algebraically nonclosed field k, we can construct a n-complement $(\mathbb{P}^1, B^+)$ over k taking a sufficiently general polynomial f in $k[x]$. (Its roots belong now to $\overline{k}$.) Such a polynomial exists: take a polynomial with simple roots. E.g., for $n = 2$, f is a quadratic polynomial with nonzero discriminant.

Theorem 1. *Let $(X/Z \ni o, B)$ be a pair with a wFt morphism $X/Z \ni o$ and a boundary B on X. Then $(X/Z \ni o, B)$ has an $\mathbb{R}$-complement if and only if $(X/Z \ni o, B)$ has n-complements for infinitely many positive integers n.*

The hard part of the theorem about existence of n-complements will be stated more precisely in Theorem 2 below and proved in Section 4. The converse statement follows from the closed property for $\mathbb{R}$-complements in Theorem 6 below and proved in Section 4.

Restrictions on complementary indices. Let I be positive integers, ε be a positive real number, v be a nonrational vector in a finite dimensional $\mathbb{R}$-linear space $\mathbb{R}^l$, and e be a direction in the rational affine span $\langle v \rangle$ of v. Usually we are looking for n-complements with n satisfying the following properties: for n there exists a rational vector $v_n \in \langle v \rangle$ such that

Divisibility: I divides n;

Denominators: nv_n is integral, that is, $nv_n \in \mathbb{Z}^l$;

Approximation: $\|v_n - v\| < \varepsilon/n$.

Anisotropic approximation:

$$\left\| \frac{v_n - v}{\|v_n - v\|} - e \right\| < \varepsilon.$$

If $\varepsilon \le 1/2$, v_n is unique and is the best approximation with denominator n. Note that $\| \ \|$ denotes the *maximal absolute value* norm [Sh03, Notation 5.16].

Most of applications of the theory of complements are based on these restrictions. The choice of n in applications depend on I, ε, v, e but also on the dimension d.

Theorem 2 (Existence of n-complements)**.** *Let I, ε, v, e be the data as in Restrictions on complementary indices above and $(X/Z \ni o, B)$ be a pair with a wFt morphism $X/Z \ni o$ and a boundary B on X. Suppose also that $(X/Z \ni o, B)$ has an $\mathbb{R}$-complement. Then $(X/Z \ni o, B)$ has n-complements for infinitely many positive integers n under Restrictions on complementary indices with the given data.*

Originally, it was expected that, in a given dimension d, there exists a finite set of complementary indices n such that the existence of an $\mathbb{R}$-complement implies the existence of an n-complement. This is not true if $d \ge 3$ (see Examples 11, (1-3)). We have only the following slightly weaker form of the expectation.

Theorem 3 (Boundedness of n-complements)**.** *Let d be nonnegative integer, $\Delta \subseteq (\mathbb{R}^+)^r, \mathbb{R}^+ = \{a \in \mathbb{R} \mid a \ge 0\}$, be a compact subset (e.g., a polyhedron) and Γ be a subset in the unite segment $[0,1]$ such that $\Gamma \cap \mathbb{Q}$ satisfies the dcc. Let I, ε, v, e be the data as in Restrictions on complementary indices. Then there exists a finite set $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Gamma, \Delta)$ of positive integers (*complementary indices*) such that*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data;*

Existence of n-complement: *if $(X/Z \ni o, B)$ is a pair with wFt $X/Z \ni o$, $\dim X \le d$, connected X_o and with a boundary B, then $(X/Z \ni o, B)$ has an n-complement for some $n \in \mathcal{N}$ under either of the following assumptions:*

(1) *$(X/Z \ni o, B)$ has a klt $\mathbb{R}$-complement; or*

(2) $B = \sum_{i=1}^r b_i D_i$ *with* $(b_1, \dots, b_r) \in \Delta$ *and additionally, for every* $\mathbb{R}$*-divisor* $D = \sum_{i=1}^r d_i D_i$ *with* $(d_1, \dots, d_r) \in \Delta$*, the pair* $(X/Z \ni o, D)$ *has an* $\mathbb{R}$*-complement, where* D_i *are effective Weil divisors (not necessarily prime); or*

(3) $(X/Z \ni o, B)$ *has an* $\mathbb{R}$*-complement and, additionally,* $B \in \Gamma$.

Addendum 1. *We can relax the connectedness assumption on* X_o *and suppose that the number of connected components of* X_o *is bounded.*

Moreover, it is enough to suppose the boundedness up to local (even formal) isomorphisms over $Z \ni o$ *of corresponding neighborhoods.*

Addendum 2. *We can relax also the assumption that the base field* k *is algebraically closed and suppose only that* $\mathrm{char} k = 0$.

Similarly, the theorem holds for G*-pairs where* G *is a finite group.*

Remark 2. (1) The set $\mathcal{N}(d, I, \varepsilon, v, e, \Gamma, r)$ is not unique by Divisibility and Approximation in Restrictions on complementary indices and depends on the data $d, I, \varepsilon, v, e, \Gamma, r$. We use for $\mathcal{N}$ notation $\mathcal{N}(d, I, \varepsilon, v, e, \Gamma, r)$ only to show parameters $d, I, \varepsilon, v, e, \Gamma, r$, on which depend the choice of $\mathcal{N}$. Actually, we need only the partial data d, I, ε, v, e and $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e)$ under the assumption (1) of the theorem.

Below we use also some other sets for boundary multiplicities instead of Γ and with a different meaning (cf. 6.10).

(2) If v is nonrational then $\langle v \rangle$ is not a point and has a direction. Otherwise there are no directions and Anisotropic approximation is void. Nonetheless the other properties hold if we take $v_n = v$ for every $n \in \mathcal{N}$ and suppose that I is sufficiently divisible. In this situation we get $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v)$.

(3) Denominators property means that v_n has rational coordinates with denominators (dividing) n. We already noticed that by Approximation property, if $\varepsilon \leq 1/2$ then v_n is the best rational approximation of v with those denominators.

(4) Our main results and applications work for any base field k of characteristic 0 and G-pairs.

For certain boundaries it is enough a single complementary index, e.g., for boundaries with only hyperstandard multiplicities [PSh08, Theorem 1.4] [B, Theorem 1.7]. Alas, it is not true in general.

Example 2 (Nonsingle index). Let d, n be two positive integers. Take the projective space $X = \mathbb{P}^d$ of dimension d with a boundary

$$B = \sum \frac{1}{n+1} D_i,$$

where D_i runs through $(d+1)(n+1)$ distinct rather general hyperplanes. Then (X, B) is a klt 0-pair, has a klt $\mathbb{R}$-complement and it is actually (X, B) itself by Example 1, (3). However, (X, B) doesn't have n-complements because

$$B^+ \geq \frac{\lfloor (n+1)B \rfloor}{n} = \frac{n+1}{n} B > B$$

by (1) of Definition 2. Hence, for $d \geq 1$, every set $\mathcal{N}(d, I, \varepsilon, v, e)$ under (1-2) of Theorem 3 has at least two indices. Similar examples exist under (3) of the theorem.

Similarly, for any finite set of positive integers $\mathcal{N}$, we can find a pair (X, B) of dimension d which has a klt $\mathbb{R}$-complement but doesn't have n-complements for all $n \in \mathcal{N}$. However X of such an example should be reducible or not connected! This is why in Theorem 3 above and Conjecture 1 below we suppose that $X/Z \ni o$ is local with connected X_o, e.g., with a contraction $X/Z \ni o$ (cf. Remark 3, (2)). (Cf. also Examples 11, (1-3).)

Remark 3. (1) Taking general hyperplane sections of Z, if they are needed, we can assume in Theorems 1-3 and in similar statements that o is a closed point of Z. However we give a more rigorous treatment below for all points o avoiding taking such sections.

(2) Under the assumption that k is algebraically closed, another assumption that $X/Z \ni o$ is a local morphism of [normal] varieties with *connected* X_o in Theorem 3 is necessary for the finiteness of $\mathcal{N}$, the boundedness of n-complements, even in the klt case (1) (cf. Addendum 2). In particular, for algebraically closed k and normal complete X/k, the natural morphism $X/o = \operatorname{Spec} k$ is always local and $X_o = X$ is connected if and only if X is irreducible. By Example 2 the irreducibility is important for boundedness.

According to Addendum 1 we can replace the connectedness of X_o by the *boundedness of connected components*: for a natural number N, Theorem 3 still holds if instead of the connectedness of X_o the fiber X_o has at most N connected components. In particular, for any given pair $(X/Z, B)$ with a wFt morphism into a quasiprojective variety Z or a local one $Z \ni o$ (resp. in étale sense), with a boundary B and over any field k, by Theorem 2 and

Proposition 9 there exists an n-complements for infinitely many n if some $\mathbb{R}$-complement exists even locally over Z. Actually, the same results expected without the wFt assumption for X/Z and $X/Z \ni o$ (see Conjecture 1).

Cf. Example 2 above.

Complements in a nonnormal case are a real challenge.

(3) Another challenge is construction of ε-n-complements [Sh04b] (see Conjecture 5 below).

Main Theorems 1-3 have many important applications. Some of them are already known but some new. Certain special applications we meet in the course of the proof of our main theorems. Among other applications – Acc of log canonical thresholds, etc – see in Section 12.

A few words about the proof of Theorems 1-3. The key part of the paper is the existence and boundedness of Theorem 3 under the existence of klt $\mathbb{R}$-complements, that is, under (1) of the theorem. See the proof in Section 10 and also at the end of Section 11. The theorem under (1) implies the theorem under (3) from which follows the theorem under (2). See the proof in Section 11. In its turn, Theorem 2 is immediate by Theorem 3 under (2) and Addendum 1 or by [BSh, Corollary 1.3]. See the proof in Section 4. Finally, Theorem 1 follows from Theorem 2 and the closed property for $\mathbb{R}$-complements. See the proof also in Section 4.

All n-complements of Theorem 3 under (1) can be obtained from n-complements for exceptional pairs by two standard construction: lifting of n-complements from lower dimensional bases of fibrations and extension of n-complements from lower dimensional lc centers. These constructions are treated respectively in 7.5 and 7.6.

Generalizations and technicalities. Both construction are applied to appropriate birational models of pairs, in particular, we use modifications which blow up divisors. Blowdowns of divisors preserves n-complements but blowups do not so. To overcome this difficulty we introduce b-n-complements in Definition 3, a birational version of n-complements. Note that every monotonic n-complement is automatically a b-n-complement for log pairs (cf. Remarks 1, (2) and 4, (1-2)). In particular, this holds for B with hyperstandard multiplicities and sufficiently divisible n. This is why we do not need such a sophistication for hyperstandard boundaries. But usually, n-complements for a boundary B with arbitrary multiplicities are not b-n-complements by Example 6 (cf. Proposition 2). Fortunately, it is true for an appropriate low approximation of B which are introduced in Hyperstandard sets in Section 3. This leads to reformulation of our main result in terms of b-n-complements and of appropriate approximations of boundaries in Theorem 16 and Addendum 57. Respectively, existence of a (klt) $\mathbb{R}$-complement is transform into existence of an $\mathbb{R}$-complement for approximations of boundaries and hidden in the proof of main results. Additionally we generalize our results for pairs with b-boundaries, including, the Birkar-Zhang pairs. Again this is not our caprice or aspiration for generalization but rather a necessity related to appearance of moduli part of adjunction divisors and the adjunction itself that will be recalled in Section 7. So, we try to use these technical staff only where they are actually needed and left to more advanced readers other possible generalizations.

In most of results we assume that $X/Z \ni o$ has wFt. Expected generalizations in the nonwFt case see in Conjecture 1.

What do we use in the proof? This is a reasonable question, especially, if we would like to understand foundations in the theory of complements and its relation to the LMMP. Actually, we use and very essentially two Birkar's results and the D-MMP for Ft X/Z:

(1) boundedness of n-complements in the hyperstandard case: $B \in \Phi = \Phi(\mathfrak{R})$, where $\mathfrak{R}$ is a finite set of rational numbers in $[0,1]$ [B, Theorem 1.7];

(2) (BBAB) boundedness of ε-lc Fano varieties [B16, Theorem 1.1]; and

(3) the D-MMP holds for every $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on every Ft X/Z [Sh96, Section 5].

Actually, we use (1) only for exceptional pairs (X,B) with a boundary $B \in \Phi$ and $\mathbb{Q}$-factorial Ft X. Moreover, arguments as in [PSh08, Section 6] and [HX] allow to suppose additionally that B has multiplicities only in a finite subset of Φ and $\rho(X) = 1$. In this situation (1) is equivalent to an effective nonvanishing:

(4) let d be a nonnegative integer and $\mathfrak{R}$ be a finite set of rational numbers in $[0,1]$ then there exists a positive integer N such that for every exceptional pair (X,B) with $B \in \mathfrak{R}$ and $\mathbb{Q}$-factorial Ft X with $\dim X = d, \rho(X) = 1$,

$$|-N(K+B)| \neq \emptyset.$$

Moreover, we need the boundedness (2) only in this situation, that is, for X as in (4). This boundedness follows from (4) and a birational boundedness in the log general case [PSh08, Section 4] [HX] [B]. The D-MMP of (3) is well-known and essentially follows form finiteness of (bi)rational 1-contractions of X/Z [ShCh, Corollary 5.5] and extension results [S, Theorem 0.1] [T, Theorem 4.1] [HM, Theorem 5.4]. Thus (4) is exactly what we need. This key result for us is nontrivial and a fundamental one.

Boundedness of lc index in Corollary 31 will be established in two steps: first, for hyperstandard boundary multiplicities and then the general case based on n-complements for arbitrary boundary multiplicities (including non-rational). For hyperstandard multiplicities, we use dimensional induction in the local case and semiexceptional lc type of Theorem 13 in the global case or again [B, Theorem 1.7].

1.1 Notation and terminology

By $X/Z \ni o$ we mean a *local morphism over* Z, where o is a point in Z. That is, X/Z is a morphism $\varphi\colon X \to Z$ into a neighborhood of the point o in Z. We consider such morphisms as germs, that is, they are *equal* if so they do over some (possibly smaller) neighborhood. Similarly, we understand statements about local morphisms: properties, conditions, constructions, etc. For instance, a prime Weil divisor D on X in this situation is such a divisor that D exists over every neighborhood of o, equivalently, $\varphi(D)$ contains o. Note that o is not necessarily closed. Another example of this kind the log canonical property (lc) in (2) of Definitions 1 and 2: it means lc of (X, D^+) over some neighborhood of o in Z, that is, every nonlc center of (X, D^+) does not intersect X_o, the central fiber.

Usually, we suppose that Z is quasiprojective.

Sometimes we use notation $(X, D) \to Z$ instead of a pair $(X/Z, D)$. A local pair $(X/Z \ni o, D)$ is *global* if $X = X_o$. Usually, we denote such a pair by (X, D). Similarly for bd-pairs.

We suppose always that X is a normal irreducible algebraic space or variety over an algebraically closed field k of characteristic 0. Respectively, Z is a quasiprojective algebraic variety or scheme over k and the neighborhoods are in the Zariski topology. Usually, we suppose that the central fiber $X_o = \varphi^{-1}o$ is connected. E.g., this holds when $X/Z \ni o$ is a contraction.

An $\mathbb{R}$-divisor D on X is an element of $\operatorname{WDiv}_{\mathbb{R}} X$, the group of Weil $\mathbb{R}$-divisors on X. Every $\mathbb{R}$-divisor D has a linear presentation in terms of distinct prime Weil divisors D_i, the standard basis of $\mathrm{WDiv}_{\mathbb{R}} X$: $D = \sum d_i D_i$, where $d_i = \mathrm{mult}_{D_i} D \in \mathbb{R}$ is the multiplicity of D in D_i.

For a pair $(X/Z \ni o, D)$ with a local morphism $X/Z \ni o$, $D = \sum d_i D_i$ is an $\mathbb{R}$-divisor on X, defined locally over $Z \ni o$[, that is, the prime divisors D_i intersect the central fiber X_o].

An $\mathbb{R}$-divisor $D = \sum d_i D_i$ on X is a *subboundary* if all $d_i \le 1$. Respectively, D is a *boundary* if all $0 \le d_i \le 1$.

For an $\mathbb{R}$-divisor $D = \sum d_i D_i$ on X and a subset $\Gamma \subseteq \mathbb{R}$, $D \in \Gamma$ denotes that every $d_i \in \Gamma$. E.g., D is a subboundary if and only if $D \in (-\infty, 1]$. Respectively, D is a boundary if and only if $D \in [0, 1]$, the unite segment. Tacite agreement that $0 \in \Gamma$ always.

$K = K_X$ denotes a canonical divisor on X. K_Y ditto on any other Y.

For a natural number n, two $\mathbb{R}$-divisors D and D' on X are *n-linear equivalent* if $nD \sim nD'$ where $\sim$ denotes the linear equivalence. Respectively, $\sim_n$ will denote n-linear equivalence.

Two $\mathbb{R}$-divisors D and D' are $\mathbb{R}$-linear equivalent if $D - D'$ is $\mathbb{R}$-principal, that is, an $\mathbb{R}$-linear combination of principal divisors. We denote the equivalence by $\sim_{\mathbb{R}}$.

In general, a linear equivalence $D \sim_Z D'$ or $D \sim D'/Z$ on X over Z is a local linear equivalences over Z, that is, $D - D' \sim 0$ on some (open) neighborhood of any fiber of X/Z. That is, $D - D'$ is locally principal over Z. This does not imply that $D - D' \sim 0$ or principal on whole X. However, if X/Z is proper and Z is quasiprojective then this is true modulo vertical (base point) free divisors on X:

$$D - D' \sim V - \varphi^* H,$$

where V is a vertical (base point) free divisor on X and H is a very ample divisor on Z. The difference is an integral linear combination of vertical free divisors. Note that V is automatically vertical, if it is free, because V is numerically trivial on every fiber of X/Z. Moreover, if X/Z is a contraction then $V = \varphi^* C$, where C is a free divisor on Z [Sh19, Proposition 3]. Thus $V - \varphi^* H = \varphi^*(C - H)$, an inverse image of a Cartier divisor from Z. The same holds for $\sim_{\mathbb{Q},Z}$ with vertical $\mathbb{Q}$-free or $\mathbb{Q}$-semiample divisors. The relative $\mathbb{R}$-linear equivalence $\sim_{\mathbb{R},Z}$ is more subtle. We usually, use this relation in the local sense, that is, $D - D'$ is $\mathbb{R}$-principal locally over Z. However, if X/Z has wFt then

$$D - D' \sim_{\mathbb{R}} V - \varphi^* H,$$

for some vertical $\mathbb{R}$-free divisor V on X and some ample divisor H on Z. Moreover, if X/Z is a contraction then $V = \varphi^* C$ for an $\mathbb{R}$-free divisor C on Z.

A linear equivalences on X over $Z \ni o$ are local by definition: linear equivalences on X over a neighborhood of o. Respectively, a *numerical* equivalence $\equiv /Z \ni o$ is the numerical equivalence with respect to a (proper) local morphism $X/Z \ni o$ (or ,more generally, $\equiv$ on every complete curve of X; cf. Nef below).

A proper local pair $(X/Z \ni o, D)$ is a 0-*pair* if (X, D) is lc and $K + D \sim_{\mathbb{R}} 0/Z \ni o$. If D is a boundary we can replace $\sim_{\mathbb{R}}$ by $\equiv /Z$ (see Remark 1, (4) above).

Nef. In the paper, if it is not stated otherwise, we suppose that the (b-)nef property of a divisor means nonnegative of its intersection with every complete curve [Sh17, Remark 1]: an $\mathbb{R}$-divisor D on X is nef if it is $\mathbb{R}$-Cartier and $(C.D) \geq 0$ for every complete curve C in X (respectively, on a stable model of X). It is the usual nefness if X is complete and respectively the relative one over a scheme or a space S if X/S is proper. Notice that if D is nef then D is nef over S for every proper $X \to S$. That nefness is typical for the b-nef property of $\mathcal{D}_{\text{mod}}$ (cf. Theorem 8 and Conjecture 3).

Fano and weak Fano types. Both types can be defined in terms of $\mathbb{R}$-complements.

A *weak Fano type* (wFt) morphism $X/Z \ni o$ is such a proper local morphism that the pair $(X/Z \ni o, 0)$ has a klt $\mathbb{R}$-complement $(X/Z \ni o, B)$ with big B. Equivalently, there exists a big boundary B on X such that $(X/Z \ni o, B)$ is a klt 0-pair. Similarly, we can define a wFt morphism X/Z. Actually, for quasiprojective Z, the latter morphism has wFt if and only if it has wFt locally over Z.

Respectively, a *Fano type* (Ft) morphism is such a morphism that the pair $(X/Z \ni o, B)$ is a klt log Fano for some boundary B on X. (In particular, $X/Z \ni o$ is proper.) Equivalently, $X/Z \ni o$ has Ft if and only if $X/Z \ni o$ has wFt and projective (cf. Example 1, (1)). The same works for X/Z.

For example, every toric morphism has wFt and Ft if it is projective.

Lemma 1. *Proper X/Z has wFt if and only if X/Z is a small birational modification of Ft Y/Z.*

Proof. Almost by definition, Ft Y/Z has a klt $\mathbb{R}$-complement with big B. By Proposition 3 below every small birational modification X/Z has a klt $\mathbb{R}$-complement with big B. Thus X/Z has wFt. Notice also that the big property is invariant under small birational modifications.

Conversely, let $(X/Z, B)$ be a klt 0-pair with big B. A required model Y/Z is an lc model of $(X/Z, (1+\varepsilon)B)$ up to possible resolution of some divisors, where ε is a sufficiently small positive real number. We can suppose that X is $\mathbb{Q}$-factorial. Use a Zariski decomposition $B = M + F$, M is big. By construction $(Y/Z, B_Y)$ is also a klt 0-pair where B_Y is the image of B on Y. Moreover, $X \dashrightarrow Y$ is a 1-contraction and contracts only prime Weil divisors D of X trivial with respect to $K+B$. They have the log discrepancy $a(X, B, D) = a(Y, B_Y, D) \in (0, 1]$. By [Sh96, Theorem 3.1] we can blow up them projectively! $\square$

Corollary 1. *Let X/Z be of wFt and D be an $\mathbb{R}$-Cartier divisor on X. Then D is semiample over Z if and only if D is nef over Z.*

Proof. It is sufficient to verify that if D is nef over Z then D is semiample over Z. If X/Z has Ft it well-known. In general, by Lemma 1 there exists a birational model Y/Z of X/Z such that $X \dashrightarrow Y$ is a small modification (isomorphism in codimension 1) and Y/Z has Ft. We can suppose also that Y is $\mathbb{Q}$-factorial. Thus D on Y is also $\mathbb{R}$-Cartier. However, the small modification may not respect nef and semiample properties.

By definition there exists a big over Z boundary B on X such that $(X/Z, B)$ is a klt 0-pair. This implies that there exists a boundary B' on X and some positive ε such that (X, B') is klt and $K + B' \sim_{\mathbb{R},Z} \varepsilon D$. The semiampleness conjecture implies the required statement. The conjecture holds for Ft X and for wFt X too. Indeed it is enough to verify that $K_Y + B'$ gives a b-contraction over Z or $(Y/Z, B')$ has a minimal model. We do not know the pleudoeffective over Z property of D because X/Z is not projective. Instead we know that $(C.D) \geq 0$ for every curve C on X over Z not passing through a subset of codimension ≥ 2. The same property holds for $K_Y + B' \sim_{\mathbb{R},Z} \varepsilon D$ on Y/Z. It is sufficient for existence of a minimal model. $\square$

Corollary 2. *Let d be a nonnegative integer. There exists a positive integer N depending only on d such that if X/Z has wFt with $\dim X = d$ and D is a nef over Z Cartier divisor on X then ND is base free over Z.*

Proof. If D is big over Z the corollary follows the relative version of [K93, Thorem 1.1]. The numerically trivial over Z case can be reduced to Ft by Lemma 1 and holds by ibid. Otherwise by Corollary 1 there exists a contraction $f\colon X \to V/Z$ and an $\mathbb{Q}$-ample divisor A on V such that $f^*A \sim_{\mathbb{Q},Z} D$. If we can replace the $\mathbb{Q}$-ample property and respectively $\sim_{\mathbb{Q},Z}$ by m-ample (that is, mA is ample Cartier) and $\sim_{m,Z}$, there m is a positive integer depending only on d, then we get a required effective semiampleness for D because V has also Ft. Below we slightly modify this idea. It is easy by the (relative) LMMP over V to get a Mori klt fibration $g\colon T \to W/V$ such that

$X \dashrightarrow T$ is a birational 1-contraction, composition of extremal divisorial contractions and flips; they preserve nef, semiample and freeness property under the birational transformation of D;

D_T, the birational transform of D on T, is nef over Z and numerically trivial over W; it is enough to verify that D_T is free over Z;

there exists a nef and big over Z $\mathbb{Q}$-Cartier divisor D_W on W and a positive integer m, depending only on d, such that $g^*D_W \sim D_T$ and mD_W is Cartier; here we use the boundedness of torsions for the relative case; $W/V, V$ have wFt.

Thus the freeness of $N'mD_T$ over Z follows from that of $N'mD_W$ in the big over Z case.

□

For a natural number l, $\mathbb{R}^l$ denotes the arithmetic $\mathbb{R}$-linear space of dimension l. It contains a lattice of *integral* vectors $\mathbb{Z}^l$, a $\mathbb{Q}$-linear subspace $\mathbb{Q}^l$ of *rational* vectors, and is defined over $\mathbb{Z}$ and over $\mathbb{Q}$:

$$\mathbb{R}^l = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^l = \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}^l \supset \mathbb{Q}^l = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^l \supset \mathbb{Z}^l.$$

In particular, we know whether an affine subspaces of $\mathbb{R}^l$ is *rational*, that is, defined by linear (not necessarily homogeneous) equations with rational or integral coefficients (rational hyperplanes). A vector $v = (v_1, \dots, v_l) \in \mathbb{R}^l$ is *nonrational* if $v \not\in \mathbb{Q}^l$, equivalently, one of its coordinates v_i in the standard basis $[e_1 =](1, 0, \dots, 0), [e_2 =](0, 1, \dots, 0), \dots, [e_l =](0, 0, \dots, 1)$ of $\mathbb{R}^l$ is nonrational.

The rational affine span $\langle v \rangle$ of a vector $v \in \mathbb{R}^l$ is the smallest rational affine subspace of $\mathbb{R}^l$ containing v. Hence v is rational if and only if $\langle v \rangle$ is v itself.

A *direction* of an affine $\mathbb{R}$-space is a unite vector in its $\mathbb{R}$-linear (tangent) space of translations.

We use a standard norm on $\mathbb{R}^l$, e.g., the *maximal absolute value* norm

$$\|v\| = \|(v_1, \dots, v_l)\| = \max \|v_i\|,$$

where $\|v_i\|$ denotes the absolute value of i-th coordinate. The norm on $\langle v \rangle$ is restricted from $\mathbb{R}^l$. So, a direction in $\langle v \rangle$ is a vector $e \in \mathbb{R}^l$ such that

$$e + \langle v \rangle = \langle v \rangle \text{ and } \|e\| = 1.$$

A direction in $\langle v \rangle$ exists exactly when v is nonrational because then $\dim_{\mathbb{R}} \langle v \rangle \geq 1$.

b-Codiscrepancy [Sh03, p. 88]. Let (X, D) be a *log* pair, that is, $K + D$ is $\mathbb{R}$-Cartier. Then $\mathbb{D} = \mathbb{D}(X, D)$ will denote its b-$\mathbb{R}$-divisor of *codiscrepancy* or *pseudo-boundary*: by definition

$$\overline{K + D} = \mathbb{K} + \mathbb{D},$$

where $\mathbb{K}$ is a canonical b-divisor of X.

2 Elementary

2.1 Two basic inequalities for the Gauss integral part.

For any two real numbers a, b,

$$\lfloor a + b \rfloor \leq \lfloor a \rfloor + \lfloor b \rfloor + 1. \tag{2.1.1}$$

Proof. Any integral shift $a \mapsto a + n, n \in \mathbb{Z}$, gives an equivalent inequality. The same holds for b. So, after appropriate shifts, we can suppose that $\lfloor a \rfloor = \lfloor b \rfloor = 0$. That is, $0 \leq a, b < 1$. Hence $a + b < 2$ and

$$\lfloor a + b \rfloor \leq 1 = \lfloor a \rfloor + \lfloor b \rfloor + 1.$$

□

For any two real numbers a, b,

$$\lfloor a + b \rfloor \geq \lfloor a \rfloor + \lfloor b \rfloor. \tag{2.1.2}$$

Proof. As in the proof of the inequality (2.1.1) we can suppose that $\lfloor a \rfloor = \lfloor b \rfloor = 0$. Then $a + b \geq 0$ and

$$\lfloor a + b \rfloor \geq 0 = \lfloor a \rfloor + \lfloor b \rfloor .$$

$\square$

Example 3 (n-complement with $\mathbb{P}^1$; cf. Example 13). The main case for existence and boundedness of n-complements in dimension 1 concerns n-complements for (global) pairs $(\mathbb{P}^1, B)$, where $B = \sum b_i P_i$ is a boundary on $\mathbb{P}^1$: $b_i \in [0, 1]$ and P_i are distinct (closed) points on $\mathbb{P}^1$. The assumption (1) in Theorem 3 in this case can be strengthen to existence on an $\mathbb{R}$-complement, equivalently,

$$\sum b_i \leq 2 = -\deg K_{\mathbb{P}^1} \text{ or } \sum{}' b_i \leq c = 2 - l, \tag{2.1.3}$$

where $0 \leq l \leq 2$ is the number of points P_i with $b_i = 1$ (lc singularities) and $\sum'$ runs for $b_i \in [0, 1)$. On the other hand, existence of an n-complement means that

$$\sum{}' \lfloor (n + 1) b_i \rfloor / n \leq c = 2 - l. \tag{2.1.4}$$

Indeed, under the inequality (2.1.4), we can take

$$B^+ = \frac{\Delta}{n} + \sum{}' \frac{\lfloor (n + 1) b_i \rfloor}{n} P_i + \sum{}'' P_i,$$

where Δ is a reduced divisor supported outside of $\operatorname{Supp} B$ with

$$2n - ln - \sum{}' \lfloor (n + 1) b_i \rfloor = 2c - \sum{}' \lfloor (n + 1) b_i \rfloor$$

points and $\sum''$ runs for $b_i = 1$. To satisfy the inequality (2.1.4) we can joint together small b_i in larger multiplicities because, for any two real numbers a and b,

$$\lfloor (n + 1)(a + b) \rfloor / n \geq \lfloor (n + 1) a \rfloor / n + \lfloor (n + 1) b \rfloor / n$$

by (2.1.2). The inequality (2.1.3) holds again. However, we need to keep our assumption that the new multiplicities also belong to $[0, 1)$. This allows to suppose that all $b_i \geq 1/2$ with at most one exception $0 < b_i < 1/2$ and we have at most 4 nonzero multiplicities b_i. In this case n-complements exits and bounded by [Sh95, Example 1.11]. Moreover, we can suppose that Restrictions on complementary indices hold.

Notice that c in inequalities (2.1.3-2.1.4) can be replaced by any rational (and even real number if omit Restrictions on complementary indices). In particular, for $c = d+1$ this implies existence and boundedness of n-complements for $(\mathbb{P}^d, B)$, where $B = \sum b_i H_i, b_i \in [0,1]$, H_i are hypersurfaces in general position and $(\mathbb{P}^d, B)$ has an $\mathbb{R}$-complement (cf. [Sh95, Example 1.11]).

2.2 Inductive bounds

For any positive integer n and any real numbers $d_1, \dots, d_n$,

$$\left\lfloor \sum_{i=1}^n d_i \right\rfloor \leq n-1+\sum_{i=1}^n \lfloor d_i \rfloor . \tag{2.2.1}$$

Proof. For $n=1$, the inequality actually become the equality

$$\lfloor d_1 \rfloor = 1-1+\lfloor d_1 \rfloor .$$

For $n \geq 2$, we use the upper bound (2.1.1) and induction:

$$\left\lfloor \sum_{i=1}^n d_i \right\rfloor \leq 1+\lfloor d_1 \rfloor + \left\lfloor \sum_{i=2}^n d_i \right\rfloor \leq 1+n-2+\sum_{i=1}^n \lfloor d_i \rfloor = n-1+\sum_{i=1}^n \lfloor d_i \rfloor .$$

□

Example 4.

$$1-l+\sum_{i=1}^l \lfloor (n+1)d_i \rfloor /n \geq \left\lfloor (n+1)(1-l+\sum_{i=1}^l d_i) \right\rfloor /n,$$

for any positive integers n, l and real numbers d_i. Indeed, the inequality is equivalent to

$$l-1+\sum_{i=1}^l \lfloor (n+1)d_i \rfloor \geq \left\lfloor \sum_{i=1}^l (n+1)d_i \right\rfloor ,$$

which follows from (2.2.1).

Another application of (2.2.1). For any positive integer n and any real number d,

$$\lfloor nd \rfloor \leq n-1+n\lfloor d \rfloor . \tag{2.2.2}$$

Proof. By the inequality (2.2.1) with $d_1 = \cdots = d_n = d$. $\square$

Using the low bound (2.1.2) and induction we get, for any nonnegative integer n and any real numbers $d_1, \ldots, d_n$,

$$\left\lfloor \sum_{i=1}^{n} d_i \right\rfloor \geq \sum_{i=1}^{n} \lfloor d_i \rfloor \tag{2.2.3}$$

and

$$\lfloor nd \rfloor \geq n \lfloor d \rfloor . \tag{2.2.4}$$

2.3 Special upper bound

For any real numbers r, s such that $r < 1$ and sr is integral,

$$\lfloor (s+1)r \rfloor \leq sr. \tag{2.3.1}$$

Proof. Since $sr \in \mathbb{Z}$ and $r < 1$,

$$\lfloor (s+1)r \rfloor = \lfloor sr + r \rfloor = sr + \lfloor r \rfloor \leq sr.$$

$\square$

[Even] if $s = n$ is a positive integer then the inequality (2.3.1) is equivalent to

$$\lfloor (n+1)r \rfloor / n \leq r$$

and may not hold unless nr is integral.

Example 5. (1) For $s = 1$ and $r = 1/2$,

$$\lfloor (1+1)1/2 \rfloor = 1 \not\leq 1(1/2) = 1/2.$$

(2) However, if $d \in \mathbb{Z}/n$ and < 1 then

$$\lfloor (n+1)d \rfloor / n \leq d < 1$$

and $= d$ if and only if $d \geq 0$.

(3) For any real number $d < 1$ and any positive integer n,

$$\lfloor (n+1)d \rfloor / n \leq 1.$$

The low bound

$$\lfloor (n+1)d \rfloor /n \geq d$$

does not work in general (cf. [PSh08, Lemma 3.5] and see the next example).

(4) There exists a closed dcc subset $\Gamma \subset [0,1] \cap \mathbb{Q}$ with the only accumulation point 1 and such that, for every positive integer n, there exists $b \in \Gamma$ with $\lfloor (n+1)b \rfloor /n < b$. For given n, consider a rational number $b(n) < n/(n+1)$ which is very close to $n/(n+1)$. Then

$$\frac{n-1}{n} = \lfloor (n+1)b(n) \rfloor /n < b(n).$$

So, $\Gamma = \{b(n) \mid n \in \mathbb{N}\}$ is a required set.

2.4 Approximations and round down

Lemma 2. *Let I, n be two positive integers, l be a nonnegative integer and b be a real number such that $I|n$, $b, l/I \in [0,1)$ and $\|b - l/I\| < 1/I(n+1)$. Then*

$$\lfloor (n+1)b \rfloor /n = \left\lfloor (n+1)\frac{l}{I} \right\rfloor /n = \frac{l}{I}.$$

Proof.

$$\left\lfloor (n+1)\frac{l}{I} \right\rfloor /n = \left\lfloor \frac{nl}{I} + \frac{l}{I} \right\rfloor /n = \frac{nl}{I}\frac{1}{n} = \frac{l}{I}$$

because $I|n$. Let $\delta = \|b - l/I\|$. Then $\delta < 1/I(n+1)$ and $b = l/I \pm \delta$. For $b = l/I + \delta$,

$$\left\lfloor (n+1)(\frac{l}{I} + \delta) \right\rfloor /n = \left\lfloor \frac{nl}{I} + \frac{l}{I} + (n+1)\delta \right\rfloor /n = \frac{nl}{I}\frac{1}{n} = \frac{l}{I}.$$

For $b = l/I - \delta$, $l \geq 1$ and

$$\left\lfloor (n+1)(\frac{l}{I} - \delta) \right\rfloor /n = \left\lfloor \frac{nl}{I} + \frac{l}{I} - (n+1)\delta \right\rfloor /n = \frac{nl}{I}\frac{1}{n} = \frac{l}{I}.$$

$\square$

Lemma 3. *Let n be a positive integer and b be a real number such that $b \in [0,1)$ and $\|b - 1\| < 1/(n+1)$. Then*

$$\lfloor (n+1)b \rfloor /n = 1.$$

Proof. Use the proof of Lemma 2 with $I = l = 1$ and $b = 1 - \delta$. □

Similarly we can verify the following estimations.

Lemma 4 ([Sh06, Lemma 4])**.** *Let n be a positive integer, b be a real number and $b_n = m/n, m \in \mathbb{Z}$, be a rational number with the denominator n such that*

$$\mid b - b_n \mid = \varepsilon/n.$$

Then

(1) $b^+ = \lfloor (n+1)b \rfloor /n \le b_n$ *if* $\varepsilon < 1 - b$*;*

(2) $b^+ \ge b_n$ *if* $\varepsilon \le b$*; and*

(3) $b^+ = b_n$ *if* $\varepsilon < \min\{b, 1-b\}$.

3 Technical

We introduce now a b-version of Definition 2. It will be crucial for our induction.

Definition 3. Let n be a positive integer and $(X/Z \ni o, D)$ be a *log* pair with a local morphism $X/Z \ni o$ and with an $\mathbb{R}$-divisor D. A pair $(X/Z \ni o, D^+)$, with the same local morphism and with a $\mathbb{Q}$-divisor D^+ on X, is a b-*n-complement* of $(X/Z \ni o, D)$ if $(X/Z \ni o, D^+)$ is an n-complement of $(X/Z \ni o, D)$ and the same hold for all crepant models of $(X/Z \ni o, D)$ and of $(X/Z \ni o, D^+)$. Equivalently, instead of (1) in Definition 2 the following b-version holds:

(1-b) for every prime b-divisor P of X,

$$d^+ \ge \begin{cases} 1, \text{ if} & d = 1; \\ \lfloor (n+1)d \rfloor /n & \text{otherwise}, \end{cases}$$

where $d = \operatorname{mult}_P \mathbb{D}$ and $d^+ = \operatorname{mult}_P \mathbb{D}^+$ are multiplicities at P of codiscrepancy b-divisors $\mathbb{D}$ and $\mathbb{D}^+$ of (X, D) and of (X, D^+) respectively.

The b-n-complement is *monotonic* if $\mathbb{D}^+ \ge \mathbb{D}$.

Remark 4. (1) $K+D, K+D^+$ are $\mathbb{R}$-Cartier because respectively $(X/Z \ni o, D)$ is a log pair and by Definition 2, (3). So, b-divisors $\mathbb{D}$ and $\mathbb{D}^+$ are well-defined. Actually $\mathbb{D}$ is well-defined and the definition is meaningful *only if* (X, D) is a log pair (cf. the remark (2) below). Moreover, the definition is *birational*: we can replace $(X/Z \ni o, D)$ by a log pair $(X'/Z \ni o, D')$ with a model $X'/Z \ni o$ of $X/Z \ni o$. Indeed, then $(X/Z \ni o, D^+)$ is a b-n-complement of $(X'/Z \ni o, D')$ if $(X/Z, \mathbb{D}^+)$ is (b-)n-complement of $(X'/Z, \mathbb{D}')$, where $\mathbb{D}' = \mathbb{D}(X', D')$ (cf. (10) in 7.5). In this situation we have induced b-n-complement $(X'/Z \ni o, \mathbb{D}^+{}_{X'})$ of $(X'/Z \ni o, D')$.

If $(X/Z \ni o, D)$ is not necessarily a log pair then we can also consider b-n-complements taking its small log pair model when such a model exists. For instance we can take a $\mathbb{Q}$-factorialization. In general, it is not expected that b-n-complements are birationally invariant, even for small birational modifications, if they are noncrepant (cf. Proposition 3 below). However, if $(X/Z \ni o, D)$ has a b-n-complement for a maximal (small) model then the b-n-complements are well-defined under small birational modifications (see Construction 2 and Statement 8).

(2) There are no reason to introduce b-$\mathbb{R}$-complements because usual $\mathbb{R}$-complements already do so. Indeed, if $(X/Z \ni o, D)$ is a log pair then (1-3) of Definition 1 are equivalent respectively to

(1) $\mathbb{D}^+ \geq \mathbb{D}$;

(2) $(X, \mathbb{D}^+)$ is lc; and

(3) $\mathbb{K} + \mathbb{D}^+ \sim_{\mathbb{R}} 0$.

We can use $\equiv$ over $Z \ni o$ instead of $\sim_{\mathbb{R}}$ if $X/Z \ni o$ has wFt (see Remark 1, (4)). Since by Proposition 3 below small birational modifications preserve $\mathbb{R}$-complement we get (3) for every small log model of $(X/Z \ni o, D)$ (when it exists). These models may have different b-codiscrepancies. In particular, the inequality (1) also holds for the largest one $\mathbb{D}^\sharp$ in Construction 2 (see Statement 8).

Example 6. (1) Let $(X/Z, D^+)$ be an n-complement of $(X/Z, D)$. Then $(X/Z, D^+)$ is a monotonic b-n-complement of itself. It is enough to verify (1-b) of Definition 3. That is, for every d^+,

$$d^+ \geq \begin{cases} 1, \text{ if} & d^+ = 1; \\ \lfloor (n+1)d^+ \rfloor / n & \text{otherwise.} \end{cases}$$

It follows from the inequality in Example 5, (2). Indeed, $d^+ \in \mathbb{Z}/n$ and ≤ 1 by Definition 2, (2-3).

The complement is monotonic: $\mathbb{D}^+ \geq \mathbb{D}^+$.

(2) Let $(X/Z, D^+)$ be a monotonic n-complement of a log pair $(X/Z, D)$. Then $(X/Z, D^+)$ is a monotonic b-n-complement of $(X/Z, D)$. This follows from Proposition 1 below. This is why we do not need to state this property explicitly for n-complements of pairs with hyperstandard boundary multiplicities [PSh08] [B] even we use it in the course of proof.

(3) Let

$$(\mathbb{A}^2 \ni P, \frac{1}{3}(L_1 + L_2 + L_3))$$

be a local pair on the affine plane $\mathbb{A}^2$ with three distinct lines L_1, L_2, L_3 passing through a point P. Then $(\mathbb{A}^2, 0)$ with $D^+ = 0$ is an 1-complement of this pair but $(A^2, 0)$ is not a b-1-complement of the pair. Indeed, let E be the exceptional divisor of the usual blowup of P. Then $d^+ = \operatorname{mult}_P \mathbb{D}^+ = -1$ but

$$d = \operatorname{mult}_P \mathbb{D}(A^2, \frac{1}{3}(L_1 + L_2 + L_3)) = 0$$

and $\lfloor 2 \cdot 0 \rfloor /1 = 0 > -1 = d^+$.

The next important for us technic were developed by Birkar and Zhang [BZ].

bd-Pairs. In general, it is a pair $(X/Z, D + \mathcal{P})$ with an $\mathbb{R}$-divisor D and another data $\mathcal{P}$

(Birkar-Zhang) $\mathcal{P}$ is a b-$\mathbb{R}$-Cartier divisor of X, defined up to $\sim_{\mathbb{R}}$;

(Alexeev) $\mathcal{P} = \sum r_i L_i$, where L_i are mobile linear systems on X and r_i are real numbers (the systems are not necessarily finite dimensional if X is not complete);

(b-sheaf) $\mathcal{P} = \sum r_i \mathcal{F}_i$, where $\mathcal{F}_i$ are invertible sheaves on a proper birational model Y/Z of X/Z and r_i are real numbers; sheaves are defined up to isomorphism.

It is easy to covert the b-sheaf data into the Birkar-Zhang data: to replace every sheaf $\mathcal{F}_i$ by a b-divisor $\overline{H_i}$, where H_i is a divisor on Y with $\mathcal{O}_Y(H_i) \simeq \mathcal{F}_i$. Similarly, we can convert the Alexeev data into the Birkar-Zhang data: to replace every linear system L_i by $\overline{H_i}$ for its (sufficiently general) element H_i

on a model Y, where the linear system is free. However, a converse does not hold in general. In the paper usually a bd-pair means a Birkar-Zhang one or a translation into it of an Alexeev or b-sheaf one (cf. [PSh08, Corollary 7.18, (ii)]). In particular, for every bd-pair $(X, D+\mathcal{P})$, a pair $(X, D+\mathcal{P}_X)$ with the trace $\mathcal{P}_X$ is well-defined. So, $\mathcal{P}_X$ is an $\mathbb{R}$-divisor defined up to $\sim_{\mathbb{R}}$.

A pair $(X/Z, D+\mathcal{P})$ is a *log* bd-pair if $(X/Z, D+\mathcal{P}_X)$ is a log pair, that is, $K+D+\mathcal{P}_X$ is $\mathbb{R}$-Cartier. So, we can control log singularities of $(X/Z, D+\mathcal{P})$ but not in a usual sense because $\mathcal{P}_X$ is defined up to $\sim_{\mathbb{R}}$. Respectively, the b-divisor of codiscrepancy $\mathbb{D} = \mathbb{D}(X, D+\mathcal{P}_X)$ is not unique but does so up to $\mathcal{P}$: $\mathbb{D}_{\mathrm{div}} = \mathbb{D} - \mathcal{P}$ is a b-divisor which depend only on $(X/Z, D+\mathcal{P})$ and is the same for every $\mathcal{P}$ up to $\sim_{\mathbb{R}}$. Note also that $\mathbb{D}_{\mathrm{div},X} = D$ is the trace. So, we can defined lc, klt, etc $(X/Z, D+\mathcal{P})$ (cf. subboundaries in [Sh92] and [Sh96, 1.1.2]). Similarly, numerical properties also can be imposed on $(X/Z, D+\mathcal{P}_X)$ and thus on $(X/Z, D+\mathcal{P})$. After that we can introduce a lot of concepts from the LMMP for those pairs. E.g., a log bd-pair $(X/Z, D+\mathcal{P})$ is (bd-)*minimal* if

$(X, D+\mathcal{P})$ is lc, or equivalently, $\mathbb{D}_{\mathrm{div}}$ is a b-subboundary, that is, for every prime b-divisor P, $\mathrm{mult}_P\, \mathbb{D}_{\mathrm{div}} \le 1$; and

$K+D+\mathcal{P}_X$ is nef over Z, or equivalently, $\mathbb{K}+\mathbb{D}$ is b-nef over Z, e.g., $\mathcal{P}$ has associated b-divisor $\mathbb{P}$ ($= \mathcal{P}$ for Birkar-Zhang pairs) and X/Z has a proper birational model Y/Z with a nef $\mathbb{R}$-divisor H such that $\mathbb{K}+\mathbb{D} = \mathbb{K}+\mathbb{D}_{\mathrm{div}}+\mathbb{P} = \overline{H}$.

In other words, singularities are controlled by the divisorial part $\mathbb{D}_{\mathrm{div}}$ and positivity properties by whole $\mathbb{D}$. It is not a new phenomenon. Since $\mathcal{P}$ has no a fixed support for the Alexeev and b-sheaf data it does not effect singularities: they only related to $\mathbb{D}_{\mathrm{div}}$. A corresponding theory for pairs with nonnegative multiplicities r_i is well-known as the MMP for Alexeev or mobile pairs [C, Defenition 1.3.1 and Section 1.3]. The Birkar-Zhang data, defined up to an $\mathbb{R}$-linear even numerical equivalence, shares the same property of singularities and a version of the LMMP.

Crepant bd-models. (Cf. crepant 0-contractions in 7.1.) We say that two bd-pairs $(X'/Z', D'+\mathcal{P}), (X/Z, D+\mathcal{P})$ are *birationally equivalent* or *crepant*, if both pairs are log bd-pairs and there exists a commutative diagram

$$\begin{array}{ccc} (X', D' + \mathcal{P}) & \dashrightarrow & (X, D + \mathcal{P}) \\ \downarrow & & \downarrow \\ Z' & \dashrightarrow & Z \end{array},$$

where the horizontal arrow $(X', D' + \mathcal{P}) \dashrightarrow (X, D + \mathcal{P})$ is a crepant proper birational isomorphism, another horizontal arrow $Z' \dashrightarrow Z$ is a proper birational isomorphism. The crepant property for the horizontal arrow is similar to the case of usual pairs and means that $D' = D_{X'} = D_{\mathrm{div},X'} = (\mathbb{D}_{\mathrm{div}})_{X'}, \mathbb{D}'_{\mathrm{div}} = \mathbb{D}(X', D' + \mathcal{P}_{X'}) = \mathbb{D}(X, D + \mathcal{P}_X) = \mathbb{D}_{\mathrm{div}}$. We say also that $(X'/Z', D' + \mathcal{P})$ is a (*crepant*) *model* of $(X/Z, D + \mathcal{P})$. Notice that crepant bd-pairs have the same b-$\mathbb{R}$-divisor $\mathcal{P}$.

Example 7. Let $\varphi\colon X' \to X$ be a birational contraction. Then the crepant bd-pair $(X', D' + \mathcal{P})$ of $(X, D + \mathcal{P})$ has D' given uniquely by equation

$$\varphi^*(K + D + \mathcal{P}_X) = D' + \mathcal{P}_{X'}.$$

Indeed, $\varphi^*(K + D + \mathcal{P}_X) = \mathbb{D}(X, D + \mathcal{P}_X)_{X'} = (\mathbb{D}_{\mathrm{div}} + \mathcal{P})_{X'} = D' + \mathcal{P}_{X'}$ and $D' = D_{X'}$.

The typical example of bd-pairs is related to log adjunction, where $D, \mathbb{D}_{\mathrm{div}}$ are respectively divisorial and b-divisorial parts of adjunction, and $\mathcal{P}$ is the (b-)moduli part of adjunction (see 7.1).

However, to have a good theory even for log bd-pairs we need more restrictions [BZ]. We follow Birkar and Zhang, and always suppose that every bd-pair satisfies

(index m) $m\mathcal{P}$ is a Cartier b-divisor for the Birkar-Zhang data and $\mathcal{P}$ is defined up to $\sim_m$, where m is the positive integer; respectively, for the Alexeev and b-sheaf data every $mr_i \in \mathbb{Z}$;

(positivity) $\mathcal{P}$ is a b-nef $\mathbb{R}$-divisor of X for the Birkar-Zhang data; respectively for the Alexeev and b-sheaf data every r_i is a nonnegative real number, every $\mathcal{F}_i$ is a nef invertible sheaf. In particular, the b-nef property over Z can be applied to proper X/Z (cf. Nef in Section 1).

Such a bd-pair $(X/Z, D + \mathcal{P})$ will be called a *bd-pair of index* m, where m is a positive integer. So, for bd-pairs of index m: $\mathcal{P}$ is b-nef, in particular, $\mathcal{P}$ is b-nef over S in the usual sense for every proper X/S.

The sum $D+\mathcal{P}$ is formal. Hence really the bd-pair is a triple $(X/Z, D, \mathcal{P})$.

If $(X/Z, D+\mathcal{P}_X)$ is not a log pair then we can take its small birational log model $(Y/Z, D+\mathcal{P}_Y)$ with the birational transform of D. b-Codiscrepancy $\mathbb{D} = \mathbb{D}(Y, D_Y + \mathcal{P})$ and $\mathbb{D}_{\mathrm{div}}$ depend on the model but $\mathcal{P}$ is same. So, in statements, where we use pairs $(X/Z, D+\mathcal{P})$, in the results which concern birational concepts, e.g., b-n-complements, we add the assumption that $(X/Z, D+\mathcal{P}_X)$ is a log pair (cf. Addendum 30 for exceptional n-complements and similar addenda for other type of n-complements).

Note for applications that if $\mathcal{P}$ is a b-sheaf than

$$m\mathcal{P} = \otimes\mathcal{F}_i^{\otimes mr_i} \simeq \mathcal{O}_Y(H)$$

is invertible nef and $\mathcal{P}_X = \overline{H}_X/m$, where H is a nef Cartier divisor on Y. Thus in this situation the b-sheaf data implies the Birkar-Zhang one. Respectively, for the Alexeev data, $H \in m\mathcal{P} = L$ and $\mathcal{P}_X = H$, the birational image of a sufficiently general H, where L is a free linear system on Y. The Alexeev data is more restrictive and actually is required for general applications (cf. Conjecture 1 and Corollary-Conjecture 1). However, for wFt or Ft X/Z the Birkar-Zhang data works very well (cf. Example 16 and Corollary 34).

For bd-pairs it is better to join $\mathcal{P}$ with the canonical divisor and consider the b-divisor $\mathbb{K}+\mathcal{P}$, respectively, $\mathcal{P}_X$ with K in the $\mathbb{R}$-divisor $K+\mathcal{P}_X$. The b-divisor $K+\mathcal{P}$ for bd-pairs of index m is defined up to $\sim_m$ and behave as $\mathbb{K}$ but not as b-Cartier divisor $\mathcal{P}$.

If $\mathcal{P}=0$ then we get usual pairs or log pairs $(X/Z, D)$ with $\mathbb{D}_{\mathrm{div}} = \mathbb{D} = \mathbb{D}(X, D)$. The same holds for models Y/Z of X/Z over which $\mathcal{P}$ is *stabilized*: $\mathcal{P} = \overline{\mathcal{P}_Y}$. In this case $(Y, D_Y + \mathcal{P})$ is lc (klt, etc) if and only if (Y, D_Y) is lc (respectively, klt, etc).

Complements can be defined for bd-pairs too. We consider as Birkar and Zhang [B, Theorem 1.10 and Definition 2.18, (2)] usually bd-pairs $(X/Z, D+\mathcal{P})$ of index m and their complements $(X/Z, D^+ + \mathcal{P})$ with other required birational models, e.g., $(X^\sharp/Z, D^\sharp_{X^\sharp} + \mathcal{P})$, such that the models will have the same birational part $\mathcal{P}$.

Definition 4. Let $(X/Z \ni o, D+\mathcal{P})$ be a bd-pair with a [proper] local morphism $X/Z \ni o$ and with an $\mathbb{R}$-divisor D on X. Another bd-pair $(X/Z \ni o, D^+ + \mathcal{P})$ with the same local morphism, same $\mathcal{P}$ and with an $\mathbb{R}$-divisor D^+ on X is called an $\mathbb{R}$-*complement* of $(X/Z \ni o, D+\mathcal{P})$ if

(1) $D^+ \geq D$;

(2) $(X, D^+ + \mathcal{P}_X)$ is lc, or equivalently, $\mathbb{D}^+_{\mathrm{div}}$ is a b-subboundary; and

(3) $K+D^++\mathcal{P}_X \sim_{\mathbb{R}} 0/Z \ni o$, or equivalently, $\mathbb{K}+\mathbb{D}^+ = \mathbb{K}+\mathbb{D}^+_{\mathrm{div}}+\mathcal{P} \sim_{\mathbb{R}} 0/Z$.

In particular, $(X/Z \ni o, D^+ + \mathcal{P})$ is a *[local relative]* 0-*bd-pair*.

The complement is *klt* if (X, D^+) is klt.

If $(X/Z \ni o, D+\mathcal{P})$ is a log bd-pair then (1) can be also restated in terms of b-divisors:

(1) $\mathbb{D}^+_{\mathrm{div}} \geq \mathbb{D}_{\mathrm{div}}$ or $\mathbb{D}^+ \geq \mathbb{D}$.

Definition 5. Let n be a positive integer, and $(X/Z \ni o, D+\mathcal{P})$ be a bd-pair with a local morphism $X/Z \ni o$ and with an $\mathbb{R}$-divisor $D = \sum d_i D_i$ on X. A bd-pair $(X/Z \ni o, D^+ + \mathcal{P})$ with the same local morphism, same $\mathcal{P}$ and with an $\mathbb{R}$-divisor $D^+ = \sum d_i^+ D_i$ on X, is an *n-complement* of $(X/Z \ni o, D + \mathcal{P})$ if

(1) for all prime divisors D_i on X,

$$d_i^+ \geq \begin{cases} 1, \text{ if} & d_i = 1; \\ \lfloor (n+1)d_i \rfloor /n & \text{otherwise;} \end{cases}$$

(2) $(X, D^+ + \mathcal{P})$ is lc or equivalently, $\mathbb{D}^+_{\mathrm{div}}$ is a b-subboundary; and

(3) $K+D^++\mathcal{P}_X \sim_n 0/Z \ni o$, or equivalently, $\mathbb{K}+\mathbb{D}^+ = \mathbb{K}+\mathbb{D}^+_{\mathrm{div}}+\mathcal{P} \sim_n 0/Z$.

The n-complement is *monotonic* if $D^+ \geq D$, or equivalently, $\mathbb{D}^+_X \geq \mathbb{D}_X$ or $\mathbb{D}^+_{\mathrm{div}} \geq \mathbb{D}_{\mathrm{div}}$, when $(X/Z \ni o, D + \mathcal{P})$ is a log bd-pair.

A log bd-pair $(X/Z \ni o, D^++\mathcal{P})$ is a b-n-complement of $(X/Z \ni o, D+\mathcal{P})$ if instead of (1) in Definition 2 the following b-version holds:

(1-b) for all prime b-divisors P of X,

$$d^+ \geq \begin{cases} 1, \text{ if} & d = 1; \\ \lfloor (n+1)d \rfloor /n & \text{otherwise,} \end{cases}$$

where $d = \mathrm{mult}_P \mathbb{D}_{\mathrm{div}}$ and $d^+ = \mathrm{mult}_P \mathbb{D}^+_{\mathrm{div}}$ are multiplicities of (the divisorial part of codiscrepancy) b-divisors $\mathbb{D}_{\mathrm{div}} = \mathbb{D} - \mathcal{P}$ and $\mathbb{D}^+_{\mathrm{div}} = \mathbb{D}^+ - \mathcal{P}$ of $(X, D + \mathcal{P})$ and of $(X, D^+ + \mathcal{P})$ respectively.

The b-n-complement is *monotonic* if $\mathbb{D}^+ \geq \mathbb{D}$ or $\mathbb{D}^+_{\mathrm{div}} \geq \mathbb{D}_{\mathrm{div}}$.

Remark 5 (Cf. [B, Theorem 1.10 and Definition 2.18, (2)]). Both complements and many other constructions meaningful for arbitrary pair $(X/Z \ni o, \mathbb{D})$ with a b-divisor $\mathbb{D}$. We use and develop these concepts only for bd-pairs $(X/Z \ni o, D+\mathcal{P})$ of index m and actually for a boundary D. Moreover, usually for n-complements we suppose that $m|n$, e.g., as in Corollary 6.

Now we can state a bd-version of Theorem 3.

Theorem 4 (Boundedness of n-complements for bd-pairs)**.** *Let d be a nonnegative integer, m be a positive integer, $\Delta \subseteq (\mathbb{R}^+)^r, \mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$, be a compact subset (e.g., a polyhedron) and Γ be a subset in the unite segment $[0,1]$ such that $\Gamma \cap \mathbb{Q}$ satisfies the dcc. Let I, ε, v, e be the data as in Restrictions on complementary indices. Then there exists a finite set $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Gamma, \Delta, m)$ of positive integers (*complementary indices*) such that*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data;*

Existence of n-complement: *if $(X/Z \ni o, B+\mathcal{P})$ is a bd-pair of index m with wFt $X/Z \ni o$, $\dim X \leq d$, connected X_o and with a boundary B, then $(X/Z \ni o, B+\mathcal{P})$ has an n-complement for some $n \in \mathcal{N}$ under either of the following assumptions:*

(1-bd) *$(X/Z \ni o, B+\mathcal{P})$ has a klt $\mathbb{R}$-complement; or*

(2-bd) *$B = \sum_{i=1}^r b_i D_i$ with $(b_1, \ldots, b_r) \in \Delta$ and additionally, for every $\mathbb{R}$-divisor $D = \sum_{i=1}^r d_i D_i$ with $(d_1, \ldots, d_r) \in \Delta$, the pair $(X/Z \ni o, D+\mathcal{P})$ has an $\mathbb{R}$-complement, where D_i are effective Weil divisors (not necessarily prime); or*

(3-bd) *$(X/Z \ni o, B+\mathcal{P})$ has an $\mathbb{R}$-complement and, additionally, $B \in \Gamma$.*

Addendum 3. *We can relax the connectedness assumption on X_o and suppose that the number of connected components of X_o is bounded.*

This theorem with Theorem 3 will be proven in Section 11. However, we start from some generalities about complements.

Proposition 1 (Cf. [Sh92, Lemma 5.3])**.** *Let $(X/Z, D), (X/Z, D')$ be two (log) pairs with divisors $D \geq D'$ and $(X/Z, D^+)$ be a (respectively* b-*)n-complement of $(X/Z, D)$. Then $(X/Z, D^+)$ is a (respectively* b-*)n-complement of $(X/Z, D')$ too.*

The same holds for monotonic (respectively b-*)n-complements.*

Addendum 4. *The same holds for $\mathbb{R}$-complements of pairs and bd-pairs.*

Addendum 5. *The same holds for bd-pairs $(X/Z, D+\mathcal{P}), (X/Z, D'+\mathcal{P})$ with the same b-part $\mathcal{P}$ and with a (*b-*)n-complement $(X/Z, D^+ + \mathcal{P})$ of $(X/Z, D)$ respectively.*

Proof. Immediate by the monotonicity of $\lfloor\ \rfloor$: if $d' \leq d < 1$ then

$$\lfloor (n+1)d' \rfloor /n \leq \lfloor (n+1)d \rfloor /n \leq 1$$

and Example 5, (3).

Similarly we treat bd-pairs of Addendum 5.

Addendum 4 is immediate by definition.

□

Corollary 3. *Let $(X/Z, D^+)$ be an n-complement of $(X/Z, D)$. Then $(X/Z, D^+)$ is an $\mathbb{R}$- and monotonic n-complement of $(X/Z, D_{[n]})$ where $D_{[n]}$ has the following multiplicities, for every prime divisor P on X,*

$$\operatorname{mult}_P D_{[n]} = \begin{cases} 1, & \text{if} \quad d = 1; \\ \lfloor (n+1)d \rfloor /n & \text{otherwise.} \end{cases}$$

Proof. Immediate by Definition 2, Proposition 1 and Example 6, (1). □

Proposition 2. *Let n be a positive integer, D be an $\mathbb{R}$-divisor with normal crossing on nonsingular X and $(X/Z, D^+)$ be an n-complement of $(X/Z, D)$. Then $(X/Z, D^+)$ is also a b-n-complement of $(X/Z, D)$.*

The same holds for a bd-pair $(X/Z \ni o, D + \mathcal{P})$ with normal crossings only for Supp D *and stable $\mathcal{P}$ over X.*

Proof. (Local verification.) The statement is meaningful because X is nonsingular and (X, D) is a log pair. We need to verify only Definition 3, (1-b).

Step 1. It is enough to verify that

$$\mathbb{D}(X, D_{[n]}) \geq (\mathbb{D}(X, D))_{[n]}, \tag{3.0.1}$$

where, for an $\mathbb{R}$-divisor $D = \sum d_i D_i$ and/or a b-$\mathbb{R}$-divisor $\mathbb{D} = \sum d_i D_i$,

$$D_{[n]} = \sum d_{i[n]} D_i$$

and the rounding $x_{[n]}$ is defined in 6.13. Indeed, by Definition 2, (1) and (3.0.1),

$$D^+ \geq D_{[n]} \text{ and } \mathbb{D}^+ = \mathbb{D}(X, D^+) \geq \mathbb{D}(X, D_{[n]}) \geq (\mathbb{D}(X, D))_{[n]}.$$

Hence Definition 3, (1-b) holds for (X, D^+) with respect to (X, D).

Step 2. It is enough to verify that, for every crepant blowups (monoidal transformations) $\varphi\colon (Y, D_Y), (Y, D_{[n],Y}) \to (X, D), (X, D_{[n]})$ of a point $o \in X$,

$$D_{[n],Y} \geq D_{Y,[n]} \tag{3.0.2}$$

holds. The blowups can be considered birationally, that is, over a neighborhood of (scheme) point o. Indeed, (3.0.1) for b-$\mathbb{R}$-divisors follows from the corresponding inequality for every prime b-divisor E. Since E can be obtained by a sequence of blowups, the required inequality in E follows from (3.0.2) by induction.

Step 3. It is enough to verify that

$$1 - m + \sum_{i=1}^{m} \lfloor (n+1)d_i \rfloor /n \geq \left\lfloor (n+1)(1 - m + \sum_{i=1}^{m} d_i) \right\rfloor /n, \tag{3.0.3}$$

where m is a positive integer and d_i are real numbers < 1. Indeed, let $l \geq 1$ be the codimension of o in X and $D = \sum_{i=1}^{l} d_i D_i$ near o, where d_i are real numbers ≤ 1, the prime divisors $D_1, \ldots, D_l$ are with normal crossings and o is the generic point of their intersection $\cap_{i=1}^{l} D_i$. It is enough to verify the inequality (3.0.2) in the only exceptional divisor E of φ. For this, we can suppose that $d_i = 1$ only for $i > m$ and $d_i < 1$ otherwise.

Then

$$\operatorname{mult}_E D_Y = 1 - l + \sum_{i=1}^{l} d_i = 1 - l + l - m + \sum_{i=1}^{m} d_i = 1 - m + \sum_{i=1}^{m} d_i.$$

So, $\operatorname{mult}_E D_{Y,[n]}$ is the right side of (3.0.3), except for the case, when $\operatorname{mult}_E D_Y = \operatorname{mult}_E D_{Y,[n]} = 1$. In the last case $m = 0$ and all $d_i = 1$. Note that in this case the left side is also 1 and (3.0.3) holds. Thus we can suppose that $m \geq 1$.

On the other hand,

$$D_{[n]} = \sum_{i=1}^{m} \frac{\lfloor (n+1)d_i \rfloor}{n} D_i + \sum_{i=m+1}^{l} D_i$$

near o. Hence

$$\operatorname{mult}_E D_{[n],Y} = 1 - l + l - m + \sum_{i=1}^{m} \lfloor (n+1)d_i \rfloor / n = 1 - m + \sum_{i=1}^{m} \lfloor (n+1)d_i \rfloor / n,$$

the left side of (3.0.3).

Finally, (3.0.3) is the inequality of Example 4 in 2.2.

The bd-pairs can be treated ditto because $\mathcal{P}$ is stable over X.

$\square$

Proposition 3. *Every small birational modification preserves* $\mathbb{R}$- *and* n-*complements for pairs and bd-pairs.*

The same holds for b-$\mathbb{R}$*-complements if* X/Z *has wFt and the small modification preserves the log pair property.*

See Remark 4, (1) about b-n-complements.

Proof. Immediate by definition.

The statement about b-$\mathbb{R}$-complements holds by Addendum 8. Indeed, if $(X/Z, D)$ has a b-$\mathbb{R}$-complement then so does $(X^\sharp, D^\sharp_{X^\sharp})$ by the addendum. On the other hand, a maximal model exists under the assumptions of Construction 2. Moreover, by the construction, the assumptions and maximal models are invariant of small birational modifications of pairs $(X/Z, D)$ or bd-pairs $(X/Z, D + \mathcal{P})$. (Cf. Remarks 4, (2).)

$\square$

Hyperstandard sets. By definition [PSh08, 3.2] every such set has the form

$$\Phi = \Phi(\mathfrak{R}) = \{1 - \frac{r}{l} \mid r \in \mathfrak{R} \text{ and } l \text{ is a positive integer}\},$$

where $\mathfrak{R}$ is a set of real numbers. Usually, we take for $\mathfrak{R}$ a finite subset of rational numbers in $[0, 1]$. Additionally, we suppose that $1 \in \mathfrak{R}$. So, $0 = 1 - 1/1 \in \Phi$. (Every standard multiplicity $1 - 1/l \in \Phi$ too.) We say that Φ is *associated* with $\mathfrak{R}$.

Such a presentation for Φ is not unique. The following presentation of hyperstandard sets is crucial for us (see Proposition 4 below). It has the form $\mathfrak{G}(\mathcal{N}, \mathfrak{R}) = \Gamma(\mathcal{N}, \Phi)$, the set *associated* with $\mathcal{N}$ and $\mathfrak{R}$ or Φ (see Proposition 5 below), with the elements

$$b = 1 - \frac{r}{l} + \frac{1}{l}\left(\sum_{n \in \mathcal{N}} \frac{m_n}{n+1}\right) = \varphi + \frac{1}{l}\left(\sum_{n \in \mathcal{N}} \frac{m_n}{n+1}\right),$$

where $r \in \mathfrak{R}$, l is a positive integer, $\varphi = 1 - r/l \in \Phi$, and m_n are nonnegative integers. Usually, we suppose that $\mathcal{N}$ is a finite set of positive integers. By construction every $b \geq 0$. Additionally we suppose that $b \leq 1$. Thus $b \in [0,1] \cap \mathbb{Q}$.

The set $\Gamma(\mathcal{N}, \Phi)$ is a dcc set of rational numbers in $[0,1]$ with the only accumulation point 1. Actually, the set is a hyperstandard by Proposition 4 and by Proposition 5 below depends only on Φ. Conversely, every hyperstandard set has this form for $\mathcal{N} = \emptyset$.

For $\Phi \subseteq \Phi'$ and $\mathcal{N} \subseteq \mathcal{N}'$, $\Gamma(\mathcal{N}, \Phi) \subseteq \Gamma(\mathcal{N}', \Phi')$ holds. By our assumptions, the minimal $\mathfrak{R} = \{1\}$ and $\mathcal{N} = \emptyset$. Thus $\Gamma(\emptyset, \{1\}) = \Phi(\{1\})$ is the standard set without 1 and it is contained in every $\Gamma(\mathcal{N}, \Phi)$. Put $\Gamma(\mathcal{N}) = \Gamma(\mathcal{N}, \{1\})$. The last set contains 0 and the fractions $m/(n+1), n \in \mathcal{N}, m \in \mathbb{Z}, 0 \leq m \leq n+1$ of the great importance for us. This is why we suppose that $1 \in \mathfrak{R}$.

Proposition 4. *Let $\mathcal{N}$ be a finite set of positive integers and $\mathfrak{R}$ be a finite set of rational numbers in $[0,1]$. Then there exists a finite set of rational numbers $\mathfrak{R}'$ in $[0,1]$ such that*

$$\Gamma(\mathcal{N}, \Phi) = \Phi(\mathfrak{R}').$$

More precisely,

$$\mathfrak{R}' = \{r - \sum_{n \in \mathcal{N}} \frac{m_n}{n+1} \mid r \in \mathfrak{R} \text{ and every } m_n \in \mathbb{Z}^{\geq 0}\} \cap [0, +\infty).$$

Proof. Take $b \in \Gamma(\mathcal{N}, \Phi)$. Then by definition $b = 1 - r'/m$, where

$$r' = r - \sum_{n \in \mathcal{N}} \frac{m_n}{n+1}.$$

(Cf. with $\overline{\mathfrak{R}}$ in [PSh08, p. 160].) Since every $m_n \geq 0$ and $r \leq 1$, $r' \leq r \leq 1$ too. On the other hand, $b \leq 1$. Hence $r' \geq 0$ and the set $\mathfrak{R}'$ is finite rational in $[0,1]$. □

Proposition 5. *Let $\mathcal{N}$ be a set of positive numbers and $\mathfrak{R}, \mathfrak{R}'$ be two sets of rational numbers in $[0,1]$ such that $\Phi = \Phi(\mathfrak{R}) = \Phi(\mathfrak{R}')$. Then $\Gamma(\mathcal{N}, \Phi) = \mathfrak{G}(\mathcal{N}, \mathfrak{R}) = \mathfrak{G}(\mathcal{N}, \mathfrak{R}')$.*

Proof. Immediate by definition and the property that every $r' \in \mathfrak{R}'$ has the form $r' = r/m$ for some positive integer m if $\mathfrak{R}$ is the minimal (actually, smallest) set such that $\Phi = \Phi(\mathfrak{R})$.

$\square$

Corollary 4 (cf. [Sh95, Lemma 2.7])**.** *Let $\mathcal{N}$ be a finite set of positive integers and $\mathfrak{R}$ be a finite set of rational numbers in $[0, 1]$. Then $\Gamma(\mathcal{N}, \Phi)$ satisfies the dcc with only one accumulation point 1. Equivalently, for every positive real number ε, the set of rational numbers*

$$\Gamma(\mathcal{N}, \Phi) \cap [0, 1 - \varepsilon]$$

is finite.

Addendum 6. *If $\mathcal{N} \neq \emptyset$ then $1 \in \Gamma(\mathcal{N}, \Phi)$.*

Proof. Immediate by definition or Proposition 4.

If $n \in \mathcal{N}$, then

$$1 = 1 - \frac{1}{1} + \frac{n+1}{n+1} \in \Gamma(\mathcal{N}, \Phi).$$

$\square$

Construction 1 (Low approximations)**.** Let $\mathcal{N}$ be a nonempty finite set of positive integers and $\mathfrak{R}$ be a finite set of rational numbers in $[0, 1]$. Then by Corollary 4 and Addendum 6, for every $b \in [0, 1]$, there exists and unique (best low approximation) largest $b' \leq b$ in $\Gamma(\mathcal{N}, \Phi)$. We denote that b' by $b_{\mathcal{N}\_\Phi}$. Respectively, for a boundary B on X, $B_{\mathcal{N}\_\Phi}$ denotes the largest boundary on X such that $B_{\mathcal{N}\_\Phi} \leq B$ and $B_{\mathcal{N}\_\Phi} \in \Gamma(\mathcal{N}, \Phi)$.

Respectively, for a boundary B_Y on Y, put $B_{Y,\mathcal{N}\_\Phi} = (B_Y)_{\mathcal{N}\_\Phi}$.

We can and will apply also this construction for the divisorial part B of a bd-pair $(X/Z, B + \mathcal{P})$, when B is a boundary.

Notation:
$b_\Phi = b_{\emptyset\_\Phi}$;
$B_\Phi = B_{\emptyset\_\Phi}$ but we suppose that $0, 1 \in \Phi$;
$b_{n\_\Phi} = b_{\{n\}\_\Phi}$;
$B_{n\_\Phi} = B_{\{n\}\_\Phi}$;
$b_{n\_0} = b_{n\_\{0\}}$;
$B_{n\_0} = B_{n\_\{0\}}$
etc

$B^\sharp$, ${B_{n\_\Phi}}^\sharp = ({\mathbb{B}_{n\_\Phi}}^\sharp)_X$; for different birational model Y of X: ${B_{n\_\Phi}}^\sharp{}_Y = ({\mathbb{B}_{n\_\Phi}}^\sharp)_Y$.

Proposition 6. $\mathcal{N} \subseteq \mathcal{N}', \Phi \subseteq \Phi' \Rightarrow B_{\mathcal{N}\_\Phi} \leq B_{\mathcal{N}'\_\Phi'}$.

Proof. Immediate by definition. □

Proposition 7. *Let B, B^+ be two boundaries on the same variety and n be a positive integer such that $B^+ \in \mathbb{Z}/n, n \in \mathcal{N}$, and B^+ satisfies (1) of Definition 2 with respect to $B_{\mathcal{N}\_\Phi}$. Then B^+ satisfies* (1) *of Definition 2 with respect to B.*

Proof. Let P be a prime divisor on the variety. Put $b^+ = \operatorname{mult}_P B^+, b = \operatorname{mult}_P B$. Then $\operatorname{mult}_P B_{\mathcal{N}\_\Phi} = b_{\mathcal{N}\_\Phi}$. By our assumptions $b^+ = m/n \in [0,1], m \in \mathbb{Z}$.

If $b = 1$ then $b_{\mathcal{N}\_\Phi} = 1$ and by our assumptions $b^+ = 1$. This gives (1) of Definition 2 for b.

If $b < 1$ then by definition $b_{\mathcal{N}\_\Phi} \leq b < 1$ too. It is enough to verify that

$$\lfloor (n+1)b \rfloor /n = \lfloor (n+1)b_{\mathcal{N}\_\Phi} \rfloor /n,$$

Put $b' = m/(n+1)$, where

$$m = \max\{m \in \mathbb{Z} \mid \frac{m}{n+1} \leq b\}.$$

Then by our assumptions and definition $b' \in \Gamma(\mathcal{N}, \Phi)$. Hence $b' \leq b_{\mathcal{N}\_\Phi} \leq b < 1$. So, by the monotonicity of $\lfloor\ \rfloor$ and by construction

$$\lfloor (n+1)b' \rfloor /n = \lfloor (n+1)b_{\mathcal{N}\_\Phi} \rfloor /n = \lfloor (n+1)b \rfloor /n,$$

in particular, the required equality. Indeed, $b' = m/(n+1) \leq b < (m+1)/(n+1)$ and

$$\lfloor (n+1)b \rfloor /n = m/n = \lfloor (n+1)b' \rfloor /n.$$

The threshold $m/(n+1)$ plays an important role in the paper through $\Gamma(\mathcal{N}, \Phi)$.

□

Corollary 5. *Let B, B^+ be two boundaries on the same variety and n be a positive integer such that $B^+ \in \mathbb{Z}/n, n \in \mathcal{N}$ and $B^+ \geq B_{\mathcal{N}\_\Phi}$. Then B^+ satisfies* (1) *of Definition 2 with respect to B.*

Proof. Immediate by Proposition 7. Indeed $B^+ \geq B_{\mathcal{N}\_\Phi}$ implies (1) of Definition 2 with respect to $B_{\mathcal{N}\_\Phi}$ by arguments in Example 6, (1) and the proof of Proposition 1.

□

Corollary 6. *Let $(X/Z, B)$ be a pair with a boundary B such that $(X/Z, B_{\mathcal{N}\_\Phi})$ has an n-complement $(X/Z, B^+)$ with $n \in \mathcal{N}$. Then $(X/Z, B^+)$ is an n-complement of $(X/Z, B)$ too.*

The same holds for bd-pairs of index m $(X/Z, B + \mathcal{P}), (X/Z, B_{\mathcal{N}\_\Phi} + \mathcal{P}), (Z/Z, B^+ + \mathcal{P})$ under assumption $m|n$.

Proof. Immediate by Proposition 7 because $B^+ \in \mathbb{Z}/n$ by (3) of Definition 2.

For bd-pairs we use (3) of Definition 5 and our assumption $m|n$ (cf. Remark 5). [Indeed, $m|n$ implies that $B^+ \in \mathbb{Z}/n$ in this case too.] □

Construction 2 (Maximal model or adding fixed components)**.** Let $(X/Z, D)$ be a pair such that

X/Z has wFt; and

$-(K + D)$ is (pseudo)effective modulo $\sim_{\mathbb{R},Z}$, or equivalently,

there exists $D^+ \geq D$ such that $K + D^+ \sim_{\mathbb{R},Z} 0$ (an nonlc $\mathbb{R}$-complement).

Warning: $(X/Z, D^+)$ is not necessarily lc and an $\mathbb{R}$-complement of $(X/Z, D)$.

Consider a Zariski decomposition $-(K + D) = M + F$, where M, F are respectively the $\mathbb{R}$-mobile and fixed parts of $-(K + B)$ over Z. Such a decomposition exists by Lemma 1 and [ShCh, Corollary 4.5]. In addition, there exists a commutative triangle

$$\begin{array}{ccc} (X, D) & \overset{\psi}{\dashrightarrow} & (X^\sharp, D^\sharp_{X^\sharp}) \\ & \searrow \quad \swarrow & \\ & Z & \end{array},$$

where ψ is a birational morphism making M $\mathbb{R}$-free [-semiample] over Z as follows. First, we make a small birational modification $X \dashrightarrow Y/Z$ such that M is $\mathbb{R}$-Cartier on Y, e.g., we can use a $\mathbb{Q}$-factorialization. By Lemma 1 we can suppose also that X/Z has Ft. Second, we can apply the M-MMP to Y/Z. This gives $X^\sharp/Z$. Put $D^\sharp_{X^\sharp} = D + F$, where D, F are birational transforms of D, F on $X^\sharp$. Constructed transformation $X \dashrightarrow X^\sharp/Z$ is small and

$$-(K_{X^\sharp} + D^\sharp_{X^\sharp}) = -(K_{X^\sharp} + D) - F = M$$

is the birational transform of M. So, $-(K_{X^\sharp} + D^\sharp_{X^\sharp})$ is $\mathbb{R}$-free [-semiample] over Z. All relative varieties in the construction and, in particular, $X^\sharp/Z$ have wFt. Finally, we can take any crepant model of $(X^\sharp/Z, D^\sharp_{X^\sharp})$ and usually denote in the same way. Every such a model is assumed to be a log pair and its morphisms to previously constructed $X^\sharp$ is rational but not necessarily regular. For those pairs $X^\sharp/Z$ has not necessarily wFt. But all pairs $(X^\sharp/Z, D^\sharp_{X^\sharp})$ are log pairs. They have the same b-$\mathbb{R}$-divisor $\mathbb{D}^\sharp = \mathbb{D}(X^\sharp, D^\sharp_{X^\sharp})$. By definition $D^\sharp_{X^\sharp} = \mathbb{D}^\sharp{}_{X^\sharp}$, where the last divisor is the trace of $\mathbb{D}^\sharp$ on $X^\sharp$. We can take the trace on any birational model of X/Z, e.g., for X itself: $D^\sharp = \mathbb{D}^\sharp{}_X = D + F$. However, $\mathbb{D}^\sharp$ is *stable* (BP) only over such birational models $X^\sharp/Z$ that $\mathbb{D}^\sharp = \mathbb{D}(X^\sharp, \mathbb{D}^\sharp_{X^\sharp})$ is the b-codiscrepancy. Equivalently, $\mathcal{M}^\sharp = \overline{-(K_{X^\sharp} + D^\sharp_{X^\sharp})}$ is stable over those models but the stabilization is different (Cartier): $\mathcal{M}^\sharp = \overline{\mathcal{M}^\sharp_{X^\sharp}}$.

Note also, that if we apply $-(K+D)$-MMP to X/Z we get (anticanonical model) $(X^\sharp/Z, D^\sharp_{X^\sharp})$ and possibly contract some divisors, e.g., components of F. To construct the model we use antiflips. So, the model is *maximal* in contrast to a minimal one. It is also maximal and even *largest* with respect to $\mathbb{D}^\sharp$. For any log pair [model] $(Y/Z, D^\sharp_Y)$ of the [bd-]pair $(X/Z, \mathbb{D}^\sharp)$,

$$\mathbb{D}(Y, D^\sharp_Y) \leq \mathbb{D}^\sharp$$

and $=$ holds exactly when $(Y/Z, D^\sharp_Y)$ is a maximal model of $(X/Z, D)$.

Equivalently,

$$\overline{M_{X^\sharp}} \leq \overline{M_Y} = \mathcal{M}^\sharp$$

and $=$ holds exactly when $(Y/Z, D^\sharp_Y)$ is maximal, where $M_{X^\sharp}, M_Y$ are birational transform of M on $X^\sharp, Y$ respectively. However, to establish this in such a more general situation it is better to use the negativity [Sh95, 2.15] (see Lemma 5 below).

The same construction works for a bd-pair $(X/Z, D + \mathcal{P})$ such that

X/Z has wFt; and

$-(K + D + \mathcal{P}_X)$ is (pseudo)effective modulo $\sim_{\mathbb{R},Z}$, or equivalently,

there exists $D^+ \geq D$ such that $K + D^+ + \mathcal{P}_X \sim_{\mathbb{R},Z} 0$.

A maximal model in this case will be a log bd-pair $(X/Z, D^\sharp_{X^\sharp} + \mathcal{P})$ with same $\mathcal{P}$. That is, $\mathbb{D}^\sharp = \mathbb{D}^\sharp_{\rm div} + \mathcal{P}$ or $\mathcal{P}^\sharp = \mathcal{P}$. We add the fixed part F only to the divisorial part: $(D + \mathcal{P}_X)^\sharp = D^\sharp + \mathcal{P}_X$. However, the $\mathbb{R}$-free over Z property holds for $-(K_{X^\sharp} + D^\sharp_{X^\sharp} + \mathcal{P}_{X^\sharp})$ but usually not for $-(K_{X^\sharp} + D^\sharp_{X^\sharp})$. The construction actually works for any b-divisor $\mathcal{P}$ under our assumptions.

By construction, the assumptions and maximal models are invariant of small birational modifications of pairs $(X/Z, D)$ or bd-pairs $(X/Z, D + \mathcal{P})$.

Proposition 8 (Monotonicity I). *Let $(X/Z, D)$ be a pair under the assumptions of Construction 2. Then for every $D^+ \geq D$ such that $K + D^+ \sim_{\mathbb{R},Z} 0$, $\mathbb{D}^+ \geq \mathbb{D}^\sharp$ and $D^\sharp \geq D$. Moreover, $\mathbb{D}^\sharp \geq \mathbb{D}$ if $(X/Z, D)$ is a log pair.*

Addendum 7. *$D \geq 0$ implies $D^\sharp \geq 0$.*

Addendum 8. *$(X/Z, D)$ has an $\mathbb{R}$-complement if and only if $(X^\sharp, D^\sharp_{X^\sharp})$ is lc, equivalently, $(X, \mathbb{D}^\sharp)$ is lc or $\mathbb{D}^\sharp$ is a b-subboundary.*

Addendum 9. *The same holds for bd-pairs.*

Proof. Immediate by definition and Construction 2. Notice for this that $\mathbb{D}^+ = \mathbb{D}(X, D^+)$ is a birational invariant of $(X/Z, D^+)$, that is, every model $(Y/Z, D^+_Y), D^+_Y = \mathbb{D}^+_Y$, of $(X/Z, D^+)$ is crepant. In Addendum 8, an $\mathbb{R}$-complement of $(X^\sharp/Z, D^\sharp_{X^\sharp})$ exists if it is lc (cf. Examples 1, (1-2)).

Similarly we can treat bd-pairs.

□

Corollary 7 (Monotonicity II). *Let $(X/Z, D)$ be a pair under the assumptions of Construction 2 and D' be an $\mathbb{R}$-divisor on X such that $D' \leq D$. Then $(X/Z, D')$ also satisfies the assumptions of Construction 2 and $\mathbb{D}'^\sharp \leq \mathbb{D}^\sharp$.*

Lemma 5 (b-Negativity). *Let $\mathcal{D}, \mathcal{D}'$ be two b-$\mathbb{R}$-divisor of a variety or space X with a proper morphism $X \to S$ to a scheme S such that*

(1) *$\mathcal{D}'$ is b-pseudoantinef;*

(2) *$\mathcal{D}'_X \geq \mathcal{D}_X$; and*

(3) *$\mathcal{D}$ is* stable *over X, that is, $\mathcal{D} = \overline{\mathcal{D}_X}$* (cf. [Sh03, Discent of divisors 5.1]).

Then $\mathcal{D}' \geq \mathcal{D}$.

Proof. The b-psedoantinef means that $\mathcal{D}' = \lim_{n\to\infty} \mathcal{D}_n$, where the limit is weak (multiplicity wise) and every $\mathcal{D}_n$ is a b-antinef. So, we can suppose that $\mathcal{D}'$ is itself b-antinef. Take any birational proper model Y/X of X over which $\mathcal{D}'$ is stable. Then $\mathcal{D}'_Y \geq \mathcal{D}_Y$ by [Sh92, Negativity 1.1]. Apply to $D = \mathcal{D}'_Y - \mathcal{D}_Y$ and to the birational contraction $Y \to X$. This implies $\mathcal{D}' \geq \mathcal{D}$ because we can take an arbitrary high model Y.

$\square$

Corollary 8 (Monotonicity III)**.** *Let $(X/S, D)$ be a log pair with proper $X \to S$ and $\mathbb{D}'$ be a (pseudoBP) b-$\mathbb{R}$-divisor of X such that*

(1) $\mathbb{K} + \mathbb{D}'$ *is b-pseudoantinef; and*

(2) $\mathbb{D}'_X \geq D$.

Then $\mathbb{D}' \geq \mathbb{D}$, where $\mathbb{D} = \mathbb{D}(X, D)$.

The same holds for a log bd-pair $(X, D + \mathcal{P})$ with

(1-bd) $\mathbb{K} + \mathbb{D}' + \mathcal{P}$ *is b-pseudoantinef; and*

and $\mathbb{D} = \mathbb{D}(X, D + \mathcal{P}) - \mathcal{P}$.

Proof. Immediate by Lemma 5 with $\mathcal{D} = \mathbb{K} + \mathbb{D}$ and $\mathcal{D}' = \mathbb{K} + \mathbb{D}'$. Assumptions (1-2) of the corollary correspond respectively to (1-2) of the statement. Since (X, D) is a log pair, $\mathcal{D}$ is defined. So, $D = \mathbb{D}_X$ and $\mathcal{D}$ is stable over X as a b-$\mathbb{R}$-Cartier divisor by definition.

Respectively, for the bd-pair, $\mathcal{D} = \mathbb{K} + \mathbb{D} + \mathcal{P}$ and $\mathcal{D}' = \mathbb{K} + \mathbb{D}' + \mathcal{P}$, that is, the b-divisorial part is the same.

Note that (1) implies the pseudoBP property of $\mathbb{D}'$, the limit of BP b-$\mathbb{R}$-divisors. $\square$

Proof of Corollary 7. Immediate by Corollary 8. Apply the corollary to a maximal model $(X^\sharp, D'^\sharp_{X^\sharp})$ with a birational 1-contraction $\psi\colon X \dashrightarrow X^\sharp$ such that $D'^\sharp_{X^\sharp} = \psi(D')$. $\square$

(Non)existence of $\mathbb{R}$-complements has the local nature. The same holds for n-complements.

Proposition 9. *Let $(X/Z \ni o, D)$ be a local pair with wFt $X/Z \ni o$ and n be a positive integer. Then the existence of an $\mathbb{R}$-complement (respectively of an n-complement) is a local property with respect to connected components of X_o, that is, in the étal topology.*

The same holds for bd-pairs $(X/Z \ni o, D + \mathcal{P})$.

Proof. (Cf. the proof of Step 8 of Theorem 19.)

Step 1. (*Reduction to a semilocal case.*) We can replace $(X/Z \ni o, D)$ by $(X/Y \ni o_1, \ldots, o_l, D)$ with wFt $X/Y \ni o_1, \ldots, o_l$ and connected fibers $X_{o_1}, \ldots, X_{o_l}$. We can suppose that X_{o_i} are connected components of X_o. Indeed, take Stein factorization $X \twoheadrightarrow Y \to Z$. Let $o_1, \ldots, o_l$ be the points on Y over o. Notice that semilocal $X/Y \ni o_1, \ldots, o_l$ has also wFt, equivalently, every local $X/Y \ni o_i$ has wFt. We need to verify that $(X/Y \ni o_1, \ldots, o_l, D)$ has an $\mathbb{R}$-complement (respectively an n-complement) if and only if every $(X/Y \ni o_i, D)$ has and $\mathbb{R}$-complement (respectively an n-complement). Actually, we need to verify only the if statement.

Step 2. ($\mathbb{R}$-*Complements.*) Use Addendum 8. Indeed, we can suppose that every $(X/Y \ni o_i, D)$ has and $\mathbb{R}$-complement, in particular, every $-(K + D)$ is effective modulo $\sim_{\mathbb{R}}$ over $Y \ni o_i$. Thus Construction 2 gives $(X^\sharp/Y \ni o_1, \ldots, o_l; D^\sharp_{X^\sharp})$ for $(X/Y \ni o_1, \ldots, o_l, D)$. So, by Addendum 8 $(X^\sharp, D^\sharp_{X^\sharp})$ is lc over $Y \ni o_1, \ldots, o_l$. Thus by the same addendum $(X/Y \ni o_1, \ldots, o_l, D)$ has an $\mathbb{R}$-complement.

Step 3. ($\mathbb{R}$-*Complements.*) Immediate by the criterion of existence for n-complements in terms of linear systems [Sh92, after Definition 5.1].

Similarly we can treat bd-pairs.

□

4 Constant sheaves

Componentwise constant sheaves. We consider constant sheaves $\mathcal{F}$ on a variety T of Abelian groups, vector spaces, monoids, convex cones, union of monoids, polyhedrons with a polyhedral decomposition and of sets. E.g., if $\mathcal{F}$ is a constant sheaf of Abelian groups A then, for every point $t \in T$, $\mathcal{F}_t = A$ and, for every connected open subset $S \subseteq T$, $\mathcal{F}_S = \Gamma(S, \mathcal{F}) = A$, where $=$ means the canonical isomorphism given by the restriction. Usually in description of such a sheaf $\mathcal{F}$ we give its sections $\mathcal{F}_t$ for closed points. We also give its global sections $\mathcal{F}_T$. For connected T, $\mathcal{F}_T = \mathcal{F}_t = A$ under the restriction.

Below by a *constant* sheaf we mean also a sheaf which is constant on every connected component of T. So, it is actually constant if T is connected. Strictly speaking these sheaves are *componentwise constant*.

We need the following constant sheaves. However, they are really constant only for families of very special varieties, only over connected components of T and under an appropriate parametrization (cf. Addendum 22 below). These sheaves will be sheaves of different nature: sheaves of sets, of Abelian groups and monoids, of $\mathbb{R}$-,$\mathbb{Q}$-linear spaces, of finite union of monoids, of cones and of polyhedrons.

Marked divisors $D_{1,t},\dots,D_{r,t}$ on a family X/T of varieties X_t, are its sections over t. Actually, they form an ordered set. For connected T, sections form an ordered set of global divisors $D_{1,T},\dots,D_{r,T}$ on X. The identification of sections is given by the restriction $D_{i,t}=D_{i,T}{}_{|X_t}$. So, divisors should be good and the family X/T too. Usually, we consider distinct prime Weil divisors $D_{i,t}$.

Abelian group generated by marked divisors $\mathfrak{D}_t=\mathfrak{D}_t(D_{1,t},\dots,D_{r,t})$. We consider this group as a subgroup of $\operatorname{WDiv} X_t$. The group $\mathfrak{D}_t$ is free Abelian of rank r with the standard basis $D_{1,t},\dots,D_{r,t}$ if the divisors are distinct prime. The corresponding constant sheaf we denote by $\mathfrak{D}=\mathfrak{D}(D_1,\dots,D_r)$. Similarly, we can define real and rational (sub)spaces $\mathfrak{D}_{\mathbb{R}},\mathfrak{D}_{\mathbb{Q}}$ with $\mathfrak{D}_{\mathbb{R},t}\subseteq \operatorname{WDiv}_{\mathbb{R}} X_t,\mathfrak{D}_{\mathbb{Q},t}\subseteq \operatorname{WDiv}_{\mathbb{Q}} X_t$ generated by divisors $D_{i,t}$; $\mathfrak{D}\subseteq\mathfrak{D}_{\mathbb{Q}}\subseteq\mathfrak{D}_{\mathbb{R}}$. If divisors $D_{i,t}$ are linearly free (in particular, distinct) in $\operatorname{WDiv} X_t$ then $\mathfrak{D}_{\mathbb{R},t}=\mathfrak{D}_t\otimes\mathbb{R},\mathfrak{D}_{\mathbb{Q},t}\otimes\mathbb{Q}$.

We will use also constant divisors

$$K_{X_t}\in\mathfrak{D},\mathcal{P}_{X,t}\in\mathfrak{D}_{\mathbb{Q}}.$$

$$\begin{aligned} K_T=K_{X/T}\in&\mathfrak{D}_T=\mathbb{Z}D_{1,T}\oplus\dots\oplus\mathbb{Z}D_{r,T};\\ &\mathfrak{D}_{\mathbb{R},T}=\mathbb{R}D_{1,T}\oplus\dots\oplus\mathbb{R}D_{r,T};\\ \mathcal{P}_X=\mathcal{P}_{X,T}\in&\mathfrak{D}_{\mathbb{Q},T}=\mathbb{Q}D_{1,T}\oplus\dots\oplus\mathbb{Q}D_{r,T}.\end{aligned}$$

Warning: usually we do not suppose that $\mathcal{P}$ exits globally but $\mathcal{P}_t$ exists for every closed $t\in T$, $\mathcal{P}_{X,t}$ exists for every $t\in T$ and $\mathcal{P}_X=\mathcal{P}_{X,T}$ exists too. However, if $\mathcal{P}$ exists globally over T then $\mathcal{P}_T=\mathcal{P}$ is meaningful and $\mathcal{P}_X$ is its trace on X.

Abelian monoid of effective divisors generated by marked divisors $\mathfrak{D}_t^+=\mathfrak{D}_t^+(D_{1,t},\dots,D_{r,t})\subseteq\operatorname{WDiv} X_t$ with elements $D_t\in\mathfrak{D}_t,\mathbb{Z}^{\ge0}$. The monoid $\mathfrak{D}_t^+$ is free Abelian of rank r with the standard basis $D_{1,t},\dots,D_{r,t}$ if the divisors are distinct prime. The corresponding constant sheaf we denote by $\mathfrak{D}^+=\mathfrak{D}^+(D_1,\dots,D_r)$. Similarly, we can define closed convex rational polyhedral (sub)cones $\mathfrak{D}_{\mathbb{R},t}^+\subseteq\mathfrak{D}_{\mathbb{R},t},\mathfrak{D}_{\mathbb{Q},t}^+\subseteq\mathfrak{D}_{\mathbb{Q},t}$ generated by divisors $D_{i,t}$.

$$
\begin{aligned}
D_{1,T},\dots,D_{r,T} \in & \mathfrak{D}_T^+ = \mathbb{Z}^{\ge 0} D_{1,T} \oplus \cdots \oplus \mathbb{Z}^{\ge 0} D_{r,T};\\
& \mathfrak{D}_{\mathbb{R},T}^+ = \mathbb{R}^+ D_{1,T} \oplus \cdots \oplus \mathbb{R}^+ D_{r,T},\ \mathbb{R}^+ = [0,+\infty);\\
& \mathfrak{D}_{\mathbb{Q},T}^+ = \mathbb{Q}^+ D_{1,T} \oplus \cdots \oplus \mathbb{Q}^+ D_{r,T},\ \mathbb{Q}^+ = \mathbb{R}^+ \cap \mathbb{Q}.
\end{aligned}
$$

Monoid of linearly effective divisors $\mathfrak{E}_t \subseteq \operatorname{WDiv} X_t$ with elements $D_t \in \mathfrak{D}_t$, which are effective modulo $\sim$. In general the monoid is not finitely generated. The corresponding constant sheaf we denote by $\mathfrak{E} = \mathfrak{E}(D_1,\dots,D_r)$. Similarly, we can define (sub)cones $\mathfrak{E}_{\mathbb{R},t} \subseteq \mathfrak{D}_{\mathbb{R},t}, \mathfrak{E}_{\mathbb{Q},t} \subseteq \mathfrak{D}_{\mathbb{Q},t}$ with elements $D_t \in \mathbb{R},\mathbb{Q}$, which are effective modulo $\sim_{\mathbb{R}},\sim_{\mathbb{Q}}$ respectively. In general those cones are not closed rational polyhedral. However, for wFt X/T, the monoid finitely generated by Corollary 9 and cones are closed convex rational polyhedral [ShCh, Corollary 4.5]. Moreover, the monoid and cones have finite decompositions into respectively finitely generated monoids, convex rational polyhedral cones (possibly not closed) such that the components correspond to the same rational 1-contraction [ShCh, Corollary 5.3].

$$
\begin{aligned}
&\mathfrak{D}_T^+ \subseteq \mathfrak{E}_T = \mathfrak{E}\, X = \{D \in \mathfrak{D}_T \mid\ D \text{ is effective modulo} \sim\};\\
&\mathfrak{D}_{\mathbb{R},T}^+ \subseteq \mathfrak{E}_{\mathbb{R},T} = \mathfrak{E}_{\mathbb{R}}\, X = \{D \in \mathfrak{D}_{\mathbb{R},T} \mid\ D \text{ is effective modulo} \sim_{\mathbb{R}}\};\\
&\mathfrak{D}_{\mathbb{Q},T}^+ \subseteq \mathfrak{E}_{\mathbb{Q},T} = \mathfrak{E}_{\mathbb{Q}}\, X = \{D \in \mathfrak{D}_{\mathbb{Q},T} \mid\ D \text{ is effective modulo} \sim_{\mathbb{Q}}\}.
\end{aligned}
$$

Important submonoids and subcones of semiample, nef, mobile divisors D_t

$$
\mathfrak{sA}_t, \mathfrak{Nef}_t, \mathfrak{M}_t \subseteq \mathfrak{E}_t,\ \mathfrak{sA}_{\mathbb{R},t}, \mathfrak{Nef}_{\mathbb{R},t}, \mathfrak{M}_{\mathbb{R},t} \subseteq \mathfrak{E}_{\mathbb{R},t},\ \mathfrak{sA}_{\mathbb{Q},t}, \mathfrak{Nef}_{\mathbb{Q},t}, \mathfrak{M}_{\mathbb{Q},t} \subseteq \mathfrak{E}_{\mathbb{Q},t}
$$

have constant sheaves $\mathfrak{sA} = \mathfrak{sA}(D_1,\dots,D_r), \mathfrak{sA}_{\mathbb{R}} = \mathfrak{sA}_{\mathbb{R}}(D_1,\dots,D_r), \mathfrak{sA}_{\mathbb{Q}} = \mathfrak{sA}_{\mathbb{Q}}(D_1,\dots,D_r)$ etc for $\mathfrak{Nef}, \mathfrak{M}, \mathfrak{E}$. Global sections are respectively

$$
\begin{aligned}
&\mathfrak{sA}_T = \{D \in \mathfrak{D}_T \mid\ D \text{ is semiample}\};\\
&\mathfrak{sA}_{\mathbb{R},T} = \{D \in \mathfrak{D}_{\mathbb{R},T} \mid\ D \text{ is semiample}\};\\
&\mathfrak{sA}_{\mathbb{Q},T} = \{D \in \mathfrak{D}_{\mathbb{Q},T} \mid\ D \text{ is semiample}\}.
\end{aligned}
$$

Etc for $\mathfrak{Nef}, \mathfrak{M}$.

Warning 1 (cf. [ShCh]). Here we use $\mathfrak{sA}, \mathfrak{Nef}, \mathfrak{M}$ and $\mathfrak{E}$ for the absolute case with multiplicities in $\mathbb{Z}$. For $\mathfrak{E}, \mathfrak{C}$ we use $\sim$, etc instead of $\equiv$. We also use absolute $\sim$, etc but not relative $\sim_T$, etc. Of course the last substitution needs appropriate parametrization (cf. Step 3 in the proof of Proposition 10 below).

lc Divisors $\mathfrak{lc}_t \subseteq \operatorname{WDiv} X$ with elements $D_t \in \mathfrak{D}_t$, such that (X_t, D_t) is lc. The corresponding constant sheaf we denote by $\mathfrak{lc} = \mathfrak{lc}(D_1, \dots, D_r)$. Similarly, we can define closed convex rational polyhedrons (possibly non-compact, e.g., $d_i \le 1$ correspond to subboundaries) [Sh92, (1.3.2)] $\mathfrak{lc}_{\mathbb{R},t} \subseteq \mathfrak{D}_{\mathbb{R},t}, \mathfrak{lc}_{\mathbb{Q},t} \subseteq \mathfrak{D}_{\mathbb{Q},t}$ with elements $D_t \in \mathbb{R}, \mathbb{Q}$ respectively, such that (X_t, D_t) is lc.

Similarly, for bd-pairs, $\mathfrak{lc}_t \mathcal{P}_t \subseteq \operatorname{WDiv} X$ with elements $D_t \in \mathfrak{D}_t$, such that $(X_t, D_t + \mathcal{P}_t)$ is lc as a bd-pair. The corresponding constant sheaf we denote by $\mathfrak{lc}\,\mathcal{P} = \mathfrak{lc}(D_1, \dots, D_r, \mathcal{P})$. Similarly, we can define closed convex rational polyhedrons [Sh92, (1.3.2)] $\mathfrak{lc}_{\mathbb{R},t} \mathcal{P}_t \subseteq \mathfrak{D}_{\mathbb{R},t}, \mathfrak{lc}_{\mathbb{Q},t} \mathcal{P}_t \subseteq \mathfrak{D}_{\mathbb{Q},t}$ with elements $D_t \in \mathbb{R}, \mathbb{Q}$ respectively, such that $(X_t, D_t + \mathcal{P}_t)$ is lc. For the constant sheaf property, it is better to suppose that the b-divisor $\mathcal{P}$ exists globally over T but this is not applicable in our paper (cf. the proof of Addendum 30 in Step 8 of the proof of Theorem 7).

Divisors with $\mathbb{R}$*-complement* $\mathfrak{C}_t \subseteq \operatorname{WDiv} X$ with elements $D_t \in \mathfrak{D}_t$, such that (X_t, D_t) has an $\mathbb{R}$-complement. The corresponding constant sheaf we denote by $\mathfrak{C} = \mathfrak{C}(D_1, \dots, D_r)$. Similarly, we can define convex sets $\mathfrak{C}_{\mathbb{R},t} \subseteq \mathfrak{D}_{\mathbb{R},t}, \mathfrak{C}_{\mathbb{Q},t} \subseteq \mathfrak{D}_{\mathbb{Q},t}$ with elements $D_t \in \mathbb{R}, \mathbb{Q}$ respectively, such that (X_t, D_t) has an $\mathbb{R}$-complement (cf. Theorem 6).

Similarly, for bd-pairs, $\mathfrak{C}_t \mathcal{P}_t \subseteq \operatorname{WDiv} X$ with elements $D_t \in \mathfrak{D}_t$, such that $(X_t, D_t + \mathcal{P}_t)$ has an $\mathbb{R}$-complement as a bd-pair. The corresponding constant sheaf we denote by $\mathfrak{C}\,\mathcal{P} = \mathfrak{C}(D_1, \dots, D_r, \mathcal{P})$. Similarly, we can define convex sets $\mathfrak{C}_{\mathbb{R},t} \mathcal{P}_t \subseteq \mathfrak{D}_{\mathbb{R},t}, \mathfrak{C}_{\mathbb{Q},t} \mathcal{P}_t \subseteq \mathfrak{D}_{\mathbb{Q},t}$ with elements $D_t \in \mathbb{R}, \mathbb{Q}$ respectively, such that $(X_t, D_t + \mathcal{P}_t)$ has an $\mathbb{R}$-complement (cf. Addendum 25).

By definition and our assumptions, for $K \in \mathfrak{D}, K + \mathcal{P}_X \in \mathfrak{D}_{\mathbb{Q}}$, respectively

$$\mathfrak{C} \subseteq -K - \mathfrak{E}_{\mathbb{R}}, \quad \mathfrak{C}\,\mathcal{P} \subseteq -K - \mathcal{P}_X - \mathfrak{E}_{\mathbb{R}}$$

(see Addenda 24 and 25). Respectively, if every X_t is $\mathbb{Q}$-factorial, then

$$\mathfrak{C} \subseteq \mathfrak{lc} \cap (-K - \mathfrak{E}_{\mathbb{R}}), \quad \mathfrak{C}\,\mathcal{P} \subseteq \mathfrak{lc}\,\mathcal{P} \cap (-K - \mathcal{P}_X - \mathfrak{E}_{\mathbb{R}})$$

(see again Addenda 24 and 25) Etc over $\mathbb{R}, \mathbb{Q}$. In general $=$ does not hold even over $\mathbb{R}$ (however, cf. the exceptional case in Step 3 of the proof of Theorem 7).

$\mathfrak{lc}_T = \{D \in \mathfrak{D}_T \mid (X, D) \text{ is lc}\}, \mathfrak{lc}_{\mathbb{R},T} = \{D \in \mathfrak{D}_{\mathbb{R},T} \mid (X, D) \text{ is lc}\}, \mathfrak{lc}_{\mathbb{Q},T} = \{D \in \mathfrak{D}_{\mathbb{Q},T} \mid (X, D)$ is

$$\mathfrak{C}_T = \{D \in \mathfrak{D}_T \mid (X/T, D) \text{ has an } \mathbb{R}\text{-complement}\}.$$

Etc over $\mathbb{R}, \mathbb{Q}$.

Notice also that $\mathfrak{lc}_\mathbb{R}, \mathfrak{C}_\mathbb{R}$ are usually not compact. However, for effective divisors D both sets $\mathfrak{lc}, \mathfrak{C}$ are compact because in this case D is a boundary, a compact condition.

Sheaf $\mathbf{Cl} = \mathbf{Cl}\, X/T$ *of Abelian class groups of Weil divisors modulo* $\sim$ with $\mathbf{Cl}_t\, X/T = \mathrm{Cl}\, X_t$ and $\mathbf{Cl}_T\, X/T = \mathrm{Cl}\, X/T$. Respectively, $\mathbf{Cl}_\mathbb{R}\, X/T, \mathbf{Cl}_\mathbb{Q}\, X/T$ with $\mathbf{Cl}_{\mathbb{R},t}\, X/T = \mathrm{Cl}_\mathbb{R}\, X_t, \mathbf{Cl}_{\mathbb{Q},t}\, X/T = \mathrm{Cl}_\mathbb{Q}\, X_t$ and $\mathbf{Cl}_{\mathbb{R},T}\, X/T = \mathrm{Cl}_\mathbb{R}\, X/T, \mathbf{Cl}_{\mathbb{Q},T}\, X/T = \mathrm{Cl}_\mathbb{Q}\, X/T$. The relative class group $\mathrm{Cl}\, X/T$ is defined modulo relative $\sim_T$. Respectively, $\mathrm{Cl}_\mathbb{R}\, X/T, \mathrm{Cl}_\mathbb{Q}\, X/T$ modulo $\sim_{\mathbb{R},T}, \sim_{\mathbb{Q},T}$.

Other important subsheaves in $\mathbf{Cl}$ include invertible sheaves, classes of semiample, nef, mobile, effective and torsion divisors

$$\mathbf{Pic} = \mathbf{Pic}\, X/T, \mathbf{sAmp} = \mathbf{sAmp}\, X/T, \mathbf{Nef} = \mathbf{Nef}\, X/T, \mathbf{Mob} = \mathbf{Mob}\, X/T, \mathbf{Eff} = \mathbf{Eff}\, X/T, \mathbf{To}$$

etc for $\mathbb{R}, \mathbb{Q}$ with

$$\mathbf{Pic}_t = \mathrm{Pic}\, X_t, \mathbf{sAmp}_t = \mathrm{sAmp}\, X_t, \mathbf{Nef}_t = \mathrm{Nef}\, X_t, \mathbf{Mob}_t = \mathrm{Mob}\, X_t, \mathbf{Eff}_t = \mathrm{Eff}\, X_t, \mathbf{Tor}_t = \mathrm{Tor}$$

and

$$\mathbf{Pic}_T = \mathrm{Pic}\, X/T, \mathbf{sAmp}_T = \mathrm{sAmp}\, X/T, \mathbf{Nef}_T = \mathrm{Nef}\, X/T, \mathbf{Mob}_T = \mathrm{Mob}\, X/T, \mathbf{Eff}_T = \mathrm{Eff}\, X/T,$$

etc for $\mathbb{R}, \mathbb{Q}$. Notice that $\mathbf{Tor}_\mathbb{R} = \mathbf{Tor}_\mathbb{Q} = 0$ is trivial. Cf. Warning 1.

Sheaves $\mathbf{Pic}, \mathbf{Tor}$ are sheaves of Abelian groups. Sheaves $\mathbf{Pic}_\mathbb{R}, \mathbf{Pic}_\mathbb{Q}$ are sheaves of respectively $\mathbb{R}$- and $\mathbb{Q}$-linear spaces. Sheaves $\mathbf{sAmp}, \mathbf{Nef}, \mathbf{Mob}, \mathbf{Eff}$ are sheaves of Abelian monoids. Their $\mathbb{R}, \mathbb{Q}$ versions are sheaves of convex cones.

Remark: Usually $\mathrm{sAmp}_\mathbb{R}\, X_t, \mathrm{Nef}_\mathbb{R}\, X_t, \mathrm{Mob}_\mathbb{R}\, X_t, \mathrm{Eff}_\mathbb{R}\, X_t$ are considered in $\mathrm{N^1X_t}$, the space of $\mathbb{R}$-divisors of X_t modulo the numerical equivalence $\equiv$ [ShCh, Section 4]. However, in the paper we usually consider wFt X_t and $\equiv$ is $\sim_\mathbb{R}$ in this situation [ShCh, Corollary 4.5].

Proposition 10. *Let $X_t, t \in T$, be a variety or an algebraic space in a bounded family of rationally connected spaces. Then $\mathrm{Cl}\, X_t$ has bounded rank r. More precisely, for appropriate parametrization X/T, every X_t has marked distinct prime divisors $D_{1,t}, \dots, D_{r,t}$ such that, for every $D \in \mathrm{WDiv}\, X_t$,*

$$D \sim d_1 D_{1,t} + \cdots + d_r D_{r,t}$$

for some $d_1, \dots, d_r \in \mathbb{Z}$.

Addendum 10. $\mathfrak{lc}, \mathfrak{lc}_{\mathbb{R}}, \mathfrak{lc}_{\mathbb{Q}}$ *are also constant.*

Addendum 11. $\mathfrak{lc}\,\mathcal{P}, \mathfrak{lc}_{\mathbb{R}}\,\mathcal{P}, \mathfrak{lc}_{\mathbb{Q}}\,\mathcal{P}$ *are constant if* $\mathcal{P}$ *defined over* T.

Addendum 12. *Sheaves* $\mathbf{Cl}\,X/T, \mathbf{Tor}\,X/T, \mathfrak{D}, \mathfrak{P}, \mathfrak{N}$ *are constant,*

$$\mathbf{Cl}\,X/T = \mathfrak{D}/\sim = \mathfrak{D}/\mathfrak{P}, \mathbf{Cl}_T\,X/T = \mathfrak{D}_T/\sim = \mathfrak{D}_T/\mathfrak{P}_T \text{ and } \operatorname{Cl} X_t = \mathfrak{D}_t/\sim = \mathfrak{D}_t/\mathfrak{P}_t,$$

$$\mathbf{Tor}\,X/T = \operatorname{Tor}(\mathbf{Cl}\,X/T) = \mathfrak{N}/\sim = \mathfrak{N}/\mathfrak{P},$$

$$\mathbf{Tor}_T\,X/T = \operatorname{Tor}(\mathbf{Cl}_T\,X/T) = \mathfrak{N}_T/\sim = \mathfrak{N}_T/\mathfrak{P}_T \text{ and } \operatorname{Tor} X_t = \operatorname{Tor}(\operatorname{Cl} X_t) = \mathfrak{N}_t/\sim = \mathfrak{N}_t/\mathfrak{P}_t$$

for every (closed) $t \in T$, *where* $\mathfrak{P}, \mathfrak{N}$ *are respectively subsheaves of principal and of numerically trivial over* T *divisors in* $\mathfrak{D}$. *Canonical isomorphisms are given by homomorphisms* $\mathfrak{D}_t \to \operatorname{Cl} X_t, D_t \mapsto D_t/\sim = D_t/\mathfrak{P}_t$. *The following sheaves*

$$\mathbf{Cl}_{\mathbb{R}}\,X/T, \mathbf{Cl}_{\mathbb{Q}}\,X/T, \mathfrak{P}_{\mathbb{R}}, \mathfrak{P}_{\mathbb{Q}}, \mathfrak{N}_{\mathbb{R}}, \mathfrak{N}_{\mathbb{Q}}$$

are also constant and

$$\mathbf{Cl}_{\mathbb{R}}\,X/T = (\mathbf{Cl}\,X/T) \otimes \mathbb{R}, \mathbf{Cl}_{\mathbb{Q}}\,X/T = (\mathbf{Cl}\,X/T) \otimes \mathbb{Q},$$

$$\mathfrak{N}_{\mathbb{R}} = \mathfrak{P}_{\mathbb{R}} = \mathfrak{N} \otimes \mathbb{R} = \mathfrak{P} \otimes \mathbb{R}, \mathfrak{N}_{\mathbb{Q}} = \mathfrak{P}_{\mathbb{Q}} = \mathfrak{N} \otimes \mathbb{Q} = \mathfrak{P} \otimes \mathbb{Q},$$

$$\mathbf{Tor}_{\mathbb{R}}\,X/T = \mathbf{Tor}_{\mathbb{Q}}\,X/T = 0.$$

Addendum 13. *If additionally every* X_t *has only rational singularities then the following sheaves*

$$\mathbf{Pic}\,X/T, \mathbf{Pic}_{\mathbb{R}}\,X, \mathbf{Pic}_{\mathbb{Q}}\,X/T, \mathfrak{Car}, \mathfrak{Car}_{\mathbb{R}}, \mathfrak{Car}_{\mathbb{Q}}$$

are also constant and

$$\mathbf{Pic}\,X/T = \mathfrak{Car}/\sim = \mathfrak{Car}/\mathfrak{P},$$

$$\mathbf{Pic}_{\mathbb{R}}\,X/T = (\mathbf{Pic}\,X/T) \otimes \mathbb{R} = \mathfrak{Car}_{\mathbb{R}}/\sim_{\mathbb{R}} = \mathfrak{Car}_{\mathbb{R}}/\mathfrak{P}_{\mathbb{R}}, \mathbf{Pic}_{\mathbb{Q}}\,X/T = (\mathbf{Pic}\,X/T) \otimes \mathbb{Q} = \mathfrak{Car}_{\mathbb{Q}}/\sim_{\mathbb{Q}} =$$

$$\mathfrak{Car}_{\mathbb{R}} = \mathfrak{Car} \otimes \mathbb{R}, \mathfrak{Car}_{\mathbb{Q}} = \mathfrak{Car} \otimes \mathbb{Q},$$

where $\mathfrak{Car}$ *is the subsheaf of Cartier or locally principal divisors in* $\mathfrak{D}$.

$$\mathbf{Pic}_T\,X/T = \mathfrak{Car}_T/\sim = \mathfrak{Car}_T/\mathfrak{P}_T \text{ and } \operatorname{Pic} X_t = \mathfrak{Car}_t/\sim = \mathfrak{Car}_t/\mathfrak{P}_t,$$

for every (closed) $t \in T$. *Canonical isomorphisms are induced by the homomorphism* $\mathfrak{D} \to \mathbf{Cl}\,X/T$. *Equivalences* $\sim_{\mathbb{Q}}, \sim_{\mathbb{R}}$ *can be replaced by* $\equiv /T$.

Proof. Step 1. We can suppose that every X_t is nonsingular, projective, the family X/T is smooth, X, T are nonsingular, irreducible. Use a resolution of singularities, reparametrization and Noetherian induction. For every closed $t \in T$, X_t is again rationally connected.

Indeed, let $Y \to X/T$ be a resolution and reparametrization such that $Y \to X$ and $Y_t \to X_t, t \in T$ are birational. We can suppose that X, T also satisfies required properties. By definition there are canonical surjections

$$\operatorname{Cl} Y/T \twoheadrightarrow \operatorname{Cl} X/T, \operatorname{Cl} Y_t \twoheadrightarrow \operatorname{Cl} X_t,$$

and the first one commutes with restrictions. Indeed, the group of Weil divisors on Y modulo $\sim$ is the group of Cartier divisors on Y modulo $\sim$ and goes canonically to the group of Weil divisors on X modulo $\sim$. The same works for $Y_t \to X_t$. Note also that the restriction of divisor commutes with its image. Since we consider relative $\operatorname{Cl} Y/T$ we need to take the quotient of $\operatorname{Cl} Y$ modulo the vertical over T divisors or to take sufficiently small T. (We can suppose that they are pullbacks of Cartier divisors from T.) Their images are also vertical over T.

The nonexceptional over X_t marked distinct prime divisors on Y_t goes to marked distinct prime divisors on X_t. Recall, that by definition of marked divisors: $D_{1,t} = D_1{}_{|Y_t}, \dots, D_{s,t} = D_s{}_{|Y_t}$ on Y_t, where $D_1, \dots, D_s$ are prime divisors on Y. Suppose that $D_1, \dots, D_s$ are required generators. Let $D_{1,t}, \dots, D_{1,r}, r \leq s$, be the only nonexceptional among them, equivalently, $D_1, \dots, D_r$ be the only nonexceptional over X among $D_1, \dots, D_s$. Then

$$\mathfrak{D}(D_1, \dots, D_s)/\sim = \mathfrak{D}(D_1, \dots, D_r)/\sim \bigoplus \mathfrak{D}(D_{r+1}, \dots, D_s) = \operatorname{Cl} Y/T = \operatorname{Cl} Y_t =$$
$$\mathfrak{D}(D_{1,t}, \dots, D_{r,t})/\sim \bigoplus \mathfrak{D}(D_{r+1,t}, \dots, D_{s,t}.$$

Note that for distinct prime divisors $D_{r+1}, \dots, D_s$ with exceptional $D_{r+1} + \dots + D_s$, $\mathfrak{D}(D_{r+1}, \dots, D_s)/\sim = \mathfrak{D}(D_{r+1}, \dots, D_s)$ (e.g., by [Sh92, 1.1]). Thus the required statements for X_t follows from that of Y_t. Note for this that $\mathfrak{D}(D_1, \dots, D_r), \mathfrak{D}(D_{1,t}, \dots, D_{r,t})$ are birational invariants with respect to blowdowns on X, X_t respectively.

For sufficiently small T we can use $\sim$ instead of $\sim_T$ (see Step 3). We can replace quotients by $\sim$ by that of $\mathfrak{P}$, the principal divisor sheaf. (For $\sim_T$ we can use locally over T principal divisors.)

In the following we suppose that $X/T, X_t$ satisfy required properties.

Step 2. $\mathbf{Cl}\, X/T = \mathbf{Pic}\, X/T$ is constant sheaf of $\operatorname{Pic} X_t$. The equation holds because X is nonsingular and all Weil divisors are Cartier. For closed points $t \in T$, X_t is rationally connected. Hence $\operatorname{Pic} X_t$ is a finitely generated Abelian group. (This follows from the finite generation of the Neron-Severi group and from the vanishing $H^0(X_t, \Omega^1_{X_t}) = 0$; e.g., see [Kol91, Corollary 3.8, Chapter IV].) This implies also that $\operatorname{Pic} X/T$ is locally constant in étal topology. Moreover, the monodromy is finite because it transforms ample divisors into ample ones of the same degree. By rational connectedness again there are only finitely many of those divisors up to $\sim$. After a base change we can suppose that the monodromy is trivial and $\mathbf{Pic}\, X/T$ is constant.

Note that if we are working over $k = \mathbb{C}$ then $\mathbf{Pic}\, X/T$ is the constant sheaf of $\operatorname{Pic} X_t = H^2(X_t, \mathbb{Z})$. The same holds in characteristic 0 by the Lefshchetz principal.

Step 3. Search for generators. Take a closed point $t \in T$ and generators $D_{1,t}, \dots, D_{r,t}$ of $\operatorname{Cl} X_t = \operatorname{Pic} X_t$. More precisely, their classes module $\sim$ are those generators. By construction and Step 2 there exists horizontal divisors $D_1, \dots, D_r$ such that $D_{1,t} \sim D_{1|_{X_t}}, \dots, D_r \sim D_{r|_{X_t}}$. Replacing every $D_{i,t}$ by the restriction $D_{i|_{X_t}}$ we get equation instead of $\sim$. Since X/T is projective, sufficiently general D_i are reduced with reduced restriction $D_{i,t}$ for every $t \in T$ and with pairwise disjoint support. When it is needed, we can take sufficiently general t, that is, a nonempty open subset in T. The prime components of D_i corresponds to the prime components of $D_{i,t}$. Take prime components of all $D_{i,t}$ we get required marked distinct prime divisors $D_{1,t}, \dots, D_{r,t}$ for every $t \in T$. (Possible with a different r.) Here we need again a finite covering of T related to the monodromy of components. By construction $D_{i,t}$ generate $\operatorname{WDiv} X_t = \operatorname{CDiv} X_t$ modulo $\sim$ for every t. This concludes the proof of the proposition and Addendum 12. By construction, for every closed $t \in T$ the natural homomorphism

$$\mathfrak{D} \to \mathfrak{D}_t / \sim = \operatorname{Cl} X_t = \operatorname{Cl} X/T$$

is surjective. Its kernel consists of divisors $D \in \mathfrak{D}$ such that $D \sim 0$ is principal modulo vertical divisors. In general, we can't remove vertical divisors. But for every given linear equivalence we can make this if we consider smaller T. Since we have a finitely generated Abelian group $\mathfrak{D}_T$, $\mathfrak{P}_T$ is finitely generated too and it is enough finitely many relations and we can get a required sufficiently small T.

Assume that Y/T is a log resolution of X/T: the exceptional divisors

$E_{i,t}$ with the birational transform of $D_{1,t},\dots,D_{r,t}$ are normal crossing. In this situation we can use the arguments of the proof of [Sh92, (1.3.2)] for Addendum 10.

Addendum 11 can be verified similarly, assuming that the log resolution is sufficiently high: $\mathcal{P}$ is stable over X. The latter means that $\mathcal{P}$ is defined over T.

Addendum 12 for **Tor** and $\mathfrak{N}$ follows from the fact that topologically or numerically trivial divisors in the rationally connected variety X_t correspond to torsions of $\mathrm{Tor}(H^2(X_t,\mathbb{Z})) = \mathrm{Tor}(\mathrm{Pic}\, X_t)$. That is, they are torsions modulo $\sim$.

Addendum 13 is immediate by Addendum 12 under the rationality of singularities. Indeed, the divisors of $\mathfrak{Car}_t$ correspond to the divisors on the resolution Y_t which are linearly trivial over X_t, equivalently, vertical Cartier on Y_t over X_t [Sh19, Propositions 2-3]. By the assumption, $\sim$ over Y_t is $\equiv /T$ up to torsions. This implies the constant property for sheaf of $\mathbb{Q}$-Cartier divisors $\mathfrak{Car}_{\mathbb{Q}} \cap \mathfrak{D}$. The Cartier divisors of the last sheaf are $\mathbb{Q}$-Cartier Weil divisors of Cartier index 1. This is an open condition with respect to T. A Noetherian induction concludes the proof. We need to use here also the boundedness of the Cartier index that follows from the boundedness of torsions.

$\square$

Theorem 5. *Let X/Z be a wFt morphism. Then the classes of effective Weil divisors in $\mathrm{Cl}\, X/Z$ form a finitely generated Abelian monoid $\mathrm{Eff}\, X/Z$. The same holds for classes effective Cartier divisors in $\mathrm{Cl}\, X/Z$.*

Addendum 14. *More precisely, there exists a finite set of classes $C_1,\dots,C_r \in \mathrm{Cl}\, X/Z$ of effective divisors with b-free mobile part over Z, in particular, their Zariski decomposition over Z is defined over $\mathbb{Z}$, and a finite set of classes $W_1,\dots,W_s \in \mathrm{Cl}\, X/Z$ of effective Weil divisors such that every class $E \in \mathrm{Cl}\, X/Z$ of an effective Weil divisor has the form*

$$E = W_j + n_1C_1 + \dots n_rC_r,\ \text{every } n_i \in \mathbb{Z}^{\geq 0}. \tag{4.0.4}$$

Addendum 15. *If X is $\mathbb{Q}$-factorial, then we can suppose additionally that every C_i is the class of an effective Cartier divisor.*

For us a different presentation of effective classes in terms of prime divisors will be more important, especially, on the level of divisors (cf. Step 6 in the proof of Theorem 7).

Finite linear presentation. Let X/Z be a morphism and $D_1, \dots, D_r$ be a finite collection of distinct prime divisors on X. They give a monoid of effective Weil divisors supported on $D_1 + \dots + D_r$

$$\mathfrak{D}^+ = \mathfrak{D}^+(D_1, \dots, D_r) = \{d_1 D_1 + \dots + d_r D_r \mid d_1, \dots, d_r \in \mathbb{Z}^{\geq 0}\} \subseteq \mathfrak{D} = \mathfrak{D}(D_1, \dots, D_r).$$

The monoid is Abelian with 0 and free, finitely generated. We say that $\mathfrak{D}^+$ is a *linear representative* if every effective divisor E on X is in $\mathfrak{D}^+$ modulo $\sim$:

$$E \sim_Z d_1 D_1 + \dots + d_r D_r \in \mathfrak{D}^+.$$

For local $X/Z \ni o$, it is enough $\sim$ instead of $\sim_Z$.

The same can be defined for Cartier divisors with effective indecomposable Cartier generators, not necessarily prime. An effective Cartier divisor D is *indecomposable*, if $D > 0$ and $D = C_1 + C_2$, where C_1, C_2 are effective Cartier then $D = C_1$ for $C_1 > 0$, otherwise $D = C_2$. However, Cartier generators may be not free in $\operatorname{CDiv} X$.

Corollary 9. *Let X/Z be a wFt morphism. Then there exists a linear representative finitely generated by prime Weil divisors monoid $\mathfrak{D}^+$. The same hold for Cartier divisors with effective indecomposable Cartier generators.*

Addendum 16. *Every ($\mathbb{R}$-linearly) effective $\mathbb{R}$-divisor E is $\mathbb{R}$-linearly equivalent to an element in $\mathfrak{D}^+_{\mathbb{R}}$. Respectively, every ($\mathbb{Q}$-linearly) effective $\mathbb{Q}$-divisor E is $\mathbb{Q}$-linearly equivalent to an element in $\mathfrak{D}^+_{\mathbb{Q}}$.*

Proof. Immediate by Theorem 5. Take as generators of $\mathfrak{D}^+$ distinct prime components D_i of effective Weil divisors E_j such that classes of E_j generate $\operatorname{Eff}(X/Z)$.

Effective Cartier generators can be replace by indecomposable ones.

The addendum is immediate. If Weil divisors $W_1, \dots, W_n$ are linearly equivalent to effective divisors $E_1, \dots, E_n \in \mathfrak{D}^+$ respectively, then

$$D = r_1 W_1 + \dots + r_n W_n \sim_{\mathbb{R}} r_1 E_1 + \dots + r_n E_n \in \mathfrak{D}^+_{\mathbb{R}}$$

for every real numbers $r_1, \dots, r_n \geq 0$. The same works for over $\mathbb{Q}$.

□

Corollary 10. *Let X/Z be a wFt morphism. Then the classes of effective exceptional divisors of X/Z in $\operatorname{Cl} X/Z$ is a finite union of free finitely generated Abelian saturated monoids of classes of effective exceptional divisors.*

Respectively, the classes of effective linearly fixed over Z divisors in $\operatorname{Cl} X/Z$ *belong to a finite union of orbits with action of free saturated monoids of classes of exceptional divisors.*

In particular, exceptional and effective linearly fixed divisors supported a finite reduced divisor.

Addendum 17. $\operatorname{Fix} X/Z = \mathfrak{F}$ *has finite support and is a finite union of orbits with action of exceptional submonoids of* $\mathfrak{Exc}^+$. *Every saturated submonoid in* $\mathfrak{Exc}^+$ *is exceptional.*

Exceptional divisors here and everywhere include birationally exceptional divisors, that is, divisors contractible by a rational 1-contraction.

Proof. A sum of (effective) exceptional divisors can be nonexceptional. For a wFt morphism or, equivalently, for a Ft morphisms X/Z the exceptional divisors belong to a union of exceptional divisors corresponding to birational 1-contractions $X \dashrightarrow Y/Z$. The exceptional divisors of such a contraction form a free Abelian group generated by the prime exceptional divisors of this contraction. Respectively, the effective exceptional divisors of this contraction form a free Abelian saturated monoid generated by the prime exceptional divisors of this contraction. The required results follows from [ShCh, Corollary 4.5 and Proposition 5.14].

An effective *linearly fixed* divisor is the only divisor in its linear system. A sum of those divisors is effective but not necessarily linearly fixed. By Corollary 9 every effective linearly fixed divisor D is supported in $D_1 + \cdots + D_r$, that is, $D = n_1 D_1 + \cdots + n_r D_r, n_i \in \mathbb{Z}^{\geq 0}$. On the other hand, nonnegative integral multiplicities n_i can be arbitrary large only for exceptional D_i. This concludes the statement about effective linearly fixed divisors.

In Addendum 17, $=$ holds because every effective fixed divisor is unique in its class modulo $\sim$. An *exceptional* submonoid of $\mathfrak{Exc}^+$ in the addendum is the monoid of effective exceptional divisors for a birational 1-contraction $X \dashrightarrow Y/Z$. Respectively, *saturation* means that if $n_1 E_1 + \cdots + n_m E_m$ belongs to the submonoid, where every $n_i \in \mathbb{N}$ and every E_i is an effective Weil divisor, then every E_i belongs to the submonoid.

∎

The proof of Theorem 5 will be reduced to the following [general fact from algebra].

Lemma 6. *Let C be a (free) Abelian monoid generated by a finite set of generators $C_1, \dots, C_r$, and S be a set with a transitive action of C and T be a subset in S closed under the action of C. Then T is finitely generated over C, that is, can be covered by finitely many orbits of C in T.*

Proof. Induction on r. The case $r = 0$ has empty set of generators and S is has at most one element.

Suppose now that $r \geq 1$ and $S \neq \emptyset$. Then by transitivity there exists an element $0 \in S$ such that every element $s \in S$ has the form

$$s = 0 + n_1 C_1 + \cdots + n_r C_r, \text{ every } n_i \in \mathbb{Z}^{\geq 0},$$

where $+$ denotes the action. We can suppose also that $T \neq \emptyset$. Then T has an element

$$t = 0 + m_1 C_1 + \cdots + m_r C_r, \text{ every } m_i \in \mathbb{Z}^{\geq 0}.$$

This element gives a big orbit $t + C$ in T. This means that for all other element

$$t' = 0 + m_1' C_1 + \cdots + m_r' C_r, \text{ every } m_i' \in \mathbb{Z}^{\geq 0}$$

in T some $m_i' < m_i$. The elements t' with fixed m_i' belong to the orbit

$$S' = 0 + m_i' C_i + C',$$

where C' is a free Abelian submonoid in C generated by C_j with $j \neq i$. By induction on r the intersection $S' \cap T$ can be covered by finitely many orbits of C'. Since we have finitely many subsets as S' in S, T can be covered by finitely many orbits of C.

$\square$

Proof of Theorem 5. Step 1. (Monoid.) If $E, E' \in \operatorname{Eff} X/Z$ are classes of effective Weil divisors D, D' then $E + E'$ is the class of effective Weil divisor $D + D'$. The zero class of $\operatorname{Eff} X/Z$ is the class of $0 \geq 0$. So, $\operatorname{Eff} X/Z$ is a Abelian monoid.

Step 2. Taking a $\mathbb{Q}$-factorialization $\varphi \colon Y \dashrightarrow X$ with a small birational modification of Lemma 1, we can suppose that X is $\mathbb{Q}$-factorial and X/Z has Ft. Since φ is a small birational transformation, it preserves required generators according to the commutative diagram

$$\begin{array}{ccc} \operatorname{Eff} Y/Z & \subset & \operatorname{Cl} Y/Z \\ \downarrow & & \downarrow \varphi_* \\ \operatorname{Eff} X/Z & \subset & \operatorname{Cl} X/Z \end{array}$$

with vertical isomorphisms.

Step 3. Generators C_i. We find generators modulo $\sim_{\mathbb{Q}}$. The classes of effective Weil divisors map naturally (on generators over $\mathbb{R}^+$) into the cone of effective divisors:

$$c\colon \operatorname{Eff} X/Z \to \operatorname{Eff}_{\mathbb{R}} X/Z \subset \operatorname{Cl}_{\mathbb{R}} X/Z = (\operatorname{Cl} X/Z) \otimes_{\mathbb{Z}} \mathbb{R},$$

a class modulo $\sim_Z$ goes to the class modulo $\sim_{\mathbb{R},Z}$ or $\equiv$ over Z. (The image generate the cone over $\mathbb{R}^+$.) The cone $\operatorname{Eff}_{\mathbb{R}} X/Z$ is closed convex rational polyhedral [ShCh, Corollary 4.5]. It has a rational simplicial cone decomposition. So, it is enough to establish the required finite generatedness of classes of effective Weil divisors over any rational simplicial cone $Q \subseteq \operatorname{Eff}_{\mathbb{R}} X/Z$. Moreover, we can suppose that Q is standard with respect to the lattice

$$(\operatorname{Cl} X/Z)/\operatorname{Tor} \subset \operatorname{Cl}_{\mathbb{R}} X/Z,$$

where Tor denotes the subgroup of torsions in $\operatorname{Cl} X/Z$. In other words, the primitive vectors $e_1, \dots, e_r$ of edges of Q generate the monoid $Q \cap ((\operatorname{Cl} X/Z)/\operatorname{Tor})$. By definition and construction, for every e_i, some positive multiple $C_i = m_i e_i$ is linear equivalent to an effective Weil divisor. By [ShCh, Proposition 5.14] we can take C_i which satisfy Addenda 14 and 15. The b-*free* assumption means that, after a small birational modification of X/Z, the mobile part of C_i become the class of a linearly free divisor over Z. In this situation, the Zariski decomposition of every C_i is defined in $\operatorname{Cl} X/Z$, in particular, over $\mathbb{Z}$. We can suppose also that the positive integers m_i are sufficiently divisible, e.g., kill torsions: for every m_i, $m_i \operatorname{Tor} = 0$. So, a lifting of C_i to $\operatorname{Cl} X/Z$ is well-defined: we identify $C_i \in Q$ with the lifting $C_i = m_i e_i' \in \operatorname{Cl} X/Z$, where $e_i' \in \operatorname{Cl} X/Z$ goes to $e_i \in Q$. Every class C_i in $\operatorname{Cl} X/Z$ is not a torsion and belongs to $\operatorname{Eff} X/Z$. The free Abelian monoid C generated by $C_1, \dots, C_r$ will be considered as a submonoid in $\operatorname{Eff} X/Z$ and in Q.

Step 4. Generators W_j. The submonoid C acts naturally on $\operatorname{Eff} X/Z, Q$ and on $c^{-1}Q$: for every element $n_1C_1 + \cdots + n_rC_r \in C$, a class $E \in \operatorname{Eff} X/Z$ goes to

$$E + n_1C_1 + \cdots + n_rC_r \in \operatorname{Eff} X/Z$$

under the action of $n_1C_1 + \cdots + n_rC_r$. Indeed, if $E \in c^{-1}Q$, then

$$c(E + n_1C_1 + \cdots + n_rC_r) = c(E) + n_1C_1 + \cdots + n_rC_r \in Q$$

and

$$E + n_1C_1 + \cdots + n_rC_r \in c^{-1}Q.$$

So, if $E \in c^{-1}Q \cap (\mathrm{Eff}\, X/Z)$, then for every $n_1C_1 + \cdots + n_rC_r \in C$,

$$E + n_1C_1 \cdots + n_rC_r \in c^{-1}Q \cap (\mathrm{Eff}\, X/Z)$$

and C acts on $c^{-1}Q \cap (\mathrm{Eff}\, X/Z)$. The required finite generatedness over Q means that there exists finitely many classes $W_1, \ldots, W_s \in c^{-1}Q \cap (\mathrm{Eff}\, X/Z)$ such that every class $E \in c^{-1}Q \cap (\mathrm{Eff}\, X/Z)$ has the form (4.0.4).

Step 5. Reduction to Lemma 6. Actually, it is enough to establish this for every orbit $E + C$ of $c^{-1}Q$. The orbits are not disjoint but finitely many of them cover $c^{-1}Q$: for finitely many $E_1, \ldots, E_l \in c^{-1}Q$,

$$c^{-1}Q = \cup_{i=1}^{l}(E_i + C).$$

Indeed, the same holds for Q with

$$E_j = n_1e_1 + \cdots + n_re_r, \text{ every } 0 \le n_i < m_i.$$

The required classes for $c^{-1}Q$ are the liftings of these E_j, that is, the elements of $c^{-1}E_j$. Every last set is finite and has as many elements as Tor.

The required finite generatedness of $(E + C) \cap (\mathrm{Eff}\, X/Z)$ for every given class $E + C$, that is, the intersection can be covered by finitely many orbits under the action of C, follows from Lemma 6.

The version with Cartier divisors can be established similarly.

□

Proposition 11. *Let $X_t, t \in T$, be a wFt variety or an algebraic space in a bounded family. Then a linear representative monoid $\mathfrak{D}_t^+$ has bounded rank r. More precisely, for appropriate parametrization X/T, every X_t has marked distinct prime divisors $D_{1,t}, \ldots, D_{r,t}$ such that for every effective $D \in$ WDiv X_t,*

$$D \sim d_1D_{1,t} + \cdots + d_rD_{r,t}$$

for some $d_1, \ldots, d_r \in \mathbb{Z}^{\ge 0}$.

Notice that these marked divisors also satisfy Proposition 10 (cf. Addendum 22 below).

Addendum 18. *The sheaf of linear representative monoids $\mathfrak{D}^+ = \mathfrak{D}^+(D_1, \ldots, D_r)$ is constant. Every $D \in \mathfrak{E}$ is linear equivalent to an effective divisor in $\mathfrak{D}_T^+$.*

Remark: $\mathfrak{D}^+ \subseteq \mathfrak{E}$ but not $=$ in general.

Addendum 19. *Sheaves* $\mathbf{Eff}\, X/T, \mathbf{Mob}\, X/T, \mathbf{Fix}\, X/T, \mathbf{Exc}\, X/T, \mathbf{Exc}^+\, X/T, \mathfrak{E}, \mathfrak{M}, \mathfrak{F}, \mathfrak{Exc}, \mathfrak{Exc}$ *are constant,*

$$\mathbf{Eff}\, X/T = \mathfrak{E} \,/ \sim = \mathfrak{E} \,/\, \mathfrak{P}, \mathbf{Eff}_T\, X/T = \mathfrak{E}_T \,/ \sim = \mathfrak{E}_T \,/\, \mathfrak{P}_T \ \text{ and } \operatorname{Eff} X_t = \mathfrak{E}_t \,/ \sim = \mathfrak{E}_t \,/\, \mathfrak{P}_t,$$

$$\mathbf{Mob}\, X/T = \mathfrak{M} \,/ \sim = \mathfrak{M} \,/\, \mathfrak{P}, \mathbf{Mob}_T\, X/T = \mathfrak{M}_T \,/ \sim = \mathfrak{M}_T \,/\, \mathfrak{P}_T \ \text{ and } \operatorname{Mob} X_t = \mathfrak{M}_t \,/ \sim = \mathfrak{M}_t \,/$$

$$\mathbf{Fix}\, X/T = \mathfrak{F}, \mathbf{Fix}_T\, X/T = \operatorname{Fix} X/T = \mathfrak{F}_T \ \text{ and } \operatorname{Fix} X_t = \mathfrak{F}_t,$$

$$\mathbf{Exc}\, X/T = \mathfrak{Exc}, \mathbf{Exc}_T\, X/T = \operatorname{Exc} X/T = \mathfrak{Exc}_T \ \text{ and } \operatorname{Exc} X_t = \mathfrak{Exc}_t,$$

$$\mathbf{Exc}^+\, X/T = \mathfrak{Exc}^+, \mathbf{Exc}_T^+\, X/T = \operatorname{Exc}^+ X/T = \mathfrak{Exc}_T^+ \ \text{ and } \operatorname{Exc}^+ X_t = \mathfrak{Exc}_t^+,$$

for every (closed) $t \in T$, *where* $\mathfrak{F}, \mathfrak{Exc}, \mathfrak{Exc}^+$ *are respectively subsheaves of (linearly) fixed, exceptional, effective exceptional divisors over* T *in* $\mathfrak{D}$ *and* $\mathbf{Fix}\, X/T, \mathbf{Exc}\, X/T, \mathbf{Exc}^+\, X/T$ *are corresponding subsheaves in* $\mathbf{Cl}\, X/T$.

The following sheaves

$$\mathbf{Eff}_{\mathbb{R}}\, X/T, \mathbf{Eff}_{\mathbb{Q}}, \mathbf{Mob}_{\mathbb{R}}\, X/T, \mathbf{Mob}_{\mathbb{Q}}\, X/T, \mathbf{Exc}_{\mathbb{R}}\, X/T, \mathbf{Exc}_{\mathbb{Q}}\, X/T, \mathbf{Exc}_{\mathbb{R}}^+\, X/T, \mathbf{Exc}_{\mathbb{Q}}^+\, X/T$$

$$\mathfrak{E}_{\mathbb{R}}, \mathfrak{E}_{\mathbb{Q}}, \mathfrak{M}_{\mathbb{R}}\, \mathfrak{M}_{\mathbb{Q}}, \mathfrak{Exc}_{\mathbb{R}}, \mathfrak{Exc}_{\mathbb{Q}}, \mathfrak{Exc}_{\mathbb{R}}^+, \mathfrak{Exc}_{\mathbb{Q}}^+$$

are also constant, generated respectively by $\mathbf{Eff}\, X/T, \mathbf{Mob}\, X/T, \mathbf{Exc}\, X/T, \mathbf{Exc}^+\, X/T, \mathfrak{E}, \mathfrak{M}, \mathfrak{Exc},$ *over* $\mathbb{R}^+, \mathbb{Q}^+$ *and*

$$\mathbf{Eff}_{\mathbb{R},T}\, X/T = \mathfrak{E}_{\mathbb{R},T} \,/ \sim_{\mathbb{R}} = \mathfrak{E}_{\mathbb{R},T} \,/\, \mathfrak{P}_{\mathbb{R},T}, \mathbf{Eff}_{\mathbb{Q},T} = \mathfrak{E}_{\mathbb{Q},T} \,/ \sim_{\mathbb{Q}} = \mathfrak{E}_{\mathbb{Q},T} \,/\, \mathfrak{P}_{\mathbb{Q},T},$$

$$\mathbf{Mob}_{\mathbb{R},T}\, X/T = \mathfrak{M}_{\mathbb{R},T} \,/ \sim_{\mathbb{R}} = \mathfrak{M}_{\mathbb{R},T} \,/\, \mathfrak{P}_{\mathbb{R},T}, \mathbf{Mob}_{\mathbb{Q},T} = \mathfrak{M}_{\mathbb{Q},T} \,/ \sim_{\mathbb{Q}} = \mathfrak{M}_{\mathbb{Q},T} \,/\, \mathfrak{P}_{\mathbb{Q},T},$$

$$\mathbf{Exc}_{\mathbb{R},T}\, X/T = \operatorname{Exc}_{\mathbb{R}} X/T = \mathfrak{Exc}_{\mathbb{R},T}, \mathbf{Exc}_{\mathbb{Q},T} = \operatorname{Exc}_{\mathbb{Q}} X/T = \mathfrak{Exc}_{\mathbb{Q},T},$$

$$\mathbf{Exc}_{\mathbb{R},T}^+\, X/T = \operatorname{Exc}_{\mathbb{R}}^+ X/T = \mathfrak{Exc}_{\mathbb{R},T}^+, \mathbf{Exc}_{\mathbb{Q},T}^+ = \operatorname{Exc}_{\mathbb{Q}} X/T = \mathfrak{Exc}_{\mathbb{Q},T}^+ .$$

Canonical isomorphisms are given by homomorphisms $\mathfrak{D}_t \to \operatorname{Cl} X_t, D_t \mapsto D_t / \sim = D_t / \mathfrak{P}_t$. *Equivalences* $\sim_{\mathbb{Q}}, \sim_{\mathbb{R}}$ *can be replaced by* $\equiv /T$.

Remark: We do not consider $\mathbf{Fix}_{\mathbb{R}}\, X/T, \mathbf{Fix}_{\mathbb{Q}}\, X/T$ and respectively $\mathfrak{F}_{\mathbb{R}}, \mathfrak{F}_{\mathbb{Q}}$ because to be fixed even under wFt is preserved for multiplicities of a divisor if and only if it is exceptional (Zariski decomposition).

For X/Z, monoids $\mathfrak{E}$ and $\mathfrak{M}$ are defined respectively as effective and mobile divisors in $\mathfrak{D}$ modulo $\sim_Z$. The corresponding classes in $\operatorname{Cl} X/Z$ are $\operatorname{Eff} X/Z$ and $\operatorname{Mob} X/Z$ respectively. But $\mathfrak{F}$ defined as divisors without $\sim_Z$ in $\mathfrak{D}$ and $\mathfrak{F} \supseteq \mathfrak{Exc}^+$. The corresponding classes belong to $\operatorname{Fix} X/Z$ and $\operatorname{Fix} X/Z = \mathfrak{F}$ under the canonical identification.

Addendum 20. *If additionally X/T is projective, that is, has Ft, then sheaves of monoids*

$$\mathbf{sAmp}\,X/T = \mathbf{Nef}\,X/T, \mathfrak{sA} = \mathfrak{Nef}$$

are constant and

$$\mathbf{sAmp}\,X/T = \mathfrak{sA}\,/\sim = \mathfrak{sA}\,/\,\mathfrak{P}, \mathbf{sAmp}_T\,X/T = \mathfrak{sA}_T\,/\sim = \mathfrak{sA}_T\,/\,\mathfrak{P}_T,$$

$$\mathbf{Nef}\,X/T = \mathfrak{Nef}\,/\sim = \mathfrak{Nef}\,/\,\mathfrak{P}, \mathbf{Nef}_T\,X/T = \mathfrak{Nef}_T\,/\sim = \mathfrak{Nef}_T\,/\,\mathfrak{P}_T\,.$$

The following sheaves of cones

$$\mathbf{sAmp}_{\mathbb{R}}\,X/T = \mathbf{Nef}_{\mathbb{R}}\,X/T, \mathbf{sAmp}_{\mathbb{Q}}\,X/T = \mathbf{Nef}_{\mathbb{Q}}\,X/T, \mathfrak{sA}_{\mathbb{R}} = \mathfrak{Nef}_{\mathbb{R}}, \mathfrak{sA}_{\mathbb{Q}} = \mathfrak{Nef}_{\mathbb{Q}}$$

are also constant, generated respectively by

$$\mathbf{sAmp}\,X/T, \mathfrak{sA}$$

over $\mathbb{R}^+, \mathbb{Q}^+$ and

$$\mathbf{sAmp}_{\mathbb{R},T}\,X/T = \mathfrak{sA}_{\mathbb{R},T}\,/\sim_{\mathbb{R}} = \mathfrak{sA}_{\mathbb{R},T}\,/\,\mathfrak{P}_{\mathbb{R},T}, \mathbf{sAmp}_{\mathbb{Q},T} = \mathfrak{sA}_{\mathbb{Q},T}\,/\sim_{\mathbb{Q}} = \mathfrak{sA}_{\mathbb{Q},T}\,/\,\mathfrak{P}_{\mathbb{Q},T},$$

$$\mathbf{Nef}_{\mathbb{R},T}\,X/T = \mathfrak{Nef}_{\mathbb{R},T}\,/\,\mathfrak{N}_{\mathbb{R},T}, \mathbf{Nef}_{\mathbb{Q},T} = \mathfrak{Nef}_{\mathbb{Q},T}\,/\,\mathfrak{N}_{\mathbb{Q},T}\,.$$

Canonical isomorphisms are given by homomorphisms $\mathfrak{D}_t \to \operatorname{Cl} X_t, D_t \mapsto D_t/\sim = D_t/\,\mathfrak{P}_t$ Equivalences $\sim_{\mathbb{Q}}, \sim_{\mathbb{R}}$ can be replaced by $\equiv /T$.

Proof. For good properties of the family $X_t, t \in T$, we need to change it or to take an appropriate parametrization. Usually we alternate the base T and cut out its closed subfamilies. We use for this a Noetherian induction. However, the key point in all proofs is a boundedness that typically means finiteness and boundedness of generators.

Step 1. We can suppose that every X_t is $\mathbb{Q}$-factorial, has Ft and there exists a polarization H_t on X_t compatible with a complement. The latter means that there exists an $\mathbb{R}$-complement (X_t, B_t) of $(X_t, 0)$ and effective $E_t \le B_t$ such that $E_t \equiv hH_t$, where h is a positive rational number, independent of t, and H_t is an ample Cartier divisor on X_t. For this we can convert our family X/T into a wFt and $\mathbb{Q}$-factorial Ft variety with $\mathbb{Q}$-factorial Ft fibers X_t by Lemma 1 and after a $\mathbb{Q}$-factorialization. Then we can construct an $\mathbb{R}$-complement and actually a klt n-complement $(X/T, B)$ of $(X/T, 0)$, a relative polarization H on X over T and an effective $\mathbb{Q}$-divisor $E \le B$ on

X such that $E \equiv hH/T$, where h is a required positive rational number. Moreover, by Proposition 10 we can suppose that K, B, E, H are constant, that is, belong to $\mathfrak{D}_{\mathbb{Q}}$. Put $B_t = B_{|X_t}, E_t = E_{|X_t}, H_t = H_{|X_t}$.

In particular, boundedness of divisors or other cycles on X_t can be measured by H or, more precisely, by H_t.

Note also that sheaves $\mathbf{Eff}, \mathbf{Mob}, \mathbf{Fix}, \mathbf{Exc}, \mathbf{Exc}^+$, their divisorial versions $\mathfrak{E}, \mathfrak{M}, \mathfrak{F}, \mathfrak{Exc}, \mathfrak{Exc}^+$ and their $\mathbb{R}, \mathbb{Q}$ versions are invariants of small birational modification. But sheaves $\mathbf{sAmp} = \mathbf{Nef}, \mathfrak{sA} = \mathfrak{Nef}$ and their $\mathbb{R}, \mathbb{Q}$ versions depend on a model of X/T; $=$ holds by [ShCh, Corollary 4.5] because every X_t has Ft.

Our main objective is to verify the constant sheaf property for $\mathfrak{E}$ and $\mathbf{Eff}\, X/T$. We will be sketchy for other sheaves. By Proposition 10 we can suppose that the sheaf of $\mathbb{R}$-linear spaces $\mathbf{Cl}_{\mathbb{R}}\, X/T$ is constant.

Step 2. Its subsheaf of closed convex rational polyhedral cones $\mathbf{Eff}_{\mathbb{R}}\, X/T$ is also constant. Indeed, $\mathrm{Eff}_{\mathbb{R}}\, X/T$ is covered by finitely many convex rational polyhedral cones (countries) $\mathfrak{P}_\varphi$ (relative geography). The cone $\mathfrak{P}_\varphi$ corresponds to a models $\varphi\colon X \dashrightarrow Y/T$, where φ is a rational 1-contraction. The class of an $\mathbb{R}$-divisor D belongs to $\mathfrak{P}_\varphi$ if its 1-contraction φ_D (D-model) is φ and the birational 1-contractions, on which D is nef, are the same (minimal D-models). Cones $\mathfrak{P}_\varphi$ are not necessarily closed. (They depend also on the minimal models.) The finiteness and existence of such a covering for $\mathrm{Eff}_{\mathbb{R}}\, X/T$ and for $\mathrm{Eff}_{\mathbb{R}}\, X_t$ see in [ShCh, Theorem 3.4 and Corollary 5.3].

To verify that $\mathbf{Eff}_{\mathbb{R}}\, X/T$ is constant it is easier to do this with the constant sheaf property of cones $\mathfrak{P}_\varphi$. For this we need to verify that the restriction $\varphi_{|\mathsf{X_t}}$ of cone $\mathfrak{P}_\varphi$ for $\mathrm{Eff}_{\mathbb{R}}\, X/T$ is $\mathfrak{P}_{\varphi_t}$, where

$$\varphi_t = \varphi_{|X_t}\colon X_t \dashrightarrow Y_t$$

and the corresponding models with nef D_t are restrictions too.

We can do this inductively with respect to models φ. We start with $\varphi = \mathrm{Id}_X$, the identical model. In this case $\mathfrak{P}_\varphi = \mathrm{Amp}_{\mathbb{R}}\, X/T$ is the cone of ample $\mathbb{R}$-divisors on X/T. The sheaf of such cones $\mathbf{Amp}_{\mathbb{R}}\, X/T$ is also constant over appropriate nonempty open subset in T. Moreover, so does its closure $\mathbf{Nef}_{\mathbb{R}}\, X/T = \mathbf{sAmp}_{\mathbb{R}}\, X/T$. The equality $=$ holds because X/T has Ft. The closure is covered by cones $\mathfrak{P}_\varphi$ with contractions $\varphi\colon X \to Y/T$ given by nef over T divisors D. Contractions $X_t \to Y_t$ are restricted from those contractions of X over T after taking an appropriate nonempty open subset in T. Both X/T and X_t have finitely many contractions. Moreover, they are bounded. This allows to remove a proper closed subset in T of contraction of X_t which are not restricted. The boundedness uses the n-complement structure. If φ is a fibration then generic fibers $X_y = \varphi^{-1}y, y \in Y$, are bounded: $(X_y, B_y - E_y)$ is a klt log Fano of bounded (local) lc index, where $B_y = B_{|X_y}, E_y = E_{|X_y}$. If φ is birational then its exceptional locus is covered by bounded curves C over Y and bounded itself. To find bounded curves C we can take ones with $-(C.K+B-E) \le 2\dim X$ [MM]. Since $(C.K+B) = 0$ then $(C.H) \le 2\dim X/h$ is bounded.

Fibers and curves are treated as effective cycles. Perhaps, we need an alteration of T to have an appropriate (constant) family of those cycles.

If $\mathrm{Eff}_{\mathbb{R}} X/T = \mathrm{Nef}_{\mathbb{R}} X/T$ we are done. Otherwise we take a cone $\mathfrak{P}_\varphi$ of the maximal dimension as $\mathrm{Nef}_{\mathbb{R}} X/T$ with the closure intersecting $\mathrm{Nef}_{\mathbb{R}} X/T$. In this case X is birational to Y under φ and $\mathfrak{P}_\varphi = \mathrm{Amp}_{\mathbb{R}} Y/T$. (Actually, we can suppose that φ is an extremal divisorial contraction or small birational modification, an extremal flop of $(X/T, B)$.) So, we get a bounded family of Ft varieties Y_t. We can repeat above arguments and extend our constant sheaf $\mathbf{Nef}_{\mathbb{R}} X/T$ by also a constant sheaf $\mathbf{Nef}_{\mathbb{R}} Y/T$. Since we have finitely many models φ and closures of their cones $\mathfrak{P}_\varphi$ are connected by the convex property of $\mathbf{Eff}_{\mathbb{R}} X/T$, we get the constant property of $\mathbf{Eff}_{\mathbb{R}} X/T$, possible taking a nonempty open subset in T. For the rest in T we use Noetherian induction.

Since all cones $\mathfrak{P}_\varphi$ are rational we proved also that $\mathbf{Eff}_{\mathbb{Q}} X/T$ is constant too. Similarly we can prove that $\mathbf{Mob}_{\mathbb{R}} X/T, \mathbf{Mob}_{\mathbb{Q}} X/T$ are constant. Sheaves $\mathbf{Exc}\, X/T = \mathfrak{Exc}, \mathbf{Exc}^+ X/T = \mathfrak{Exc}^+$ and their $\mathbb{R}, \mathbb{Q}$ versions are also constant. They are union of corresponding monoids of exceptional locus for birational models φ. For $=$ we need a marked divisors $D_{i,t}$ as in Proposition 10. Note that $\mathfrak{Exc}_t, \mathfrak{Exc}_t^+$ are defined as divisors without $\sim$ in $\mathfrak{D}_t$. The same marked divisors work for the constant sheaf property with other properties of $\mathfrak{E}_{\mathbb{R}}, \mathfrak{M}_{\mathbb{R}}$ and their $\mathbb{Q}$ versions.

By the way we proved that sheaves $\mathbf{Nef}_{\mathbb{R}} X/T = \mathbf{sAmp}_{\mathbb{R}} X/T$ and their $\mathbb{Q}$ version are constant too. The marked divisors as above can be used to prove that $\mathfrak{Nef}_{\mathbb{R}} = \mathfrak{sA}_{\mathbb{R}}$ and their $\mathbb{Q}$ versions are constant and satisfy required properties.

The other sheaves of integral divisors need more sophisticated technique of the proof of Theorem 5. A Zariski decomposition is not useful here since it is not integral usually. On the other hand, the decomposition into a fixed and a mobile parts is not stable (Zariski) under multiplication. Thus we can't reduce the constant property for $\mathbf{Eff}\,X/T$ to that of $\mathbf{Fix}\,X/T$ and of $\mathbf{Mob}\,X/T$. Cf. also Step 4.

Step 3. $\mathbf{Eff}\,X/T$ is constant. We consider $\mathbf{Eff}\,X/T$ as a sheaf of submonoids in $\mathbf{Cl}\,X/T$. We use the proof of Theorem 5 with minor [modifications] improvements (cf. Lemma 6).

By Proposition 10, $\mathbf{Cl}\,X/T$ is constant and there exists a natural constant sheaf homomorphism

$$c\colon\ \mathbf{Cl}\,X/T \to \mathbf{Cl}_{\mathbb{R}}\,X/T$$

induced by $\otimes\mathbb{R}$. We consider a constant rational simplicial cone $Q \subseteq \mathbf{Eff}_{\mathbb{R}}\,X/T \subseteq \mathbf{Cl}_{\mathbb{R}}\,X/T$ as in Step 3 of the proof of Theorem 5. We can suppose also that Q lies in the closure of a cone $\mathfrak{P}_{\varphi}$ in $\mathbf{Eff}_{\mathbb{R}}\,X/T$. We suppose also that $C_i \in Q$ are free on a nef model for C_i (cf. Addendum 14). We take also constant classes W_j corresponding to W_j of Step 4 in the proof of Theorem 5. Notice for this that $\mathbf{Cl}_{\mathbb{R},T}\,X/T = \mathrm{Cl}_{\mathbb{R}}\,X/T$ by Proposition 10. We state that restrictions $C_{i,t} = C_{i}{}_{|X_t}, W_{j,t} = W_{j}{}_{|X_t}$ generates $c^{-1}Q \cap (\mathrm{Eff}\,X_t)$, possibly after taking a nonempty open subset in T. This implies that $c^{-1}Q \cap (\mathbf{Eff}\,X/T)$ is constant and $\mathbf{Eff}\,X/T$ is constant too.

For this we consider orbits $W+C$, where $c(W)$ belongs to integral points of Q. Since the kernel of c is finite and Q is covered by finitely many orbits $c(W)+C$, the inverse image $c^{-1}Q$ is covered by finitely many orbits $W+C$ too. We verify that, for every t in some nonempty open subset in T, every orbit $W_t + C_t$ has only restricted classes of Eff X/T in the intersection with Eff X_t, where $W_t = W_{|X_t}, C_t = C_{|X_t}$. This implies required statement about generators. If it is not true, for some W, there exist *infinitely* many elements $W_t + n_1C_{1,t} + \dots + n_rC_{r,t} \in \mathrm{Eff}\,X_t$ with nonnegative integers n_i which are not restricted from elements of Eff X/T. That is, there exist infinitely many nonnegative integral vectors $(n_1,\dots,n_r)$ and some $t \in T$ for every such vector that $W_t + n_1C_{1,t} + \dots + n_rC_{r,t} \in \mathrm{Eff}\,X_t$ is not restricted from elements of Eff X/T. Otherwise, we can remove finitely many closed proper subsets in T corresponding to vectors $(n_1,\dots,n_r)$ of nonrestricted elements. The effective divisors with the linear equivalence class $W_t + n_1C_{1,t} + \dots + n_rC_{r,t}$ are bounded. (By definition Eff X_t consists of the classes of effective Weil divisors modulo $\sim$.) Replacing W by $W + m_1C_1 + \dots + m_rC_r$ with a fixed nonnegative integral vector $(m_1,\dots,m_r)$ and interchanging classes C_i we can suppose that nonnegative integral vectors $(n_1,\dots,n_r)$ has the form $(n_1,\dots,n_q,0,\dots,0), q \le r$, where $n_i, i = 1,\dots,q$, are arbitrary large: for every nonnegative integral vector $(m_1,\dots,m_q)$ there exists $(n_1,\dots,n_q)$ with every $n_i \geq m_i$. This is impossible after removing a proper closed subset in T.

Indeed, consider the contraction $X' \to Y/T$ corresponding to the nef model of φ. By construction if the class of linear equivalence $W_t + n_1 C_{1,t} + \cdots + n_q C_{q,t} \in \operatorname{Eff} X_t$ then it has an effective divisor D_t on X_t. So, a sufficiently general fiber $X'_{t,y}, y \in Y$, of X'/Y has an effective divisor $D_{t,y} = D_{t|X'_{t,y}}$ in the class of linear equivalence $W_{t,y} = W_{t|X'_{t,y}}$. In other words, the class W_t has a representative modulo $\sim$ with the effective restriction $D_{t,y}$, that is, the horizontal part of the representative is effective. Those representative are bounded as cycles on X'/T. If the corresponding closed subset in T is proper then we can remove this set with all classes $W + n_1 C_1 \cdots + n_q C_q$ and there are no nonrestricted classes. Thus we can assume that, for every $t \in T$, W_t has a representative modulo $\sim$ such that its horizontal part over Y is effective. Adding $n_1 C_{1,t} + \cdots + n_q C_{q,t}$ with sufficiently large $n_1, \ldots, n_q$ we get $W_t + n_1 C_{1,t} + \cdots + n_q C_{q,t}$ in $\operatorname{Eff} X_t$. Indeed, $n_1 C_{1,t} + \cdots + n_q C_{q,t}$ is the pull back of a very ample divisor over T from Y. Thus $W_t + n_1 C_{1,t} + \cdots + n_q C_{q,t}$ is the restriction of $W + c_1 C_1 \cdots + c_q C_q \in \operatorname{Eff} X/T$, a contradiction.

Theorem 5 implies that $\mathbf{Eff}\, X/T$ is finitely generated. Thus we can find marked divisors D_i and construct $\mathfrak{E} \subset \mathfrak{D}$ with surjection $\mathfrak{E} \twoheadrightarrow \mathbf{Eff}\, X/T$. The sheaf $\mathfrak{E}$ is also constant.

Step 4. $\mathbf{Mob}\, X/T$ is constant by the similar arguments as for $\mathbf{Eff}\, X/T$ in Step 3. This allows to construct the constant sheaf $\mathfrak{M}$ with required properties.

$\mathbf{Fix}\, X/T = \mathfrak{F}$ is constant too by Addendum 17.

Finally, we can replace $\sim_T$ by $\sim$ for smaller T in relations such as $\mathbf{Eff}\, X/T = \mathfrak{E}/\sim$ because $\mathcal{P}$ is finitely generated. However, for the sheaf $\mathbf{Eff}\, X/T$ it is better to use $\sim_T$. The same holds for other relations and over $\mathbb{R}, \mathbb{Q}$.

□

Proposition 12. *Let $X_t, t \in T$, be a wFt variety or an algebraic space in a bounded family. Then for appropriate parametrization X/T sheaves $\mathfrak{C}, \mathfrak{C}_{\mathbb{R}}, \mathfrak{C}_{\mathbb{Q}}$ are constant.*

Addendum 21. *$\mathfrak{C}\mathcal{P}, \mathfrak{C}_{\mathbb{R}}\mathcal{P}, \mathfrak{C}_{\mathbb{Q}}\mathcal{P}$ are also constant if $\mathcal{P}$ is defined over T (globally).*

Addendum 22. *We can suppose that all our standard sheaves are constant simultaneously. Moreover, they are bounded or, equivalently, finitely generated.*

Example 8 (cf. Step 8 Addendum 30 in Proof of Theorem 7). Consider a family $(X_t, D'_t + \mathcal{P}_t), t \in T$, of exceptional wFt pairs, not assuming that $\mathcal{P}$ is defined over T, but assuming that the pairs $(X_t, D'_t + \mathcal{P}_{X_t})$ form a family with $\mathcal{P}_{X_t} = (\mathcal{P}_t)_{X_t}$. Then sheaves $\mathfrak{C}_t\,\mathcal{P}_t \cap \{D_t \in \mathfrak{D}_t \mid D_t \geq D'_t\}, \mathfrak{C}_{\mathbb{R},t}\,\mathcal{P}_t \cap \{D_t \in \mathfrak{D}_{\mathbb{R},t} \mid D_t \geq D'_t\}, \mathfrak{C}_{\mathbb{Q},t}\,\mathcal{P}_t \cap \{D_t \in \mathfrak{D}_{\mathbb{Q},t} \mid D_t \geq D'_t\}$ are well-defined and constant; additionally $\mathfrak{C}_{\mathbb{R},t}\,\mathcal{P}_t \cap \{D_t \in \mathfrak{D}_{\mathbb{R},t} \mid D_t \geq D'_t\}$ is closed convex rational polyhedral. Indeed, $(X_t, D_t + \mathcal{P}_t)$ is also exceptional if and only if $-(K_{X_t} + D_t + \mathcal{P}_{X_t}) \in \mathfrak{E}_{\mathbb{R},t}$, equivalently, has an $\mathbb{R}$-complement. However, the inclusion assumption is constant and polyhedral by Addendum 19 and [ShCh, Corollary 4.5]. We assume here that $K_{X_t}, \mathcal{P}_{X_t} \in \mathfrak{D}_t$ are also constant.

Notice that $\mathfrak{C}_t\,\mathcal{P}_t, \mathfrak{C}_{\mathbb{R},t}\,\mathcal{P}_t, \mathfrak{C}_{\mathbb{Q},t}\,\mathcal{P}_t$ are possibly nonconstant and $\mathfrak{C}_{\mathbb{R}}\,\mathcal{P}$ is not well-defined if $\mathcal{P}$ is not defined over T.

Proof of Proposition 12. It is enough to be constant for $\mathfrak{C}_{\mathbb{R}}$. To establish this we can use the proof of Theorem 6 below. In particular, we can suppose that the 1-contractions $\varphi_t \colon X_t \dashrightarrow Y_t$ and cones $\mathfrak{P}_{\varphi_t}$ are constant. In this situation, Addendum 10 implies that $\mathfrak{C}_{\mathbb{R}}$ is constant.

Respectively, Addendum 11 implies Addendum 21.

Addendum 22 is immediate by proofs of Propositions 10, 11 and their Addenda. First, we begin from the constant property of $\mathbf{Cl}\,X/T$ and its $\mathbb{R}, \mathbb{Q}$ versions. Second, we add its constant subsheaves $\mathbf{Eff}\,X/T, \mathbf{Mob}\,X/T$ etc and their $\mathbb{R}, \mathbb{Q}$ versions. Third, we pick up marked divisors $D_1, \ldots, D_r$ in classes of $\mathbf{Cl}\,X/T$ as generators of $\mathbf{Eff}\,X/D$. This gives the constant sheaf $\mathfrak{D}$, adds its constant subsheaves $\mathfrak{E}, \mathfrak{M}$ etc and their $\mathbb{R}, \mathbb{Q}$ versions. We can suppose that marked divisors are prime and generate $\mathbf{Cl}\,X/T$. Forth, we can include into generators prime exceptional and prime components of fixed divisors. Here we use the finiteness of Addendum 17. Finally, for projective X_t, we can add the constant subsheaves $\mathbf{sAmp}\,X/T, \mathbf{Nef}\,X/T, \mathfrak{sA}, \mathfrak{Nef}$ and their $\mathbb{R}, \mathbb{Q}$ versions. We add constant divisors $K, \mathcal{P}_{X_t}$ etc, finitely many other constant divisors or their classes.

The same works for families of bd-pairs $(X_t, D_t + \mathcal{P}_t)$, e.g., for sheaves as $\mathfrak{lc}_{\mathbb{R}}\,\mathcal{P}, \mathfrak{C}_{\mathbb{R}}\,\mathcal{P}$.

$\square$

Lemma 7. *Let $\mathfrak{P}$ be a compact rational polyhedron in $\mathfrak{E}_{\mathbb{R}}$. Then there exists a positive integer J such that, for any rational divisor $D \in \mathfrak{P}$ with $qD \in \mathbb{Z}$, where q is a positive integer, JqD is effective modulo $\sim$.*

Addendum 23. *If $\mathfrak{E}$ is also constant then, for a constant compact rational polyhedron $\mathfrak{P}_t \subseteq \mathfrak{E}_{\mathbb{R},t}$, there exists a positive integer J such that the lemma hold for same J for every $D_t \in \mathfrak{P}_t$ (possibly after reparametrization).*

Proof. Step 1. Reduction to the case when $\mathfrak{P} = [D_1, \ldots, D_r]$ is a simplex. Decompose the polyhedron into finitely many simplices $\mathfrak{P}_1, \ldots, \mathfrak{P}_l$. Take $J = J_1 \ldots J_r$ where J_i is a required integer for $\mathfrak{P}_i$.

Step 2. Since every vertex D_i is a rational in $\mathfrak{E}_{\mathbb{R}}$, there exists a positive integer J_1 such that $J_i D_i \sim E_i$ and E_i is an effective Weil divisor. In particular, every $J_i D_i \in \mathbb{Z}$.

Step 3. We contend that $J = J_1^2 \ldots J_r^2$ is a required integer. Indeed, take $D \in [D_1, \ldots, D_r]$ with $qD \in \mathbb{Z}$ for a positive integer q. Then $D = w_1 D_1 \cdots + w_r D_r$ with $0 \le w_1, \ldots, w_r \in \mathbb{Q}, w_1 + \cdots + w_r = 1$. Then by Step 2 every $J_1 \ldots J_r q w_i$ is a nonnegative integer and

$$JqD = Jqw_1D_1 + \cdots + Jqw_rD_r \sim (Jqw_1/J_1)E_1 + \cdots + (Jqw_r/J_r)E_r \geq 0 \text{ and } \in \mathbb{Z}.$$

Step 4. The addendum holds for constant vertices $D_{1,t}, \ldots, D_{r,t}$. Indeed, every $J_i D_{i,t}$ is constant too and $J_i D_{i,t} \sim E_{i,t}$ for some $E_{i,t} \in \mathfrak{E}_t$. (We do not assume that $E_{i,t}$ is constant. However, it is true for an appropriate parametrization by Addendum 22 if X/T has wFt and the collection of marked divisors is sufficiently large.)

□

Theorem 6. *Let X/Z be a wFt morphism and $D_1, \ldots, D_r$ be a finite collection of distinct prime divisors on X. Then the set*

$$\mathfrak{C}_{\mathbb{R}} = \{D \in \mathfrak{D}_{\mathbb{R}} \mid (X/Z, D) \text{ has an } \mathbb{R}\text{-complement}\}$$

is a closed convex rational polyhedron in $\mathfrak{D}$.

Addendum 24. *If $K \in \mathfrak{D}$ then*

$$\mathfrak{C}_{\mathbb{R}} \subseteq (-K - \mathfrak{E}_{\mathbb{R}}).$$

Moreover, if X is $\mathbb{Q}$-factorial then

$$\mathfrak{C}_{\mathbb{R}} \subseteq \mathfrak{lc}_{\mathbb{R}} \cap (-K - \mathfrak{E}_{\mathbb{R}}).$$

In general, $=$ does not hold but it is true for divisors D such that $-(K+D)$ is nef over Z (cf. the proof below).

Addendum 25. *For any b-divisor $\mathcal{P}$ of X, the same holds for the set*

$$\mathfrak{C}_{\mathbb{R}}\,\mathcal{P} = \{D \in \mathfrak{D}_{\mathbb{R}} \mid (X/Z, D+\mathcal{P}) \text{ has an } \mathbb{R}\text{-complement}\}.$$

In Addendum 24, K should be replace by $K+\mathcal{P}_X$.

Actually, we need the last statement especially for bd-pairs $(X/Z, D+\mathcal{P})$.

Proof. Convexity of $\mathfrak{C}_{\mathbb{R}}$ is by definition: [if $(X/Z, D_1), (X/Z, D_2)$ have respectively $\mathbb{R}$-complements $(X/Z, D_1^+), (X/Z, D_2^+)$ and $a_1, a_2 \in [0,1], a_1+a_2 = 1$, then $(X/Z, a_1D_1^+ + a_2D_2^+)$ is an $\mathbb{R}$-complement of $(X/Z, a_1D_1 + a_2D_2)$].

It is enough to verify the statement for a sufficiently large collection of divisors $D_1, \dots, D_r$. In particular, we can suppose that some $K \in \mathfrak{D}$.

Step 1. Reduction to the nef and lc properties. The cone decomposition $\mathfrak{P}_\varphi$ of $\operatorname{Eff}_{\mathbb{R}} X/Z$ [ShCh, Theorem 3.4 and Corollary 5.3] (see also Step 2 in the proof of Proposition 11) induces a convex rational polyhedral cone decomposition of $\mathfrak{E}_{\mathbb{R}}$. We use the same notation for cones $\mathfrak{P}_\varphi$ of $\mathfrak{E}_{\mathbb{R}}$ as for $\operatorname{Eff}_{\mathbb{R}} X/Z$. It is enough to establish the polyhedral property of $\mathfrak{C}_{\mathbb{R}}$ for its intersection

$$\mathfrak{C}_{\mathbb{R}} \cap (-K - \overline{\mathfrak{P}_\varphi})$$

with the closure of every cone $\mathfrak{P}_\varphi$ of the decomposition. Indeed, $\mathfrak{P}_\varphi$ corresponds to a rational 1-contraction $\varphi\colon X \dashrightarrow Y/Z$ (and its birational nef models). By definition, for every $D \in \mathfrak{C}_{\mathbb{R}}$, $-(K+D) \in \mathfrak{E}_{\mathbb{R}}$ holds, or equivalently, $D \in -K - \mathfrak{E}_{\mathbb{R}}$. Moreover, a birational 1-contraction $\psi\colon X \dashrightarrow X^\sharp/Z$, on which D is nef, is one of maximal models of Construction 2. By Addendum 8, if $D \in (-K - \overline{\mathfrak{P}_\varphi})$ then, for the birational transform $D^\sharp_{X^\sharp}$ of divisor D on $X^\sharp$, $(X/Z, D)$ has an $\mathbb{R}$-complement if and only if $(X^\sharp/Z, D^\sharp_{X^\sharp})$ is lc. By construction, $-(K_{X^\sharp} + D^\sharp_{X^\sharp})$ is nef over Z. On the other hand, mapping of

$$-K - \overline{\mathfrak{P}_\varphi} \to -K_{X^\sharp} - \mathfrak{Nef}_{\mathbb{R}}\, X^\sharp/Z, D \to D^\sharp_{X^\sharp},$$

is affine over $\mathbb{Q}$ with a polyhedral image, where $\mathfrak{D}_{\mathbb{R}}X^\sharp$ is generated by the birational images $D_{i,X^\sharp}$ of nonexceptional divisors D_i on $X^\sharp$. Thus $\mathfrak{C}_{\mathbb{R}} \cap (-K - \mathfrak{Nef}_{\mathbb{R}})$ is the preimage of $\mathfrak{lc}_{\mathbb{R}}\, X^\sharp \cap (-K_{X^\sharp} - \mathfrak{Nef}_{\mathbb{R}}\, X^\sharp/Z)$ and we need to verify that the latter is closed rational polyhedral. Notice also that K goes birationally to $K_{X^\sharp} \in \mathfrak{D}X^\sharp$.

Step 2. We can consider only the case $\overline{\mathfrak{P}_\varphi} = \mathfrak{Nef}_\mathbb{R}$ with $\varphi = \mathrm{Id}_X$. In this situation

$$\mathfrak{C}_\mathbb{R} \cap (-K - \mathfrak{Nef}_\mathbb{R}) = \mathfrak{lc}_\mathbb{R} \cap (-K - \mathfrak{Nef}_\mathbb{R}).$$

In other words, for nef $-(K+D)$ over Z, $(X/Z, D)$ has an $\mathbb{R}$-complement if and only if (X, D) is lc (cf. Example 1, (1)). (In this case (X, D) is a log pair.) But $\mathfrak{lc}_\mathbb{R}$ is a closed convex rational polyhedron [Sh92, (1.3.2)]. Thus the above intersection is closed convex rational polyhedron too.

Step 3. Addendum 24 follows from the effective property of $-(K+D)$ noticed in Step 1. The rest of the addendum follows from the lc property of $\mathbb{R}$-complements [if $(X/Z, D)$ has an $\mathbb{R}$-complement and (X, D) is a log pair then (X, D) is lc].

The pairs with a b-divisor $\mathcal{P}$ can be treated similarly.

□

Proof of Theorem 2. Immediate by [BSh, Corollary 1.3] and the rational polyhedral property Theorem 6. (Cf. Step 5 in the proof of Theorem 7 below.) For simplicity, we can use here a $\mathbb{Q}$-factorialization of X and Proposition 3. □

Lemma 8. *Let c be a real number. Then every real number b has a sufficiently close approximation by $\lfloor (n+c)b \rfloor /n$ for every sufficiently large (positive) real number n:*

$$\lim_{n\to+\infty} \lfloor (n+c)b \rfloor /n = b.$$

Proof. Indeed,

$$|(n+c)b - \lfloor (n+c)b \rfloor| < 1.$$

Hence

$$\lim_{n\to+\infty} \lfloor (n+c)b \rfloor /n = \lim_{n\to+\infty} (n+c)b/n = b.$$

□

Proof of Theorem 1. The easy part of the theorem is about the existence of $\mathbb{R}$-complements when n-complements $(X/Z \ni o, B^+)$ exist. This part is immediate by Theorem 6 and approximation of b by $\lfloor (n+1)b \rfloor /n$ in Lemma 8 with $c = 1, n \in \mathbb{Z}$. Notice also that by Corollary 3 $(X/Z \ni o, B^+)$ is an $\mathbb{R}$- and monotonic n-complement of $(X/Z, B_{[n]})$ (cf. $x_{[n]}$ in 6.13).

Conversely, the existence of n-complements, when an $\mathbb{R}$-complement exists, is immediate by Theorem 2.

□

5 Exceptional complements: explicit construction

Exceptional pairs. They can be defined in terms of $\mathbb{R}$-complements. A pair (X, D) is called *exceptional* if it has an $\mathbb{R}$-complement and every its $\mathbb{R}$-complement is klt.

The same definition works [for pairs $(X, \mathbb{D})$ with an arbitrary b-divisor $\mathbb{D}$, in particular,] for bd-pairs $(X, D + \mathcal{P})$ (of index m).

Note that exceptional pairs can be defined for local pairs $(X/Z \ni o, D)$ but, under the klt assumption, the only exceptional pairs are global. (For local exceptional pairs we should allow *exceptional* nonklt singularities; see [Sh92, §7].)

Exceptionality is related to boundedness.

Corollary 11. *Let d be a nonnegative integer and $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0, 1]$. The exceptional pairs (X, B) with Ft X of dimension d and with $B \in \Phi$ are bounded.*

The boundedness of pairs includes the boundedness of B with multiplicities.

Addendum 26. *The same holds for bd-pairs $(X, B + \mathcal{P})$ of [fixed] index m. In this case the boundedness includes additionally the boundedness of $\mathcal{P}_X$ modulo $\sim_m$.*

Q. Is the b-divisor $\mathcal{P}$ itself bounded modulo $\sim_m$ in higher dimensions? It is true for surfaces because $C'^2 \le C^2 - 1$ for the birational transform C' of a curve C on a surface under the monoidal transformation in a point of C [IShf, 6.1, (8)].

Proof. Follows from two fundamental results of Birkar: n-complements with hyperstandard multiplicities and BBAB [B, Theorem 1.8] [B16, Theorem 1.1].

Can be reduced to the crucial case: exceptional klt $(X, 0)$ [B, Theorem 1.3]. In this case, it is enough to establish the nonvanishing: there exists a positive integer n, depending only on the dimension $d = \dim X$ such that $h^0(X, -nK) \ne 0$. After that we can use the boundedness of Hacon-Xu [HX, Theorem 1.3].

The boundedness of boundary B means that of $\operatorname{Supp} B$ and the finiteness of multiplicities of B. This holds because B is effective and its multiplicities are hyperstandard. Indeed, $B = b_1D_1+\cdots+b_rD_r$ and every $b_i \ge 0$. Moreover, $b_i = 0$ or $b_i \ge c$, where

$$c = \min\{b \in \Phi \setminus 0\}$$

is a positive rational number because Φ is a dcc set. Since $-K-B$ is effective modulo $\sim_{\mathbb{R}}$, $\deg(-K-B) \ge 0$ and $\deg B \le -\deg K$, where the degree deg is taken with respect to a bounded polarization of X. Hence $\operatorname{Supp} B$ is bounded because $\deg K$ is bounded. Since (X, B) is exceptional, the set of possible b_i is finite by Corollary 4. Otherwise an n-complement (X, B), with some b_i close to 1, gives a nonklt $\mathbb{R}$-complement of (X, B) [B, Theorem 1.8], a contradiction.

Similarly for bd-pairs of index m with bounded X, $\mathcal{P}_X$ is also bounded:

$$\mathcal{P} \sim_m E_1 - E_2,$$

where E_1, E_2 are two bounded effective $\mathbb{Q}$-divisors. By construction $E_1, E_2 \in \mathbb{Z}/m$. The class of $\mathcal{P}_X$ modulo $\sim_{\mathbb{R}}$ is a bounded point in the closed convex rational polyhedral cone $\operatorname{Eff}_{\mathbb{R}}(X)$. (Moreover, the class belongs to the mobile cone $\operatorname{Mob}_{\mathbb{R}}(X)$.) Indeed, by definition $(X, B + \mathcal{P}_X)$ has an $\mathbb{R}$-complement. Thus $-(K + B + \mathcal{P}_X)$ is effective modulo $\sim_{\mathbb{R}}$ and $\deg(B + \mathcal{P}_X) \le -\deg K$. Since $\mathcal{P}$ is b-nef, $\mathcal{P}_X$ is effective (even mobile) modulo $\sim_{\mathbb{R}}$ [ShCh, Corollary 4.5]. This implies that the divisorial part B is bounded as above. This gives also the boundedness of the class of $\mathcal{P}_X$ with respect to the polarization. The class of $m\mathcal{P}_X$ is integral. Thus $m\mathcal{P}$ modulo $\sim_{\mathbb{R}}$ is $E_1' - E_2'$, where E_1', E_2' are two bounded Weil divisors on X. Here we use Addenda 12, 19 and 22. In particular, for bounded family of (w)Ft varieties, sheaves $\mathbf{Cl}, \mathbf{Eff}, \mathbf{Mob}, \mathbf{Tor}$ are constant and modulo $\mathbf{Tor}$ are subsheaves of $\mathbf{Cl}_{\mathbb{R}}$. The classes of $K, B, \mathcal{P}_X, E_1', E_2'$ and of the polarization in $\operatorname{Cl}_{\mathbb{R}} X$ are restrictions on X of corresponding sections of $\mathbf{Cl}_{\mathbb{R}}$. Moreover, $\sim_{\mathbb{R}}$ can be replaced by $\sim$ and torsions, that is, Weil divisors D such with $nD \sim 0$. The torsions are bounded on bounded wFt varieties by Addendum 12 that gives required presentation with $E_1 = E_1'/m, E_2 = E_2'/m$ modulo torsions.

$\square$

Lemma 9. *Let $\lambda\colon V \to W$ be a linear or just continuous map of finite dimensional $\mathbb{R}$-linear spaces with norms and e be a direction in V such that $\lambda(e) \neq 0$. Then, for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that if e' is another direction in V with $\|e' - e\| < \delta$ then*

(1) $\lambda(e') \neq 0$ *too; and*

(2)
$$\left\| \frac{\lambda(e')}{\|\lambda(e')\|} - \frac{\lambda(e)}{\|\lambda(e)\|} \right\| < \varepsilon.$$

In other words, close directions go to close ones.

Proof. The map λ is continuous. Hence $\lambda(e') \neq 0$ for every $e' \in W$ with $\|e' - e\| < \delta'$.

The map
$$\frac{\lambda(e')}{\|\lambda(e')\|} - \frac{\lambda(e)}{\|\lambda(e)\|}$$
is also continuous for $e' \in W$ with $\|e' - e\| < \delta'$. Moreover,
$$\lim_{e' \to e} \frac{\lambda(e')}{\|\lambda(e')\|} - \frac{\lambda(e)}{\|\lambda(e)\|} = 0.$$
This gives required $0 < \delta \leq \delta'$.

$\square$

Theorem 7 (Exceptional n-complements)**.** *Let d be a nonnegative integer, I, ε, v, e be the data as in Restrictions on complementary indices, and $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0,1]$. Then there exists a finite set $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi)$ of positive integers such that*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data.*

Existence of n-complement: *if (X, B) is a pair with wFt X,* $\dim X = d$*, with a boundary B, with an $\mathbb{R}$-complement and with exceptional (X, B_Φ) then (X, B) has an n-complement (X, B^+) for some $n \in \mathcal{N}$.*

Addendum 27. $B^+ \geq B_{n\_\Phi}{}^\sharp \geq B_{n\_\Phi}$.

Addendum 28. *nB^+ is Cartier, if it is $\mathbb{Q}$-Cartier and X has Ft.*

Q. Does the same hold for wFt varieties with bounded Ft models?

Addendum 29. *(X, B^+) is a monotonic n-complement of itself and of $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$, and is a monotonic b-$n$-complement of itself and of $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp), (X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$, if $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$ are log pairs respectively.*

Addendum 30. *The same holds for bd-pairs $(X, B+\mathcal{P})$ of index m with $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi, m)$. That is,*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data and $m|n$.*

Existence of n-complement: *if $(X, B+\mathcal{P})$ is a bd-pair of index m with wFt X, $\dim X = d$, with a boundary B, with an $\mathbb{R}$-complement and with exceptional $(X, B_\Phi+\mathcal{P})$ then $(X, B+\mathcal{P})$ has an n-complement $(X, B^+ + \mathcal{P})$ for some $n \in \mathcal{N}$.*

Addenda 27-28 hold literally. In Addenda 29 $(X, B^+ + \mathcal{P})$ is a monotonic n-complement of itself and of $(X, B_{n\_\Phi}+\mathcal{P}), (X, {B_{n\_\Phi}}^\sharp+\mathcal{P})$, and is a monotonic b-$n$-complement of itself and of $(X, B_{n\_\Phi}+\mathcal{P}), (X, {B_{n\_\Phi}}^\sharp+\mathcal{P}), (X^\sharp, {B_{n\_\Phi}}^\sharp{}_{X^\sharp} + \mathcal{P})$, if $(X, B_{n\_\Phi}+\mathcal{P}_X), (X, {B_{n\_\Phi}}^\sharp+\mathcal{P}_X)$ are log bd-pairs respectively.

Proof. Step 1. Construction of an appropriate family of pairs. We construct a bounded family of pairs with marked divisors such that every pair of the theorem will be in the family and the theorem holds for every (typical) pair (X, B) in the family if and only if it holds in any other pair of the connected component of the family with (X, B). The boundedness implies the finiteness of typical pairs which can be chosen. This reduces the theorem to the case where X is a fixed $\mathbb{Q}$-factorial Ft variety and B is a boundary such that

(1) $B \geq B'$, where $B' \in \Phi$ is also fixed, (X, B') is exceptional; and

(2) (X, B) has an $\mathbb{R}$-compliment.

Taking a $\mathbb{Q}$-factorialization and by Lemma 1 we can suppose that every X has Ft and is $\mathbb{Q}$-factorial. Indeed, we use for this a small birational modification and it preserves all required complements by Proposition 3. However, this works only for a projective factorialization for Addendum 28. So, by Corollary 11 the pairs (X, B_Φ) associated with pairs (X, B) of the theorem are bounded. It is enough to consider one connected family of pairs $(X_t, B_{\Phi,t}), t \in T$. Hence we can suppose that $B_{\Phi,t} = B'_t \in \Phi$ is fixed or constant in the following sense. There exists a positive integer r and rational numbers $b'_1, \ldots, b'_h \in \Phi$ such that every pair $(X_t, B_{\Phi,t})$ in the family has *marked* distinct prime divisors $D_{1,t}, \ldots, D_{h,t}$ and

$$B_{\Phi,t} = B'_t = b'_1 D_{1,t} + \cdots + b'_h D_{h,t} \in \mathfrak{D}_{\mathbb{R},t}.$$

(1) holds by our assumptions for every pair (X, B') in our family. We contend that construction of n-complements is [uniform][constant for an appropriate parametrization [family] (see Section 4 and cf. Proposition 12), that is, if an n-complement with required properties exists for one pair (X_t, B_t) then the same holds with same n for every other pair of the family with the same boundary. Possibly we need an appropriate reparametrization of Addendum 22.

By Proposition 10, Addendum 12 and Proposition 11 we can suppose that the collection [ordered set] of marked divisors $D_1, \ldots, D_h$ is sufficiently large and the parametrization is appropriate. This means that, for every pair (X_t, B'_t) in our family,

$$K_{X_t} \sim d_1 D_{1,t} + \cdots + d_h D_{h,t}$$

for some $d_1, \ldots, d_h \in \mathbb{Z}$ independent on t; so, we can suppose that a canonical divisor is constant:

$$K_{X_t} = d_1 D_{1,t} + \cdots + d_h D_{h,t} \in \mathfrak{D}_t [= \mathfrak{D}(D_{1,t}, \ldots, D_{h,t})];$$

and

every effective Weil divisor on X_t is linearly equivalent to an element of a constant monoid of effective Weil divisors $\mathfrak{D}_t^+ = \mathfrak{D}^+(D_{1,t}, \ldots, D_{h,t})$, the monoid of effective Weil divisors generated by $D_1, \ldots, D_h$.

Actually, we need slightly more: Proposition 12 and Addendum 19. We can assume this and that some other (standard) sheaves are also constant by Addendum 22.

The reduction to one (typical) pair will be done in Step 3 in the special case when B is supported on $D_1 + \cdots + D_h$, that is, $B \in \mathfrak{D}^+$. The general case will be treated in Step 7, that is, an n-complement of (X, B) under (1-2) with $n \in \mathcal{N}$ of Step 2 below can be constructed in terms of an n-complement of the special case. But before we reformulate the existence of n-complements in terms of linear systems. Note also the dependence of $\mathcal{N}$ on X instead of $d = \dim X$ in Step 2. Since we have finitely many typical (X, B') of dimension d, the total finite set $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi)$ is a finite union of $\mathcal{N}(X, B', I, \varepsilon, v, e, \Phi)$.

Step 2. (Reduction to linear complements.) Let (X, B') be pair as in Step 1, that is, under assumption (1). Suppose that there exists a finite set of positive integers $\mathcal{N} = \mathcal{N}(X, B', I, \varepsilon, v, e, \Phi)$ such that, every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices and, for every boundary B on X under (1-2), there exists $n \in \mathcal{N}$ such that

(3)
$$\lfloor (n+1)B \rfloor / n \geq B_{n\_\Phi};$$

and

(4)
$$-nK - \lfloor (n+1)B \rfloor$$

is linear equivalent to an effective Weil divisor E (cf. $\overline{D}$ after [Sh92, Definition 5.1]).

Then (X, B) has an n-complement. The same holds for the bd-pairs (X, B) of index $m|n$ with (possibly different) $\mathcal{N} = \mathcal{N}(X, B' + \mathcal{P}_X, I, \varepsilon, v, e, \Phi, m)$.

Indeed, (X, B) has an n-complement (X, B^+) with required properties, including
$$B^+ \geq B_{n\_\Phi}. \tag{5.0.5}$$
Take
$$B^+ = \lfloor (n+1)B \rfloor / n + E/n.$$
We verify only that (X, B^+) is an n-compliment of (X, B). The addenda will be explained later in Step 8. By construction and (4)
$$n(K + B^+) = nK + \lfloor (n+1)B \rfloor + E \sim 0.$$
This implies (3) of Definition 2. By (3) and Proposition 6, since $B^+ \geq B_{n\_\Phi} \geq B_\Phi \geq B'$ and (X, B') is exceptional by (1), (X, B^+) is lc, moreover, klt. This implies (2) and (1) of Definition 2 by Corollary 5.

Step 3. Reduction to one pair in the special case. Suppose that $(X, B) = (X_t, B_t)$ for some $t \in T$. The special case means that $B \in \mathfrak{D}^+$ and $B_t \in \mathfrak{D}^+_{\mathbb{R},t}$.

The pair (X_t, B'_t) satisfies (1). By construction every other pair (X_s, B'_s) also satisfies (1). We need to verify the following. Let $B_s \in \mathfrak{D}^+_{\mathbb{R},s}$ be a boundary under assumptions (1-2), then (X_s, B_s) satisfies (3-4) for some $n \in \mathcal{N}$ of Step 2. In particular, by Step 2 (X_s, B_s) has also a required n-complement.

By our assumptions and construction $B_s = b_1 D_{1,s} + \cdots + b_h D_{h,s}$ satisfies (1-2). Hence $B = B_t = b_1 D_{1,t} + \cdots + b_h D_{h,t}$ also satisfies (1-2). Indeed, we can also treat divisors B, B', K and D_i as constant ones $B_T, B'_T, K_{X/T}, D_{i,T}$ in $\mathfrak{D}_{\mathbb{R},T}$ as in Addendum 12. Then (1) means that $B_s \geq B'_s$ and implies

$$B_T \geq B'_T \text{ and } B_t = b_1 D_{1,t} + \cdots + b_h D_{h,t} \geq B'_t = b'_1 D_{1,t} + \cdots + b'_h D_{h,t}.$$

However, (2) is more advanced and requires the constant cone $\mathfrak{C}_{\mathbb{R}}$, the cone of divisors $D \in \mathfrak{D}_{\mathbb{R}}$ with $\mathbb{R}$-complements (see Proposition 12). Since it is constant, (2) for B_s implies that of for B_t. Actually, in the exceptional case under (1) it is enough the effective property: $\mathfrak{C}_{\mathbb{R},t} \cap \{B_t \geq B'_t\} = (-K_{X_t} - \mathfrak{E}_{\mathbb{R},t}) \cap \{B_t \geq B'_t\}$ means (2) under (1).

Now according to our assumptions, for (X_t, B_t), there exists $n \in \mathcal{N}$ such that B_t satisfies (3-4). Hence B_s also satisfies (3-4). Indeed, since $B_t, B_{n\_\Phi,t} \in \mathfrak{D}_{\mathbb{R},t}$, (3) is independent on t. Similarly, (4) independent on t by Addendum 19 because $K_{X_t}, \lfloor (n+1)B_t \rfloor \in \mathfrak{D}_t$ and $-nK_{X_t} - \lfloor (n+1)B_t \rfloor \in \mathfrak{E}_t$. Actually, we can choose $E_t \in \mathfrak{D}_t^+$ and $B_t^+ \in \mathfrak{D}_{\mathbb{Q},t}^+$.

Note that the constructed complement under (3-4) in Step 2 satisfies $B_s^+ \geq B_{s,n\_\Phi}$.

Step 4. Preparatory for the special case. Below we consider fixed $(X, B') = (X_t, B'_t)$.

By Lemma 7 there exists a positive integer J such that,

(5) for every positive integer q, if $B_q \in \mathfrak{D}_{\mathbb{R}}$ under (1-2) and such that $qB_q \in \mathbb{Z}$, then

$$-JqK - JqB_q$$

is linear equivalent to an effective Weil divisor E.

Actually, B_q is a boundary because (X, B') is exceptional. By Step 1, we can suppose that $K \in \mathfrak{D}$, integral. So, (1-2) mean that

$$B_q \in \mathfrak{C}_{\mathbb{R}} \cap \{B_q \geq B'\},$$

a compact rational polyhedron in $\mathfrak{C}_{\mathbb{R}}$ [ShCh, Corollary 4.5]. Recall that $\mathfrak{C}_{\mathbb{R}} \cap \{B \geq B'\} = (-K - \mathfrak{E}_{\mathbb{R}}) \cap \{B \geq B'\}$. We apply Lemma 7 to $\mathfrak{E}_{\mathbb{R}} \cap \{D = -K - B_q \leq -K - B'\}$. Since by Addendum 19 the polyhedral is constant over T, there exists J independent on $t \in T$ by Addendum 23.

Consider divisors $B \in \mathfrak{C}_{\mathbb{R}} \cap \{B \geq B'\}$. The multiplicities

$$b_{n\_\Phi} = \operatorname{mult}_P B_{n\_\Phi},$$

where P is a prime divisor on X and n is a positive integer, are not accumulated to 1 for all i. Indeed, since (X,B') is exceptional, B is a (klt) boundary. On the other hand, by definition

$$b_{n\_\Phi}=1-\frac{r}{l}+\frac{m}{l(n+1)},$$

where $r\in\mathfrak{R}$, m is a nonnegative integer and l,n are positive integers. We can always find a boundary $B''\in\Phi$ between B_{n_Φ} and B': $B\geq B_{n\_\Phi}\geq B''\geq B'$. For every prime divisor P on X,

$$b''=\begin{cases}1-\frac{r}{l'}, \text{ if } & b_{n\_\Phi}\geq 1-\frac{1}{l'}\geq b'=\operatorname{mult}_P B' \text{ for some positive integer } l',\\ b' & \text{otherwise},\end{cases}$$

where $b''=\operatorname{mult}_P b''$. By the klt property of B, $r>0$. By construction (X,B'') is exceptional. Hence l' and l are bounded by Corollary 11 and b_{n_Φ} are not accumulated to 1. So, the set $\mathfrak{R}'$ of rational numbers

$$r'=1-\frac{r}{l}$$

for all multiplicities b_{n_Φ} is finite and independent of n. However, n and m can be large.

Step 5. (Linear complements: the special case.) To find a required set $\mathcal{N}=\mathcal{N}(X,B',I,\varepsilon,v,e,\Phi)$ of Step 2 we use Diophantine approximations in the $\mathbb{R}$-linear space $\mathfrak{D}_\mathbb{R}=\mathfrak{D}_\mathbb{R}(D_1,\dots,D_h)$. The special case means that we consider only boundaries $B\in\mathfrak{D}_\mathbb{R}$. For those boundaries $B_\Phi,B_{n\_\Phi}\in\mathfrak{D}_\mathbb{R}$ too. For every such B under (1-2), we find $n\in\mathcal{N}$ such that (3-4) holds.

Fix for this a sufficiently divisible positive integer N: $I|N, N\mathfrak{R}'\subset\mathbb{Z}$ and $J|N$, where J of Step 4. We also suppose that $N\geq 2$. Take a positive real number δ such that $\delta<1/4N$, $\delta<\varepsilon/N$, the δ-neighborhood of B in $\langle B\rangle$ lies in $\mathfrak{C}_\mathbb{R}\cap\{B\geq B'\}$ and δ satisfies Lemma 9 for the projection $\operatorname{pr}\colon\mathfrak{D}_\mathbb{R}\times\mathbb{R}^l\to\mathbb{R}^l$ and a direction e' in $\langle(B,v)\rangle$ which goes to

$$e=\frac{\operatorname{pr}(e')}{\|\operatorname{pr}(e')\|}.$$

We assume also that

$$\delta\leq\frac{\min\{b_i,1-b_j\mid i,j=1,\dots,l, \text{ and } b_i>0\}}{3N}.$$

[Warning: the minimum for all $1-b_j$ but not for all b_i.] The minimum is positive because every $b_j<1$ by the klt property of $\mathbb{R}$-complements.

Such δ exists because $\mathfrak{C}_{\mathbb{R}} \cap \{B \geq B'\}$ is a compact rational polyhedron and the minimum is positive. Note also that e' exists because the restriction of projection

$$\langle (B,v)\rangle \to \langle v\rangle$$

is surjective. Then by [BSh, Corollary 1.3] there exists a positive integer q and a pair of vectors $B_q \in \langle B\rangle, v_q \in \langle v\rangle$ such that

(6) $qB_q \in \mathbb{Z}, qv_q \in \mathbb{Z}^l$;

(7) $\|B_q - B\|, \|v_q - v\| < \delta/q$; and

(8)
$$\left\| \frac{(B_q,v_q)-(B,v)}{\|(B_q,v_q)-(B,v)\|} - e' \right\| < \delta.$$

The approximation (B_q,v_q) applies to the vector (B,v) in $\langle (B,v)\rangle \subseteq \mathfrak{D}_{\mathbb{R}} \times \mathbb{R}^l$ with the direction e' in $\langle (B,v)\rangle$.

Put $n = Nq$ and $v_n = v_q$. Then v_n satisfies required properties of Restrictions on complementary indices: Divisibility by the divisibility of N, Denominators by (6), Approximation by (7) and Anisotropic approximation by (8) and Lemma 9 respectively. (The lemma is applied to the direction e' and another direction

$$\frac{(B_q,v_q)-(B,v)}{\|(B_q,v_q)-(B,v)\|}$$

in $\mathfrak{D}_{\mathbb{R}} \times \mathbb{R}^l$.) To avoid different v_n for the same n, it would be better to suppose that $\varepsilon < 1/2$. Instead, we assumed above that $\delta < 1/4N \leq 1/4$. Then v_n independent of B and (X,B') too.

Now we verify that, for every boundary $B'' \in \mathfrak{D}_{\mathbb{R}}$, in the δ/q-neighbourhood of B in $\mathfrak{C}_{\mathbb{R}} \cap \{B \geq B'\}$, n satisfies the properties (3-4). In other words, the properties holds for $B'' \in \mathfrak{D}_{\mathbb{R}}$ if $\|B''-B\| < \delta/q$, $B'' \geq B'$ and (X,B'') has an $\mathbb{R}$-complement. Since $\mathfrak{C}_{\mathbb{R}} \cap \{B \geq B'\}$ is compact it has a finite cover by those neighbourhoods and $\mathcal{N}$ is the finite set of numbers n for the neighbourhoods of such a covering.

However we start from the property

(9)
$$\lfloor (n+1)B'' \rfloor / n = B_q.$$

Equivalently, for every $i = 1, \ldots, h$,

$$\lfloor (n+1)b''_i \rfloor /n = b_{i,q},\ b''_i = \mathrm{mult}_{D_i} B'' \text{ and } b_{i,q} = \mathrm{mult}_{D_i} B_q.$$

By construction and (6), $b_{i,q} \in \mathbb{Z}/q \subseteq \mathbb{Z}/n$. By construction and (7)

$$\|b''_i - b_{i,q}\| \le \|b''_i - b_i\| + \|b_i - b_{i,q}\| < \frac{2\delta}{q}, b_i = \mathrm{mult}_{D_i} B. \tag{5.0.6}$$

Case $b_i = 0$. By construction and (7),

$$\|b_{i,q}\| = \|b_{i,q} - b_i\| < \frac{\delta}{q} = \frac{N\delta}{n} < \frac{1}{4n} < \frac{1}{q}.$$

Hence by (6) $b_{i,q} = 0$. By construction $b''_i \ge 0$. The inequality (5.0.6) implies

$$\|b''_i\| = \|b''_i - b_{i,q}\| < \frac{2\delta}{q} < \frac{2N\delta}{n} < \frac{1}{2n} \le \frac{1}{n+1}.$$

Thus

$$\lfloor (n+1)b''_i \rfloor /n = 0 = b_{i,q}.$$

Case $b_i > 0$. By Lemma 4 it is enough to verify the inequality

$$\|b''_i - b_{i,q}\| < \frac{\min\{b''_i, 1 - b''_i\}}{n}.$$

By construction, our assumptions and the positivity of b_i

$$b''_i > b_i - \frac{\delta}{q} \ge \min\{b_i, 1 - b_j \mid i,j = 1, \ldots, l, \text{ and } b_i > 0\} - \frac{\delta}{q} \ge 3N\delta - \frac{\delta}{q} \ge 2N\delta.$$

Similarly,

$$1 - b''_i > 1 - b_i - \frac{\delta}{q} \ge 3N\delta - \frac{\delta}{q} \ge 2N\delta.$$

Recall, that $1 - b_i > 0$ because (X, B) is klt. Both inequalities gives

$$\min\{b''_i, 1 - b''_i\} > 2N\delta.$$

Hence by the inequality (5.0.6)

$$\|b''_i - b_{i,q}\| < \frac{2\delta}{q} = \frac{2N\delta}{n} < \frac{\min\{b''_i, 1 - b''_i\}}{n}.$$

Property (3): By (9), (3) means

$$B_q \geq B''_{n\_\Phi}$$

or, equivalently, for every $i = 1, \ldots, l$,

$$b_{i,q} \geq b''_{i,n\_\Phi}.$$

By definition

$$b''_i \geq b''_{i,n\_\Phi}.$$

So, it is enough to disprove the case

$$b''_i \geq b''_{i,n\_\Phi} > b_{i,q}.$$

By the inequality (5.0.6) this implies the inequalities

$$0 < b''_{i,n\_\Phi} - b_{i,q} < \frac{2\delta}{q}.$$

The low estimation (5.0.8) of the difference below gives a contradiction. For this we use the following form of numbers under consideration:

$$b''_{i,n\_\Phi} = r + \frac{m_i}{l_i(n+1)} \text{ and } b_{i,q} = r + \frac{m}{n},$$

where $r = 1 - r_i/l_i \in \mathfrak{R}'$ [(not necessarily positive for general $\mathfrak{R}$ [PSh08, 3.2])], l_i is a positive integer, m_i is a nonnegative integer and m is an integer. Such an integer m exists because by construction $r \in \mathfrak{R}' \subset \mathbb{Z}/n$ and $b_{i,q} \in \mathbb{Z}/n$ (see the choice of N and (9) above). The inequality $b''_{i,n\_\Phi} > b_{i,q}$ implies that

$$\frac{m_i}{l_i(n+1)} > \frac{m}{n}.$$

Hence $l_i m < m_i$. Since $l_i m$ is an integer,

$$l_i m \leq m_i - 1 \text{ and } b_{i,q} = r + \frac{m}{n} = r + \frac{l_i m}{l_i n} \leq r + \frac{m_i - 1}{l_i n}.$$

Note also that the assumption $\mathfrak{R} \subset [0,1]$ implies that $r_i \leq 1$ and

$$r = 1 - \frac{r_i}{l_i} \geq 1 - \frac{1}{l_i}.$$

On the other hand, $b''_{i,n\_\Phi} \leq b''_i < 1$. So,

$$\frac{m_i}{l_i(n+1)} = b''_{i,n\_\Phi} - r < 1 - 1 + \frac{1}{l_i} = \frac{1}{l_i}, \quad \frac{m_i}{n+1} < 1$$

and $m_i \leq n$ because m_i is integer. Hence

$$\frac{m_i - 1}{n} < \frac{m_i}{n+1} \text{ and } b_{i,q} \leq r + \frac{m_i - 1}{l_i n} < r + \frac{m_i}{l_i(n+1)} = b''_{i,n\_\Phi}.$$

Thus by the inequality (5.0.6)

$$\frac{m_i}{l_i(n+1)} - \frac{m_i - 1}{l_i n} = r + \frac{m_i}{l_i(n+1)} - r - \frac{m_i - 1}{l_i n} < \frac{2\delta}{q}.$$

That is,

$$\frac{1}{n}(\frac{1}{l_i} - \frac{m_i}{l_i(n+1)}) = \frac{m_i}{l_i(n+1)} - \frac{m_i - 1}{l_i n} < \frac{2\delta}{q} = \frac{2N\delta}{n}.$$

Hence

$$\frac{1}{l_i} - \frac{m_i}{l_i(n+1)} < 2N\delta. \tag{5.0.7}$$

On the other hand, by construction and the inequality (5.0.6)

$$1 - \frac{r_i}{l_i} + \frac{m_i}{l_i(n+1)} = r + \frac{m_i}{l_i(n+1)} = b''_{i,n\_\Phi} \leq b''_i < b_i + \frac{2\delta}{q}.$$

Since $r_i \leq 1$ and $1 - b_i > 0$, $1 - b_i \geq 3N\delta$, the last inequality implies

$$\begin{aligned}\frac{1}{l_i} - \frac{m_i}{l_i(n+1)} &= 1 - \frac{r_i}{l_i} + \frac{1}{l_i} - 1 + \frac{r_i}{l_i} - \frac{m_i}{l_i(n+1)} = 1 - \frac{r_i}{l_i} + \frac{1}{l_i} - b''_{i,n\_\Phi} \\ &> 1 - \frac{r_i}{l_i} + \frac{1}{l_i} - b_i - \frac{2\delta}{q} \geq 1 - b_i - \frac{2\delta}{q} > 3N\delta - \frac{2\delta}{q}. \end{aligned} \tag{5.0.8}$$

This gives by the inequality (5.0.7)

$$3N\delta - \frac{2\delta}{q} < 2N\delta.$$

Equivalently,

$$Nq < 2$$

a contradiction because $N \geq 2, q \geq 1$.

Property (4) for B'' is immediate by (5). Indeed, B_q satisfies (1-2) by (7) and our assumptions, $qB_q \in \mathbb{Z}$ by (6), and by (9) and assumptions

$$-nK - \lfloor (n+1)B'' \rfloor = -nK - nB_q = \frac{N}{J}(-JqK - JqB_q)$$

is linear equivalent to an effective Weil divisor.

Finally, in this step it is not necessarily to suppose that X is irreducible. It is enough a bound on the number of irreducible components of X. In particular, we can take finitely many B in $\mathfrak{C}_{\mathbb{R}} \cap \{B \geq B'\}$ if the number of boundaries is bounded. Indeed, constructions of $\mathcal{N}$ and of n-complements are combinatorial: we work with the lattice $\mathfrak{D}$ and approximations.

Step 6. Preparatory for the general case. There exists a homomorphism

$$\mu\colon\ \mathrm{WDiv}\, X \to \mathfrak{D}$$

of the free Abelian group $\mathrm{WDiv}\, X$ of Weil divisors to the lattice $\mathfrak{D}$. For every prime Weil divisor P on X, put $\mu(P) = E$, where $E \in \mathfrak{D}$ such that $P \sim E$. Since P is effective we can suppose that $E \in \mathfrak{D}^+$, that is, also effective. Moreover, for the generators D_i, we take $\mu(D_i) = D_i$, that, is μ is a projection on $\mathfrak{D}$. That is, for every divisor D in $\mathfrak{D}$, $\mu(D) = D$. In particular, $\mu(K) = K$. By definition and construction, for every $D \in \mathrm{WDiv}\, X$, $\mu(D) \sim D$.

Since μ is a homomorphisms of groups, for every two divisors D, D' on X and every integer n, $\mu(D + D') = \mu(D) + \mu(D')$ and $\mu(nD) = n\mu(D)$.

Below we always assume that, for every prime Weil divisor P on X, $\mu(P)$ is effective. So, we obtain a homomorphism

$$\mu\colon\ \mathrm{EffWDiv}\, X \to \mathfrak{D}^+$$

of the free Abelian monoids [$\mathrm{EffWDiv}\, X$ effective Weil divisors on X].

For an $\mathbb{R}$-divisor $D = \sum d_i Q_i$, where Q_i are distinct prime Weil divisors, put

$$\mu(D) = \sum d_i \mu(Q_i).$$

This gives a natural $\mathbb{R}$-linear extension

$$\mu\colon\ \mathrm{WDiv}_{\mathbb{R}}\, X \to \mathfrak{D}_{\mathbb{R}} = \mathfrak{D} \otimes \mathbb{R}$$

of the above homomorphism of free groups. In particular, for every $r \in \mathbb{R}$ and every $D \in \mathrm{WDiv}_{\mathbb{R}}\, X$, $\mu(rD) = r\mu(D)$ holds. For every $\mathbb{R}$-divisor D in

$\mathfrak{D}_{\mathbb{R}}$, $\mu(D) = D$. In particular, $\mu(B') = B'$. Note also that, in the important for us situation of Step 7, $B_\Phi = B'$ and $\mu(B_\Phi) = B_\Phi$ but $\mu(B)_\Phi \neq \mu(B_\Phi)$ is possible if B is not supported in $D_1 + \cdots + D_h$.

Taking nonnegative real numbers d_i we obtain a homomorphism of monoids

$$\mu\colon \operatorname{EffWDiv}_{\mathbb{R}} X \to \mathfrak{D}_{\mathbb{R}}^+,$$

[where $\operatorname{EffWDiv}_{\mathbb{R}} X$ is the cone of effective $\mathbb{R}$-divisors and $\mathfrak{D}_{\mathbb{R}}^+$ is the cone of effective $\mathbb{R}$-divisors supported on $D_1+\cdots+D_h$]. It is an $\mathbb{R}^+$-homomorphism, that is, for every $r \in \mathbb{R}^+ = [0,+\infty)$ and every $D \in \operatorname{EffWDiv}_{\mathbb{R}} X$, $rD \in \operatorname{EffWDiv}_{\mathbb{R}} X$ and $\mu(rD) = r\mu(D)$ holds. Note also the following monotonicity

(10) if $D, D' \in \operatorname{WDiv}_{\mathbb{R}} X$ and $D \geq D'$, then $\mu(D) \geq \mu(D')$.

However, μ is not unique and not canonical.

We use in Step 7 the following monotonicity: for any $\mathbb{R}$-divisor D

$$\lfloor (n+1)\mu(D) \rfloor \geq \mu(\lfloor (n+1)D \rfloor). \tag{5.0.9}$$

Both sides are supported in $D_1 + \cdots + D_h$. So, it is enough to verify the inequality for the multiplicities in every D_i. Let $D = \sum d_j Q_j$, where Q_j are distinct prime Weil divisors and

$$\mu(Q_j) = \sum_{i=1}^h n_{i,j} D_i,$$

where $n_{i,j}$ are nonnegative integers. Then

$$\mu(D) = \sum d_j \mu(Q_j) = \sum_{i=1}^h (\sum n_{i,j} d_j) D_i \text{ and } \operatorname{mult}_{D_i} \mu(D) = \sum n_{i,j} d_j.$$

Respectively,

$$\operatorname{mult}_{Q_j} \lfloor (n+1)D \rfloor = \lfloor (n+1) \operatorname{mult}_{Q_j} D \rfloor = \lfloor (n+1) d_j \rfloor,$$

$$\mu(\lfloor (n+1)D \rfloor) = \sum_{i=1}^h (\sum n_{i,j} \lfloor (n+1) d_j \rfloor) D_i$$

and

$$\operatorname{mult}_{D_i} \mu(\lfloor (n+1)D \rfloor = \sum n_{i,j} \lfloor (n+1) d_j \rfloor.$$

Hence by inequalities (2.2.3-2.2.4), for nonnegative integers n_i and real d_i,

$$\left\lfloor (n+1)(\sum n_i d_i) \right\rfloor \geq \sum n_i \lfloor (n+1)d_i \rfloor$$

and

$$\operatorname{mult}_{D_i} \lfloor (n+1)\mu(D) \rfloor = \lfloor (n+1) \operatorname{mult}_{D_i} \mu(D) \rfloor = \left\lfloor (n+1)(\sum n_{i,j} d_j) \right\rfloor \geq$$

$$\sum n_{i,j} \lfloor (n+1)d_j \rfloor = \operatorname{mult}_{D_i} \mu(\lfloor (n+1)D \rfloor).$$

By definition and construction, for every $D \in \operatorname{WDiv}_{\mathbb{R}} X$, $\mu(D) \sim_{\mathbb{R}} D$. In particular, if (X, D^+) is an $\mathbb{R}$-complement of (X, D) then

$$\mu(D^+) \sim_{\mathbb{R}} D^+ \sim_{\mathbb{R}} -K.$$

Moreover, $\mu(D), \mu(D^+) \in \mathfrak{D}_{\mathbb{R}}$. By the monotonicity (10), $\mu(D^+) \geq \mu(D)$. Hence $(X, \mu(D^+))$ is an $\mathbb{R}$-complement of $(X, \mu(D))$ if $(X, \mu(D^+))$ is lc. Indeed,

$$K + \mu(D^+) = \mu(K) + \mu(D^+) = \mu(K + D^+) \sim_{\mathbb{R}} K + D^+ \sim_{\mathbb{R}} 0.$$

If additionally $D = B$ is a boundary then $B^+ = D^+$ is a boundary and, under the lc property of $(X, \mu(B)), (X, \mu(B^+))$, $\mu(B), \mu(B^+)$ are boundaries too.

In conclusion to this step, if (X, B) has an $\mathbb{R}$-complement (X, B^+) and $B \geq B'$, then $(X, \mu(B^+))$ is lc and is an $\mathbb{R}$-complement of $(X, \mu(B))$. Indeed, $B^+ \geq B'$ and $\mu(B^+) \geq \mu(B') = B'$ because $B' \in \mathfrak{D}_{\mathbb{R}}$. By the exceptional property of (X, B'), $(X, \mu(B^+))$ is klt.

Step 7. (Linear complements: the general case.) We consider (X, B) under (1-2). Moreover, we can suppose in this case that $B_\Phi = B'$. Put

$$B'' = \sum_{i=1}^{h} b_i D_i, \text{ every } b_i = \operatorname{mult}_{D_i} B.$$

Note that $D_1, \ldots, D_h$ include all prime divisors P on X with

$$b_{P,\Phi} = \operatorname{mult}_P(B_\Phi) = (\operatorname{mult}_P b)_\Phi > 0.$$

Indeed, $B_\Phi \in \mathfrak{D}_{\mathbb{R}}$.

By Step 6 and construction both divisors $\mu(B), B'' \in \mathfrak{D}_{\mathbb{R}}$ and satisfy (1-2). Thus by Step 5 we have $n \in \mathcal{N}$ which satisfies required properties (3-4) simultaneously for $\mu(B)$ and B''. We verify now that (3-4) holds also for B and the same n.

Property (3): (Here we need n only for B''.) By definition b_{P,n_Φ} is the largest number

$$1 - \frac{r}{l} + \frac{m}{l(n+1)} \leq b_P = \operatorname{mult}_P B,$$

where P is a prime Weil divisor of X, $r \in \mathfrak{R}$, l is a positive integer and m is a nonnegative integer.

Case $1 - r/l > 0$. Then, for some $i = 1, \dots, l$, $P = D_i$ and $b_P = b_i = \operatorname{mult}_{D_i} B''$. Hence (3) holds for this prime Weil divisor P.

Case $1 - r/l = 0$. Then $l = 1, r = 1$ and $b_{P,n\_\Phi} = m/(n+1)$. By definition

$$b_P \geq \frac{m}{n+1}.$$

So,

$$\lfloor (n+1) b_P \rfloor / n \geq \left\lfloor (n+1) \frac{m}{n+1} \right\rfloor / n = \frac{m}{n} > \frac{m}{n+1} = b_{P,n\_\Phi}.$$

Property (4): (Here we need n only for $\mu(B)$.) Let E be an effective Weil divisor such that

$$E \sim -nK - \lfloor (n+1)\mu(B) \rfloor.$$

On the other hand, by (5.0.9)

$$E' = \lfloor (n+1)\mu(B) \rfloor - \mu(\lfloor (n+1)B \rfloor)$$

is an effective Weil divisor. Thus by Step 6

$$-nK - \lfloor (n+1)B \rfloor \sim -nK - \mu(\lfloor (n+1)B \rfloor) =$$

$$-nK - \lfloor (n+1)\mu(B) \rfloor + \lfloor (n+1)\mu(B) \rfloor - \mu(\lfloor (n+1)B \rfloor) =$$

$$-nK - \lfloor (n+1)\mu(B) \rfloor + E' \sim E + E'.$$

By construction $E + E'$ is also effective.

Step 8. (Addenda.) By Step 2, $B^+ \geq B_{n\_\Phi}$. So, Proposition 8 implies Addendum 27.

Addendum 28 follows from Definition 2, (3) and from the boundedness of *canonical index*, that is, the index of canonical divisors K_{X_t} for a bounded family X/T of Ft varieties. Note that X is $\mathbb{Q}$-Gorenstein if nB^+ is $\mathbb{Q}$-Cartier. The index of K on X gives the bound on the index of all $K_{X_t}, t \in T$. If N of Step 5 will be divisible by the last index then n will be sufficiently divisible and satisfies the addendum.

Addendum 29 is standard. For n-complements, it follows from Addendum 27 and from Definition 2 for $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$. For b-$n$-complements, it follows from Example 6, (1) and from Proposition 1 for $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$, if respectively they are log pairs. Note also that $(X/Z \ni o, B^+)$ is a log pair by definition.

For $(X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$, we use Addendum 27 and Proposition 8. The pair $(X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ is a log one by definition. The statement is meaningful because the n-complement (X, B^+) induces an n-complement for a special maximal model $(X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ with a small transformation $X \dashrightarrow X^\sharp$ (cf. Proposition 3) and then for any other its crepant model by Remark 4, (1).

In this step b-n-complements of $(X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ is our final destination. They give all other complements in Addendum 29 and an n-complements of (X, B) as well. However, in many other situation we first construct a b-n-complement for $(X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$. The former is actually induced from this root (exceptional) complement.

Addendum 30 about bd-pairs $(X, B + \mathcal{P})$ can be treated as the case of usual pairs with minor improvement as follows. First, (1-2) should be replaced by

(1-bd) $B \geq B'$, where $B' \in \Phi$ is also fixed and $(X, B' + \mathcal{P})$ is exceptional with fixed $\mathcal{P}_X$; and

(2-bd) $(X, B + \mathcal{P})$ has an $\mathbb{R}$-compliment.

Note for this that by Corollary 11 and Addendum 26 we can suppose that $\mathcal{P}_X$ is fixed too and corresponds to a constant divisor $\mathcal{P}_{X,t}, t \in T$. (We do not suppose the existence of entire constant b-divisor $\mathcal{P}_t$.)

We can construct n-complements using also linear complements with same (3-bd)=(3) and modified

(4-bd) $m|n$ and
$$-nK - n\mathcal{P}_X - \lfloor (n+1)B \rfloor$$
is linear equivalent to an effective Weil divisor E.

In this situation we modify the effective property: $\mathfrak{C}_{\mathbb{R},t}\mathcal{P}_t \cap \{B_t \geq B'_t\} = (-K_{X_t} - \mathcal{P}_{X,t} - \mathfrak{C}_{\mathbb{R},t}) \cap \{B_t \geq B'_t\}$ means (2-bd) under (1-bd). This is a constant compact rational polyhedron and we can use Lemma 7 again. There exists a positive integer J such that, $m|J$ and

(5-bd) for every positive integer q, if $B_q \in \mathfrak{D}_{\mathbb{R}}$ under (1-2-bd) and such that $qB_q \in \mathbb{Z}$, then
$$-JqK - Jq\mathcal{P}_X - JqB_q$$
is linear equivalent to an effective Weil divisor E.

Since $m|J$, $Jq\mathcal{P}_X$ is integral and well-defined modulo ~ 0. The rest of proof for usual pairs is related to boundary computations and works literally.

Addendum 28 for bd-pairs follows from Definition 5, (3) and from the boundedness of *bd-semicanonical index*, that is, the index of bd-semicanonical divisors $K_{X_t} + \mathcal{P}_{X,t}$ for a bounded family X/T of Ft varieties.

□

6 Adjunction correspondence

Let r be a real number and l be a positive integer. The *adjunction correspondence* with *parameters* r, l is the following transformations of two real numbers b, d
$$d = 1 - \frac{r}{l} + \frac{b}{l} \quad \text{(direct)} \tag{6.0.10}$$
and
$$b = r - l + ld \quad \text{(inverse)}. \tag{6.0.11}$$
The correspondence is 1-to-1. How the correspondence is related to the divisorial part of adjunction for morphism see Section 7. In this section we establish the basic properties of this correspondence (almost) without any relation adjunction.

6.1 Linearity

Both transformations (6.0.10) and (6.0.11) are linear respectively for variables b and d.

Proof. Immediate by (6.0.10) and (6.0.11).

□

6.2 Monotonicity

Both transformations (6.0.10) and (6.0.11) are monotonically strictly increasing: for corresponding b_1, d_1 and b_2, d_2,

$$b_1 > b_2 \Leftrightarrow d_1 > d_2.$$

Proof. Immediate by (6.0.10) and (6.0.11) and positivity of l.

□

6.3 Rationality

Both transformations (6.0.10) and (6.0.11) are rational linear functions if r is rational. So, for rational r and corresponding b, d,

$$b \in \mathbb{Q} \Leftrightarrow d \in \mathbb{Q}.$$

6.4 Direct log canonicity (klt) (cf. [PSh08, Lemma 7.4, (iii)])

If $b \leq r$ ($< r$) then the corresponding $d \leq 1$ (respectively < 1). Note that, for adjunction, $b \leq r$ means lc of b, that is, $b \leq 1$ (see 7.2). So, if d is a multiplicity of a divisor D in a prime divisor P of X then (X, D) is lc near P. Note also that if additionally $r \leq 1$ then $b \leq 1$ too.

Proof. By Monotonicity

$$d = 1 - \frac{r}{l} + \frac{b}{l} \leq 1 - \frac{r}{l} + \frac{r}{l} = 1 \text{ (respectively } < 1).$$

□

6.5 Inverse log canonicity (klt) (cf. [PSh08, Lemma 7.4, (iii)])

If $d \leq 1$ ($d < 1$) then the corresponding $b \leq r$ (respectively $< r$). So, if additionally $r \leq 1$ then $b \leq 1$ (respectively < 1). Note that $r \leq 1$ holds for adjunction of morphisms by construction (see 7.2).

Proof. By Monotonicity

$$b = r - l + ld \leq r - l + l = r \text{ (respectively } < r).$$

□

6.6 Direct positivity (cf. [PSh08, Lemma 7.4, (i)])

If $b \geq 0$ and $r \leq l$ then the corresponding $d \geq 0$. The second assumption always holds for $r \leq 1$ because $l \geq 1$.

Proof. By Monotonicity and positivity of l

$$d = 1 - \frac{r}{l} + \frac{b}{l} \geq 1 - \frac{r}{l} \geq 0$$

□

However, the inverse positivity does not hold in general. E.g., if $0 \leq d \ll 1$ then $b \approx r - l$ and < 0 for $l > r$ which is typical for adjunction.

6.7 Direct boundary property (cf. [PSh08, Lemma 7.4, (iv)])

If $0 \leq b \leq r \leq 1$ then the corresponding $d \in [0,1]$, that is, $0 \leq d \leq 1$.

Proof. Immediate by Direct log canonicity and positivity. □

6.8 Direct hyperstandard property (cf. [PSh08, Proposition 9.3 (i)])

Let $\mathfrak{R}, \mathfrak{R}''$ be two finite subsets of rational numbers in $[0,1]$ such that $1 \in \mathfrak{R}, \mathfrak{R}''$. Then

$$\mathfrak{R}' = \{r' - l(1-r) \mid r \in \mathfrak{R}'', r' \in \mathfrak{R}, l \in \mathbb{Z}, l > 0 \text{ and } r' - l(1-r) \geq 0\}. \quad (6.8.1)$$

is also a finite subset of rational numbers in $[0,1]$ and $1 \in \mathfrak{R}'$.

For every parameters r, l such that $r \in \mathfrak{R}''$ and $b \in \Phi = \Phi(\mathfrak{R})$ such that $b \leq r$,

$$d \in \Phi' = \Phi(\mathfrak{R}'),$$

where d corresponds to b.

If additionally $\mathcal{N}$ is a (finite) set of positive integers then, for every $b \in \Gamma(\mathcal{N}, \Phi)$ such that $b \leq r$,

$$d \in \Gamma(\mathcal{N}, \Phi'),$$

where d corresponds to b.

Proof. Rationality of elements of $\mathfrak{R}'$ is immediate by definition. Since $r' \geq 0$ and $r \leq 1$, the set $\mathfrak{R}'$ is a subset of $[0,1]$. Notice for this also that elements of $\mathfrak{R}'$ are nonnegative by definition.

If $r = 1$ then $r' - l(1-r) = r' \in \mathfrak{R}$. Thus the set of such numbers is finite and 1 belongs to it.

If $r < 1$ then $1 - r > 0$. Thus the set of numbers $r' - l(1-r) \geq 0$ with $r' \in \mathfrak{R}$ is finite. Recall that l is a positive integer.

By definition and our assumptions

$$d = 1 - \frac{r}{l} + \frac{b}{l},$$

where $r \in \mathfrak{R}''$ and l is a positive integer. On the other hand, if $b \in \Gamma(\mathcal{N}, \Phi)$ then

$$b = 1 - \frac{r'}{l'} + \frac{1}{l'}(\sum_{n\in\mathcal{N}} \frac{m_n}{n+1}),$$

where $r' \in \mathfrak{R}$, l' is a positive integer and $m_n, n \in \mathcal{N}$, are nonnegative integers (if $\mathcal{N}$ is infinite then almost all $m_n = 0$). Hence

$$d = 1 - \frac{r}{l} + \frac{1 - \frac{r'}{l'} + \frac{1}{l'}(\sum_{n\in\mathcal{N}} \frac{m_n}{n+1})}{l} = 1 - \frac{r' - l'(1-r)}{ll'} + \frac{1}{ll'}(\sum_{n\in\mathcal{N}} \frac{m_n}{n+1}).$$

By our assumptions $b \leq r$. Thus by Direct log canonicity 6.4, $d \leq 1$. Thus $r' - l'(1-r) \geq 0, \in \mathfrak{R}'$ by (6.8.1) and $d \in \Gamma(\mathcal{N}, \Phi')$. If particular, if $\mathcal{N} = \emptyset$ then $d \in \Phi'$.

□

6.9 n_Φ inequality

Under assumptions and notation of 6.8, let b_1, d' be real numbers such that $0 \leq b_1 \leq r$ and

$$d' \geq d_{1,n\_\Phi'}, \tag{6.9.1}$$

where d_1 corresponds to b_1 by (6.0.10). Then

$$b_{1,n\_\Phi} \leq b', \tag{6.9.2}$$

where b' corresponds to d' by (6.0.11).

Proof. By 6.7, $d_1 \in [0,1]$ because $r \leq 1$ by 6.8. Thus the assumption (6.9.1) and the conclusion (6.9.2) are meaningful. By definition

$$b_{1,n\_\Phi} \leq b_1 \text{ and } b_{1,n\_\Phi} \in \Gamma(n,\Phi).$$

Hence by 6.2

$$d'_1 \leq d_1,$$

where d'_1 corresponds to b_{1,n_Φ} by (6.0.10). On the other hand, $d'_1 \in \Gamma(n,\Phi')$ by 6.8. Thus again by definition

$$d'_1 \leq d_{1,n\_\Phi'}.$$

This implies that

$$d'_1 \leq d'$$

by the assumption (6.9.1). Again 6.2 implies the required inequality (6.9.2). □

6.10 Direct dcc

Let $\Gamma \subset [0,1]$ be a dcc set and $\mathfrak{R}''$ be a finite subset in $[0,1]$. The the corresponding set

$$\Gamma' = \{1 - \frac{r}{l} + \frac{b}{l} \mid r \in \mathfrak{R}'', l \in \mathbb{Z}, l > 0, b \in \Gamma \text{ and } r \geq b\}$$

also satisfies dcc. Note that the last assumption equivalent to $1 - \frac{r}{l} + \frac{b}{l} \leq 1$ (cf. Direct log canonicity 6.4).

Proof. Since r, b are bounded and $r - b \geq 0$, any strictly decreasing sequence in Γ' has finite possibilities for l. Hence the finiteness of $\mathfrak{R}''$ and the dcc for Γ imply the dcc for Γ'.

□

6.11 Direct hyperstandard property for adjunction on divisor (cf. [Sh92, Lemma 4.2])

For any subset $\Gamma \subseteq [0,1]$, put

$$\widetilde{\Gamma} = \{1 - \frac{1}{l} + \sum_i \frac{l_i}{l} b_i \leq 1 \mid l, l_i \text{ are positive integers and } b_i \in \Gamma\} \cup \{1\}.$$

Then $0,1 \in \widetilde{\Gamma}, \Gamma \subseteq \widetilde{\Gamma}$ and $\widetilde{\widetilde{\Gamma}} = \widetilde{\Gamma}$. Transition form Γ to $\widetilde{\Gamma}$ corresponds to lc adjunction on a divisor [Sh92, Corollary 3.10].

Let $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a (finite) set of (rational) numbers $\mathfrak{R}$ in $[0,1], 1 \in \mathfrak{R}$, and $\mathcal{N}$ be a (finite) set of positive integers. Then

$$\Gamma\widetilde{(\mathcal{N},\Phi)} = \mathfrak{G}\widetilde{(\mathcal{N},\mathfrak{R})} = \Gamma(\mathcal{N},\widetilde{\Phi}) = \mathfrak{G}(\mathcal{N},\overline{\mathfrak{R}} \cup \{0\}),$$

in particular, $\widetilde{\Phi} = \Phi(\overline{\mathfrak{R}} \cup \{0\})$, where

$$\overline{\mathfrak{R}} = \{r_0 - \sum_i (1-r_i) \mid r_i \in \mathfrak{R}\} \cap [0,1].$$

The set $\overline{\mathfrak{R}}$ is (rational) finite, if $\mathfrak{R}$ is (rational) finite, and same as $\overline{\mathfrak{R}}$ in [PSh08, p. 160] and $\overline{\overline{\mathfrak{R}}} = \overline{\mathfrak{R}}$. (If $1 \notin \mathfrak{R}$, we need to replace $\overline{\mathfrak{R}}$ by $\overline{\mathfrak{R} \cup \{1\}}$.) Note also that $\widetilde{\Phi}$ depends only on Φ and above $\overline{\mathfrak{R}}$ is one of possible (finite) sets in $[0,1]$ to determine $\widetilde{\Phi}$ (see Example 9 below and cf. Proposition 5).

Proof. It is enough to verify that $\mathfrak{G}\widetilde{(\mathcal{N},\mathfrak{R})} = \mathfrak{G}(\mathcal{N},\overline{\mathfrak{R}} \cup \{0\})$. Then by definition, for $\mathcal{N} = \emptyset$, $\Phi = \mathfrak{G}(\emptyset,\mathfrak{R}) = \Phi(\mathfrak{R})$ and $\widetilde{\Phi} = \mathfrak{G}\widetilde{(\emptyset,\mathfrak{R})} = \mathfrak{G}(\emptyset,\overline{\mathfrak{R}} \cup \{0\}) = \Phi(\overline{\mathfrak{R}} \cup \{0\})$. Additionally $\mathfrak{G}\widetilde{(\mathcal{N},\mathfrak{R})} = \mathfrak{G}(\mathcal{N},\overline{\mathfrak{R}} \cup \{0\}) = \Gamma(\mathcal{N},\widetilde{\Phi})$ by definition and Proposition 5.

Take $b \in \mathfrak{G}\widetilde{(\mathcal{N},\mathfrak{R})}$. First, we verify that $b \in \mathfrak{G}(\mathcal{N},\overline{\mathfrak{R}} \cup \{0\})$.

Step 1. We can suppose that $b < 1$. Then by definition

$$b = 1 - \frac{1}{l} + \sum_{i=0}^{s} \frac{l_i}{l}(1 - \frac{r_i}{m_i} + \frac{1}{m_i}\sum_{n\in\mathcal{N}} \frac{l_{i,n}}{n+1}),$$

where l, l_i, m_i are positive integers, $l_{i,n}$ are nonnegative integers and $r_i \in \mathfrak{R}$. Otherwise, the only possible case $b = 1$. But then $1 = 1 - 0/1 \in \mathfrak{G}(\emptyset, 0) \subseteq \mathfrak{G}(\mathcal{N},\overline{\mathfrak{R}} \cup \{0\})$.

More precisely, we verify that $b < 1$ belongs to $\mathfrak{G}(\mathcal{N},\overline{\mathfrak{R}}) \subseteq \mathfrak{G}(\mathcal{N},\overline{\mathfrak{R}} \cup \{0\})$, We can suppose that $m_0 \geq m_1 \geq \cdots \geq m_s$.

Step 2. $m_i = 1$ for all $i \geq 1$. Otherwise, $m_1 \geq 2$. Then $m_0, m_1 \geq 2$ and

$$b \geq 1 - \frac{1}{l} + \frac{1}{l}(1 - \frac{1}{2}) + \frac{1}{l}(1 - \frac{1}{2}) = 1,$$

a contradiction. Here we use the inequality $r_i \leq 1$ for every i.

Step 3. If $m_0 = m \geq 2$ then $l_0 = 1$. Otherwise, $l_0 \geq 2$ and

$$b \geq 1 - \frac{1}{l} + \frac{2}{l}(1 - \frac{1}{2}) = 1.$$

Hence we have the following two cases.
Case 1. $m_i = 1$ for all $i \geq 1, m_0 = m \geq 2$ and $l_0 = 1$. So,

$$\begin{aligned} b =& 1 - \frac{1}{l} + \frac{1}{l}(1 - \frac{r_0}{m} + \frac{1}{m}\sum_{n \in \mathcal{N}} \frac{l_{0,n}}{n+1}) + \sum_{i=1}^{s} \frac{l_i}{l}(1 - r_i + \sum_{n \in \mathcal{N}} \frac{l_{i,n}}{n+1}) = \\ & 1 - \frac{r_0 - \sum_{i=1}^{s} m l_i (1 - r_i)}{lm} + \frac{1}{lm}(\sum_{n \in \mathcal{N}} \frac{l_{0,n} + \sum_{i=1}^{s} m l_i l_{i,n}}{n+1}) \end{aligned}$$

belongs to $\mathfrak{G}(\mathcal{N}, \overline{\mathfrak{R}})$. Indeed, $0 < r_0 - \sum_{i=1}^{s} m l_i (1 - r_i) \leq r_0 \leq 1$ and belongs to $\overline{\mathfrak{R}}$ because respectively $1 > b$ and every $r_i \leq 1$.
Case 2. Every $m_i = 1$. Then

$$\begin{aligned} b = 1 - \frac{1}{l} + \sum_{i=0}^{s} \frac{l_i}{l}(1 - r_i + \sum_{n \in \mathcal{N}} \frac{l_{i,n}}{n+1}) = \\ 1 - \frac{1 - \sum_{i=0}^{s} l_i (1 - r_i)}{l} + \frac{1}{l} \sum_{n \in \mathcal{N}} \frac{\sum_{i=0}^{s} l_i l_{i,n}}{n+1} \end{aligned}$$

also belongs to $\mathfrak{G}(\mathcal{N}, \overline{\mathfrak{R}})$ because $1 \in \mathfrak{R}$ and $1 - \sum_{i=0}^{s} l_i (1 - r_i) \in \overline{\mathfrak{R}}$.
Step 4. Conversely, take $b \in \mathfrak{G}(\mathcal{N}, \overline{\mathfrak{R}} \cup \{0\})$. Now, we verify that $b \in \widetilde{\mathfrak{G}(\mathcal{N}, \mathfrak{R})}$. By definition $b \leq 1$.
Case 1. $b = 1$. Then $b = 1 = 1 - 1/l + 1/l \in \widetilde{\mathfrak{G}(\emptyset, \{1\})} \subset \widetilde{\mathfrak{G}(\mathcal{N}, \mathfrak{R})}$ because $1 \in \mathfrak{R}$.
Case 2. $b < 1$. Then by definition

$$b = 1 - \frac{r}{l} + \frac{1}{l} \sum_{n \in \mathcal{N}} \frac{l_n}{n+1},$$

where $r \in \overline{\mathfrak{R}} \cup \{0\}$ and $r > 0$. Hence $r \in \overline{\mathfrak{R}}$ and

$$b = 1 - \frac{r_0 - \sum_{i=1}^{s}(1 - r_i)}{l} + \frac{1}{l} \sum_{n \in \mathcal{N}} \frac{l_n}{n+1} = 1 - \frac{1}{l} + \frac{1}{l} \sum_{i=0}^{s}(1 - r_i) + \frac{1}{l} \sum_{n \in \mathcal{N}} \frac{l_n}{n+1}.$$

Note now that every

$$1 - r_i = 1 - \frac{r_i}{1} \in \Phi = \Gamma(\emptyset, \Phi) = \mathfrak{G}(\emptyset, \mathfrak{R}) \subseteq \Gamma(\mathcal{N}, \Phi) = \mathfrak{G}(\mathcal{N}, \mathfrak{R})$$

and

$$\frac{1}{n+1} = 1 - \frac{1}{1} + \frac{1}{n+1} \in \Gamma(\mathcal{N}) = \mathfrak{G}(\mathcal{N}, \{1\}) \subseteq \Gamma(\mathcal{N}, \Phi) = \mathfrak{G}(\mathcal{N}, \mathfrak{R}).$$

So, $b \in \widetilde{\mathfrak{G}(\mathcal{N}, \mathfrak{R})}$.

□

Example 9. As an exception, take $\mathfrak{R} = \emptyset$. Then $\overline{\mathfrak{R}} = \Phi = \Phi(\mathfrak{R}) = \emptyset$ too and

$$\widetilde{\Phi} = \{1 - \frac{1}{l} \mid l \text{ is a positive integer}\} \cup \{1\}, = \Phi(\{0, 1\}),$$

the standard set. However, $\widetilde{\Phi} = \Phi(\overline{\mathfrak{R}'})$ for $\mathfrak{R}' \subseteq \{1/l \mid l \text{ is a positive integer}\}$ if and only if $\mathfrak{R}' = \{1, 1/2\}$.

6.12 Main inequality

For any positive integers n, l, any real number d, and any rational number r such that $nr \in \mathbb{Z}$ and $r \leq 1$,

$$r - l + l \lfloor (n+1)d \rfloor / n \geq \lfloor (n+1)(r - l + ld \rfloor / n. \tag{6.12.1}$$

Proof. The inequality (6.12.1) is equivalent to

$$nr - nl + l \lfloor (n+1)d \rfloor \geq \lfloor (n+1)(r - l + ld \rfloor = \lfloor (n+1)(r + ld) \rfloor - (n+1)l$$

or to

$$nr + l + l \lfloor (n+1)d \rfloor \geq \lfloor (n+1)(r + ld) \rfloor. \tag{6.12.2}$$

For $r = 1$, the inequality (6.12.2) has the form

$$n + l + l \lfloor (n+1)d \rfloor \geq \lfloor (n+1)(1 + ld) \rfloor = n + 1 + \lfloor l(n+1)d \rfloor,$$

that is,

$$l - 1 + l \lfloor (n+1)d \rfloor \geq \lfloor l(n+1)d \rfloor.$$

It follows from the inequality (2.2.2).

For $r < 1$, by upper bounds (2.1.1), (2.3.1) and (2.2.2)

$$\begin{aligned} \lfloor (n+1)(r + ld) \rfloor \leq & 1 + \lfloor (n+1)r \rfloor + \lfloor l(n+1)d \rfloor \leq \\ & 1 + nr + l - 1 + l \lfloor (n+1)d \rfloor = nr + l + l \lfloor (n+1)d \rfloor, \end{aligned}$$

that completes the proof of (6.12.2).

□

6.13 Inverse $\lfloor (n+1) - \rfloor /n$-monotonicity

For a real number x, put

$$x_{[n]} = \begin{cases} 1, \text{ if} & x = 1; \\ \lfloor (n+1)x \rfloor /n & \text{otherwise}; \end{cases}$$

Under the assumptions of 6.12 suppose additionally that $d \leq 1$. Then

$$b_{[n]} \leq b^{[n]}$$

where real numbers $b, b^{[n]}$ correspond respectively to $d, d_{[n]}$ according to (6.0.11).

Proof.
Case 1. $d < 1$. Immediate by (6.12.1). Indeed, $b < 1$ by 6.5 and our assumptions.
Case 2. $d = 1, r < 1$. Then $d_{[n]} = 1 = d$ and $b = b^{[n]} = r$ by (6.0.11). The required inequality follows from our assumptions by Example 5, (2).
Case 3. $d = r = 1$. As in Case 2 $b = b^{[n]} = r$. Since $r = 1$, $b_{[n]} = 1 = b^{[n]}$.
□

6.14 Inverse inequality (1) of Definition 2

Under the assumptions and notation of 6.12-6.13, for corresponding b^+, d^+ according to (6.0.11),

$$\begin{aligned} b_{[n]} \leq & b^+ \leq r \leq 1 \\ & \Uparrow \\ d_{[n]} \leq & d^+ \leq 1. \end{aligned}$$

Proof. The second inequality of the top row holds by 6.5 and the third one by our assumptions.
Since $d_{[n]} \leq 1$, $d \leq 1$ holds (cf. Remark 1, (2)). Hence by 6.13 and 6.2

$$b_{[n]} \leq b^{[n]} \leq b^+.$$

□

7 Adjunction

We recall some basic facts about (log) adjunction or subadjunction in the Kawamata terminology [K98].

7.1 Adjunction for 0-contractions [PSh08, Section 7]

Let $f\colon X \to Z$ be a contraction of normal algebraic varieties or spaces and D be an $\mathbb{R}$-divisor on X such that

(1) (X, D) is lc generically over Z;

(2) D is a boundary generically over Z or, equivalently, D^{h} is a boundary, where D^{h} denotes the horizontal part of D with respect to f; and

(3) $K + D \sim_{\mathbb{R},Z} 0$ (cf. 7.3, (3) and [PSh08, Construction 7.5]), in particular, $K + D$ is $\mathbb{R}$-Cartier and (X, D) is a log pair.

Then there exist two $\mathbb{R}$-divisors on Z:

the *divisorial part of adjunction* D_{div} of $(X, D) \to Z$ [PSh08, Construction 7.2]; and

the *moduli part of adjunction* D_{mod} of $(X, D) \to Z$ [PSh08, Construction 7.5]

such that the following generalization of Kodaira formula

$$K + D \sim_{\mathbb{R}} f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) \qquad (7.1.1)$$

holds. It is also called the *(log) adjunction formular* for $(X, D) \to Z$. The pair $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ is a log pair and $f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}})$ is well-defined. The pair $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ is called the *adjoint pair* of $(X, D) \to Z$. The divisor D_{div} is unique but D_{mod} is defined up to $\sim_{\mathbb{R}}$.

The adjunction has birational nature. We say that two contractions $f\colon (X, D) \to Z, f'\colon (X', D') \to Z'$ are *birationally equivalent* or *crepant* if there exists a commutative diagram

$$\begin{array}{ccc} (X', D') & \dashrightarrow & (X, D) \\ f' \downarrow & & \downarrow f \\ Z' & \dashrightarrow & Z \end{array},$$

where the horizontal arrow $(X', D') \dashrightarrow (X, D)$ is a crepant proper birational isomorphism, another horizontal arrow $Z' \dashrightarrow Z$ is a proper birational isomorphism and f' satisfies the same assumptions as f. In particular, $(X, D), (X', D')$ are birationally equivalent or crepant if $(X/X, D), (X'/X', D')$ does so. If X is complete then this also means that $(X/\mathrm{pt.}, D), (X'/\mathrm{pt.}, D')$ does so. Notice that $D' = D_{X'} = \mathbb{D}(X, D)_{X'}$ and (X', D') is lc generically over Z' automatically. (The property to be a boundary for D'^{h} or, equivalently, to be effective for D'^{h} can be omitted.) We say also that $f' \colon (X', D') \to Z'$ is a (*crepant*) *model* of $f \colon (X, D) \to Z$. In this situation the adjoint pair $(Z', D'_{\mathrm{div}} + D'_{\mathrm{mod}})$ is defined and $(Z', D'_{\mathrm{div}} + D'_{\mathrm{mod}}) \dashrightarrow (Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ is also a crepant proper birational isomorphism. This allows to define two b-$\mathbb{R}$-divisors $\mathbb{D}_{\mathrm{div}}$ and $\mathcal{D}_{\mathrm{mod}}$ of Z [PSh08, Remark 7.7] such that

$$D_{\mathrm{div}} = (\mathbb{D}_{\mathrm{div}})_Z, D_{\mathrm{mod}} = (\mathcal{D}_{\mathrm{mod}})_Z \text{ and } D'_{\mathrm{div}} = (\mathbb{D}_{\mathrm{div}})_{Z'}, D'_{\mathrm{mod}} = (\mathcal{D}_{\mathrm{mod}})_{Z'}.$$

Of course we use that same $\sim_{\mathbb{R}}$ for f' as in (7.1.1) and suppose that $K_{X'} = (\mathbb{K})_{X'}$ and $K_{Z'} = (\mathbb{K}_Z)_{Z'}$, where $\mathbb{K}_Z$ is a canonical b-divisor of Z. Otherwise $D'_{\mathrm{mod}} \sim_{\mathbb{R}} (\mathcal{D}_{\mathrm{mod}})_{Z'}$. Indeed, for every prime b-divisor Q of Z there exist a model f' of f and a prime divisor P on X' such that $f'(P) = Q$ is a divisor on Z'. (For details see [PSh08, Section 7].) Respectively, for b-$\mathbb{R}$-divisors, (7.1.1) become

$$\mathbb{K} + \mathbb{D} \sim_{\mathbb{R}} f^*(\mathbb{K}_Z + \mathbb{D}_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}}) \tag{7.1.2}$$

and $(Z, D_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$ become the *adjoint* log *bd-pair* of $(X, D) \to Z$, that is, $\mathbb{D}_{\mathrm{div}} = \mathbb{D}(Z, D_{\mathrm{div}} + D_{\mathrm{mod}}) - \mathcal{D}_{\mathrm{mod}}$. Actually, in our applications, $\mathcal{D}_{\mathrm{mod}}$ will be a b-nef b-$\mathbb{Q}$-divisor and $(Z, D_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$ will be a bd-pair of some positive integral index (see Theorem 8 and cf. Conjecture 3).

For a bd-pair $(X, D + \mathcal{P})$ and its contraction $f \colon (X, D + \mathcal{P}) \to Z$ under assumptions

(1-bd) $(X, D + \mathcal{P})$ is lc generically over Z;

(2-bd) D is a boundary and $\mathcal{P}$ is b-nef generically over Z; and

(3-bd) $K + D + \mathcal{P}_X \sim_{\mathbb{R},Z} 0$, in particular, $K + D + \mathcal{P}_X$ is $\mathbb{R}$-Cartier and $(X, D + \mathcal{P})$ is a log bd-pair.

Then there exist the following $\mathbb{R}$-divisors on Z and b-$\mathbb{R}$-divisors of Z [F18, Section 4]:

the $\mathbb{R}$-divisor, the *divisorial part of adjunction* $(D+\mathcal{P})_{\mathrm{div},Z}$ of $(X, D+\mathcal{P}) \to Z$;

respectively, b-$\mathbb{R}$-divisor $(D+\mathcal{P})_{\mathrm{div}}$, e.g., $((D+\mathcal{P})_{\mathrm{div}})_Z = (D+\mathcal{P})_{\mathrm{div},Z}$;

the $\mathbb{R}$-divisor, *moduli part of adjunction* $(D+\mathcal{P})_{\mathrm{mod},Z}$ of $(X, D+\mathcal{P}) \to Z$; and,

respectively, b-$\mathbb{R}$-divisor $(D+\mathcal{P})_{\mathrm{mod}}$, e.g., $((D+\mathcal{P})_{\mathrm{mod}})_Z = (D+\mathcal{P})_{\mathrm{mod},Z}$;

such that the following generalizations of Kodaira formula

$$K + D + \mathcal{P}_X \sim_{\mathbb{R}} f^*(K_Z + (D+\mathcal{P})_{\mathrm{div},Z} + (D+\mathcal{P})_{\mathrm{mod},Z})$$

and

$$\mathbb{K} + \mathbb{D}(X, D+\mathcal{P}_X) \sim_{\mathbb{R}} f^*(\mathbb{K}_Z + (D+\mathcal{P})_{\mathrm{div}} + (D+\mathcal{P})_{\mathrm{mod}})$$

hold. It is also called the (*log*) *adjunction formular* for $(X, D+\mathcal{P}) \to Z$. The bd-pair $(Z, (D+\mathcal{P})_{\mathrm{div},Z} + (D+\mathcal{P})_{\mathrm{mod}})$ is a log bd-pair and $f^*(K_Z + (D+\mathcal{P})_{\mathrm{div},Z} + (D+\mathcal{P})_{\mathrm{mod},Z})$ is well-defined. The pair $(Z, (D+\mathcal{P})_{\mathrm{div},Z} + (D+\mathcal{P})_{\mathrm{mod}})$ is called the *adjoint bd-pair* of $(X, D+\mathcal{P}) \to Z$. Indeed, $(D+\mathcal{P})_{\mathrm{div}} = \mathbb{D}(Z, (D+\mathcal{P})_{\mathrm{div},Z} + (D+\mathcal{P})_{\mathrm{mod},Z}) - (D+\mathcal{P})_{\mathrm{mod}}$. The b-$\mathbb{R}$-divisor $(D+\mathcal{P})_{\mathrm{div}}$ is unique but b-$\mathbb{R}$-divisor $(D+\mathcal{P})_{\mathrm{mod}}$ is defined up to $\sim_{\mathbb{R}}$. Actually, in our applications, $(D+\mathcal{P})_{\mathrm{mod}}$ will be a b-nef b-$\mathbb{Q}$-divisor and $(Z, (D+\mathcal{P})_{\mathrm{div},Z} + (D+\mathcal{P})_{\mathrm{mod}})$ will be a bd-pair of some positive integral index (see Addendum 31 and cf. Conjecture 3).

Adjunction for $(X, D+\mathcal{P}) \to Z$ also has birational nature. In this situation, we say that two contractions $f\colon (X, D+\mathcal{P}) \to Z$, $f'\colon (X', D'+\mathcal{P}) \to Z'$ are *birationally equivalent* or *crepant* if there exists a commutative diagram

$$\begin{array}{ccc} (X', D'+\mathcal{P}) & \dashrightarrow & (X, D+\mathcal{P}) \\ f' \downarrow & & \downarrow f \\ Z' & \dashrightarrow & Z \end{array},$$

where the horizontal arrow $(X', D'+\mathcal{P}) \dashrightarrow (X, D+\mathcal{P})$ is a crepant proper birational isomorphism, another horizontal arrow $Z' \dashrightarrow Z$ is a proper birational isomorphism and f' satisfies the same assumptions as f. Notice that $D' = D_{X'} = \mathbb{D}(X, D+\mathcal{P}_X)_{\mathrm{div},X'} - \mathcal{P}_{X'}, \mathbb{D}(X', D'+\mathcal{P}_{X'}) = \mathbb{D}(X, D+\mathcal{P}_X)$ and $(X', D'+\mathcal{P})$ is lc generically over Z' automatically. (The property to be a boundary for D'^{h} or, equivalently, to be effective for D'^{h} can be omitted.) We say also that $f'\colon (X', D'+\mathcal{P}) \to Z'$ is a (*crepant*) *model* of $f\colon (X, D+\mathcal{P}) \to Z$.

In this situation the adjoint bd-pair $(Z', (D + \mathcal{P})_{\mathrm{div},Z'} + (D + \mathcal{P})_{\mathrm{mod}})$ is defined, $(Z', (D+\mathcal{P})_{\mathrm{div},Z'} + (D+\mathcal{P})_{\mathrm{mod}}) \dashrightarrow (Z, (D+\mathcal{P})_{\mathrm{div},Z} + (D+\mathcal{P})_{\mathrm{mod}})$ is a crepant proper birational isomorphism and

$$(D + \mathcal{P})_{\mathrm{div},Z'} = ((D + \mathcal{P})_{\mathrm{div}})_{Z'}, (D + \mathcal{P})_{\mathrm{mod},Z'} = ((D + \mathcal{P})_{\mathrm{mod}})_{Z'}.$$

We can modify above concepts, e.g., birational equivalence or to be crepant, to the relative situation. However, adjunction formulae (7.1.1-7.1.2) and muduli part of adjunction have absolute nature and defines up to $\sim_{\mathbb{R}}$. For bd-pairs we assume the same for $\mathcal{P}$. (Usually we omit S if $S = \mathrm{pt}$.)

The following result about the moduli part of adjunction holds in much more general situation [Sh13]. Actually we expect more [PSh08, Conjecture 7.13] (see also Conjecture 3). But we need the result only under stated assumptions. On the other hand, the nef property is assumed here in the weak sense (see Nef in Section 1). In particular, the property applies to the situation, where X is complete or it has a proper morphism $X \to S$ to a scheme or a space S compatible with the contraction $X \to Z$, equivalently, $X \to Z$ is a morphisms over S and Z/S is proper.

Theorem 8 ([A04])**.** *Under notation and assumptions of 7.1 suppose additionally that D is an effective* $\mathbb{Q}$-divisor *generically over Z. Then $\mathcal{D}_{\mathrm{mod}}$ is a b-$\mathbb{Q}$-divisor of Z, defined up to a $\mathbb{Q}$-linear equivalence. Moreover, $\mathbb{K}_Z + \mathbb{D}_{\mathrm{div}}$ is a b-$\mathbb{R}$-Cartier divisor of Z and $\mathcal{D}_{\mathrm{mod}}$ is b-nef.*

Addendum 31 ([F18, Theorem 4.1])**.** *The same holds for every contraction $(X, D + \mathcal{P}) \to Z$ as in 7.1 under the additional assumptions of the theorem and the assumption that $(X, D + \mathcal{P})$ is a bd-pair of some positive integral index.*

[Remark: the theorem holds either without the generic $\mathbb{Q}$-assumption over Z, but with a b-$\mathbb{R}$-divisor $\mathcal{D}_{\mathrm{mod}}$ of Z, or without the generic effective property over Z. However, the effective property is very important when D has horizontal nonrational multiplicities [Sh13] [Sh08].

About notation: For b-$\mathbb{R}$-Cartier divisors we use mathcal, e.g., $\mathcal{D}_{\mathrm{mod}}, \mathcal{P}$. For b-$\mathbb{R}$-divisors with BP we use mathbb: e.g., $\mathbb{D}$. However, $\mathbb{D} + \mathcal{P}, -\mathbb{K}$ have BP and $\mathbb{K} + \mathbb{D}$ is b-$\mathbb{R}$-Cartier.]

Proof. See proofs in [A04] and [F18]. Notice only that the necessary assumption for the existence of $\mathcal{D}_{\mathrm{mod}}$ that $K + D \sim_{\mathbb{R}} f^*L$ for some $\mathbb{R}$-Cartier divisor L on Z [PSh08, Construction 7.5] holds by (3) of 7.1. Similarly, in the addendum we use (3-bd) of 7.1.

□

7.2 Adjunction correspondence of multiplicities

Let $f\colon X \to Z$ be a proper surjective morphism, e.g., a contraction as in 7.1. Then, for every *vertical over* Z prime b-divisor P of X, its image $Q = f(P)$ as a *prime* b-divisor is well-defined. For this consider a model $f'\colon X' \to Z'$ as in 7.1 of f such that P is a divisor on X' and $f'(P) = f(P) = Q$ is a divisor on Z'. Since f is proper surjective, it is also *surjective on prime* b-*divisors*, that is, for every prime b-divisor Q of Z there exists a vertical prime b-divisor P of X such that $f(P) = Q$. Indeed, for a birational model f' as above, P is any divisorial irreducible component of $f'^{-1}Q$, equivalently, P, Q are prime divisors on X', Z' respectively and $f'(P) = Q$. The prime b-divisors P, Q such that $f(P) = Q$ will be called *corresponding* with respect to f.

Let $f\colon (X, D) \to Z$ be a morphism as in 7.1. Consider prime b-divisors P, Q of X, Z respectively such that $f(P) = Q$. Put

$$r = \operatorname{mult}_P(D' + c_Q f'^*Q) \text{ and } l = \operatorname{mult}_P f'^*Q,$$

where f' is the above model of f, $D' = D_{X'} = (\mathbb{D}(X, D))_{X'}$, c_Q is the log canonical threshold as in [PSh08, Construction 7.2] and f'^*Q is well-defined generically over Q. Then r is a real number ≤ 1, because $(X', D_{X'} + c_Q f'^*Q)$ generically lc over Q and near P, and l is a positive integer such that

$$d_Q = 1 - \frac{r}{l} + \frac{d_P}{l} \tag{7.2.1}$$

and

$$d_P = r - l + l d_Q. \tag{7.2.2}$$

This is exactly the adjunction correspondence (6.0.10-6.0.11) between $b = d_P = \operatorname{mult}_P D' = \operatorname{mult}_P \mathbb{D}$ and $d = d_Q = \operatorname{mult}_Q \mathbb{D}_{\mathrm{div}}$. The parameters $r = r_P, l = l_P$ depend only on P but not on a model f' (cf. Proposition 13, (4)). They will be called *adjunction constants* of $(X, D) \to Z$ at P. Similar constants r, l can be defined for a morphism $(X, D + \mathcal{P}) \to Z$ as in 7.1.

If (X, D) is lc (klt) then $d_P \le r$ ($< r$ respectively). Conversely, (X, D) is lc (klt over Z) if $d_P \le r$ ($< r$ respectively) for every vertical prime b-divisor P of X. The relative klt over Z means $\operatorname{mult}_P \mathbb{D} < 1$ but only for vertical P.

The same holds for adjunction of $(X, D + \mathcal{P}) \to Z$ as in 7.1.

Proof. By definition and construction

$$r = \operatorname{mult}_P(D' + c_Q f'^*Q) = \operatorname{mult}_P D' + c_Q \operatorname{mult}_P f'^*Q = d_P + c_Q l$$

and $c_Q = (r - d_P)/l$. On the other hand, by definition

$$d_Q = 1 - c_Q = 1 - \frac{r - d_P}{l} = 1 - \frac{r}{l} + \frac{d_P}{l}.$$

This proves (7.2.1). The latter implies (7.2.2).

It is enough to establish the independence of r, l of f' in the case when Z is a curve [PSh08, Remark 7.3]; if Z is a point then every prime b-divisor P of X is horizontal over Z. But for a curve Z, the divisor f'^*Q can be replaces by its Cartier b-divisor. Then r, l are independent of a proper birational model X' of X over Z.

If (X, D) is lc (klt) then by definition $c_Q \geq 0$ (> 0 respectively). Hence $r \geq d_P$ (respectively $> d_P$) because $l > 0$. The converse can be established similarly.

The same works for adjunction of $(X, D + \mathcal{P}) \to Z$.

$\square$

[Remark: r_P is not a multiplicity of a (b-)divisor at P, that is, $\sum r_P P$ is not a (b-)divisor! However it is a (b-)divisor for P/Q (cf. 7.5).]

7.3 Index of adjunction

Let $f\colon (X, D) \to Z$ be a 0-contraction as in 7.1 and I be a positive integer. We say that the 0-contraction has an *adjunction index I* if

(1) $\mathbb{K} + \mathbb{D} \sim_I f^*(\mathbb{K}_Z + \mathbb{D}_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$, in particular, $K + D \sim_I f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}})$;

(2) I is an lc index of $(X/Z \ni \eta, D)$ and $I\mathbb{D}$ is b-Cartier generically over η, where $\eta \in Z$ is the generic point of Z; actually $\mathcal{D}_{\mathrm{mod}}$ is defined modulo I_η, the generic index;

(3) $I\mathcal{D}_{\mathrm{mod}}$ is b-Cartier and $\mathcal{D}_{\mathrm{mod}}$ is defined modulo $\sim_I$, in particular, $\mathbb{K} + \mathbb{D} \sim_{I,Z} f^*(\mathbb{K}_Z + \mathbb{D}_{\mathrm{div}})$ (cf. 7.1, (3) and Corollary-Conjecture 1); and

(4) $Ir_P \in \mathbb{Z}$ for every adjunction constant r_P of $(X, D) \to Z$.

Actually, (1) is equivalent to (2) and, by Proposition 13 and the reduction in Step 1 of the proof of Theorem 9 below, (3) implies (1-2) and (4).

Respectively, a 0-contraction $f\colon (X, D + \mathcal{P}) \to Z$ of a bd-pair $(X, D + \mathcal{P})$ as in 7.1 has an *adjunction index I* if

(0-bd) $(X, D + \mathcal{P})$ is a bd-pair of index I, in particular, X, Z are complete or proper over some scheme S, e.g., $(X, D + \mathcal{P})$ is a bd-pair of index m and $m|I$;

(1-bd) $\mathbb{K}+\mathbb{D}(X, D+\mathcal{P}_X) \sim_I f^*(\mathbb{K}_Z+(D+\mathcal{P})_{\mathrm{div}}+(D+\mathcal{P})_{\mathrm{mod}})$, in particular, $K + D + \mathcal{P}_X \sim_I f^*(K_Z + (D + \mathcal{P})_{\mathrm{div},Z} + (D + \mathcal{P})_{\mathrm{mod},Z})$;

(2-bd) $I(D + \mathcal{P})_{\mathrm{mod}}$ is b-Cartier and $(D + \mathcal{P})_{\mathrm{mod}}$ is defined modulo $\sim_I$;

(3-bd) I is an lc index of $(X/Z \ni \eta, D+\mathcal{P}_X)$ and $I\mathbb{D}(X, D+\mathcal{P}_X)$ is b-Cartier generically over η, where $\eta \in Z$ is the generic point of Z; and

(4-bd) $Ir_P \in \mathbb{Z}$ for every adjunction constant r_P of $(X, D + \mathcal{P}) \to Z$.

Corollary 12 below implies that under assumptions in 7.1, if D^{h} is a $\mathbb{Q}$-divisor, then $(X, D) \to Z$ has some adjunction index (cf. Corollary 31). We need a similar result for certain families of 0-contractions.

Theorem 9. *Let d be a nonnegative integer and Γ be a dcc set of rational numbers in $[0, 1]$. Then there exists a positive integer $I = I(d, \Gamma)$ such that every 0-contraction $f\colon (X, D) \to Z$ as in 7.1 with wFt $X/Z, \dim X = d$ and $D^{\mathrm{h}} \in \Gamma$ has the adjunction index I.*

Addendum 32. *$(Z, D_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$ is a log bd-pair of index I.*

Addendum 33. *There are two finite sets of rational numbers $\Gamma^{\mathrm{h}}(d) \subseteq \Gamma$ and $\mathfrak{R}'' = \mathfrak{R}''(d, \Gamma) \subset [0, 1]$ such that, for every 0-contraction $(X, D) \to Z$ in as the theorem,*

$D^{\mathrm{h}} \in \Gamma^{\mathrm{h}}(d)$; and

every nonnegative adjunction constant r_P of $(X, D) \to Z$ belongs $\mathfrak{R}''$.

Addendum 34. *Let Γ'' be another dcc in $[0, 1]$, e.g., $\Gamma'' = \Gamma$. Then there exists a dcc set $\Gamma' \subset [0, 1]$ such that, for every 0-contraction $(X, D) \to Z$ in as the theorem and with lc (X, D),*

$$D^{\mathrm{v}} \in \Gamma'' \Rightarrow D_{\mathrm{div}} \in \Gamma'.$$

Γ' depends on d, Γ, Γ'' and is rational if Γ'' is rational.

Addendum 35. *Let $\mathfrak{R}$ be a finite set of rational numbers in $[0,1]$. Then there exists a finite set of rational numbers $\mathfrak{R}' \subset [0,1]$ such that, for every (finite) set of integers $\mathcal{N}$ and every 0-contraction $(X,D) \to Z$ in as the theorem with lc (X,D),*

$$D^{\mathrm{v}} \in \Gamma(\mathcal{N},\Phi) \Rightarrow D_{\mathrm{div}} \in \Gamma(\mathcal{N},\Phi'),$$

where $\Phi = \Phi(\mathfrak{R}), \Phi' = \Phi(\mathfrak{R}')$ are hyperstandard sets associated with $\mathfrak{R},\mathfrak{R}'$ respectively. The set $\mathfrak{R}'$ depends on d,Γ and $\mathfrak{R}$.

Addendum 36. *The same holds for every contraction $(X,D+\mathcal{P}) \to Z$ as in 7.1 under the additional assumptions of the theorem and the assumption that $(X,D+\mathcal{P})$ is a bd-pair of index m. In this situation $I = I(d,\Gamma,m), \mathfrak{R}'' = \mathfrak{R}''(d,\Gamma,m), \Gamma(d,m)$ depend also on m and $\Gamma',\mathfrak{R}'$ depend respectively on d,Γ,Γ'',m and $d,\Gamma,\mathfrak{R},m$. The adjoint bd-pair in Addendum 32 $(Z,(D+\mathcal{P})_{\mathrm{div},Z}+(D+\mathcal{P})_{\mathrm{mod}})$ is a log bd-pair of index I. Additionally $m|I$.*

Proof. (Hyperstandard case.) For the general case see Section 12.

Step 1. (Reduction to relative tlc singularities.) There exists a model $f'\colon (X',D') \to Z'$ of $(X,D) \to Z$ and a boundary B on X' such that

(1) $\mathcal{D}_{\mathrm{mod}}$ is stable over Z': $\mathcal{D}_{\mathrm{mod}} = \overline{\mathcal{D}_{\mathrm{mod},Z'}}$, the Cartier closure over Z' [Sh96, Example 1.1.1];

and

(2) $B^{\mathrm{h}} = D^{\mathrm{h}}$ and $(X'/Z',B)$ is a 0-pair with tlc (toroidal log canonical) singularities.

The latter means that $(X'/Z',B)$ is toroidal near lc but not klt points, in particular, (X',B') is lc, and with vertical reduced boundary: $B^{\mathrm{v}} \in \{0,1\}$. Additionally, it means that $Z \setminus \operatorname{Supp} B_{\mathrm{div}}$ is nonsingular and, for every nonsingular hypersurface H in Z, $(X',B'+f'^*H)$ is lc over $Z \setminus \operatorname{Supp} B_{\mathrm{div}}$ too (equisingularity). (Actually, it is enough that $(X'/Z',B)$ has such a toroidal model.) For the property (2), it is enough weak semistable reduction and relative LMMP (see also [B12, Theorem 1.1]). For (1), Z' should be sufficiently high over Z by Theorem 8.

By [PSh08, Remark 7.5.1] (cf. Proposition 13), $(X'/Z',B)$ has the same adjunction index. Thus for simplicity we can suppose that $(X'/Z',B) = (X/Z,D)$.

Step 2. (Adjunction index.) By (2), $D_{\mathrm{div}} \in \{0,1\}$ (reduced) and (Z, D_{div}) has lc index 1, that is, $K_Z + D_{\mathrm{div}}$ is Cartier. Moreover, by (2) and Corollary 31 there exists a positive integer $I = I(d, \Gamma)$ and a finite set of rational numbers $\Gamma^{\mathrm{h}}(d)$ depending on d and Γ, such that

(3) $K + D \sim_{I,Z} 0$; and

(4) $D^{\mathrm{h}} \in \Gamma^{\mathrm{h}}(d)$.

Note that in this section we can prove Corollary 31 and the theorem only for hyperstandard sets, that is, assuming $\Gamma = \Phi(\mathfrak{R})$, where $\mathfrak{R}$ is a finite set of rational numbers in $[0,1]$, possibly, different from $\mathfrak{R}$ of Addendum 35. In this situation $\Gamma^{\mathrm{h}}(d) = \Gamma^{\mathrm{h}}(d, \mathfrak{R})$ depends only on d and $\mathfrak{R}$. Indeed, the proof of Corollary 31 uses boundedness of n-complements. In the case $\dim Z \geq 1$ we can use dimensional induction. In the case $\dim Z = 0$, again we can use dimensional induction as in [PSh08, 4.13] if (X, D) is lc but not klt. Finally, if $\dim Z = 0$ and (X, D) is klt then (X, D) is bounded by Corollary 11 because $D = D^{\mathrm{h}} \in \Phi$. A different approach can use [B, Theorem 1.8] [HX, Theorem 1.3] [HLSh, Theorem 1.6].

We can apply Corollary 31 because the construction in Step 1 does not change the following properties of $(X, D) \to Z$: it is still 0-contraction as in 7.1 with wFt $X/Z, \dim X = d$ and $D^{\mathrm{h}} \in \Gamma$. By construction the 0-pair $(X/Z, D)$ is maximal lc over $\operatorname{Supp} D_{\mathrm{div}}$. Thus by Corollary 31 or [B, Theorem 1.8] (3) holds over a neighborhood of $\operatorname{Supp} D_{\mathrm{div}}$. By (4), $D^{\mathrm{h}} \in \Gamma^{\mathrm{h}}(d)$. Actually, by (3), $I\Gamma^{\mathrm{h}}(d) \in \mathbb{Z}$ and $\Gamma^{\mathrm{h}}(d)$ is a finite set of rational numbers in $[0,1] \cap (\mathbb{Z}/I)$. Again by (2), for a nonsingular hyperplane H through any point z of $Z \setminus \operatorname{Supp} D_{\mathrm{div}}$, $(X/Z, D + f^*H)$ is lc over $Z \setminus D_{\mathrm{div}}$. (It is sufficiently to take effective Cartier H such that (Z, H) is lc near z.) By the same reason as above, (3) holds locally over z because f^*H is Cartier. Hence (3) holds over Z.

[Remark:] By (3), there exists a Cartier divisor L on Z such that $I(K + D) \sim f^*L$. (An additional property of the adjunction index I which follows from (1) of 7.3 for appropriate D and on an appropriate model of $(X, D) \to Z$.) Thus (1-3) of 7.3 hold by (1) and because $K_Z + D$ is Cartier (cf. [PSh08, Construction 7.5]). Since the 0-pair $(X/Z, D)$ is maximal lc (locally) over $\operatorname{Supp} D_{\mathrm{div}}$ and $K + D$ has Cartier index I, $Ir_P \in \mathbb{Z}$ for every prime b-divisor P over $\operatorname{Supp} D_{\mathrm{div}}$, that is, (4) of 7.3 over $\operatorname{Supp} D_{\mathrm{div}}$. Adding f^*H as above we can establish (4) of 7.3 everywhere over Z. Actually in this situation, just (1) of 7.3 implies (2-4) of 7.3 because $K_Z + D_{\mathrm{div}}$ is Cartier and $K + D$ has Cartier index I. (In general, (3) implies (1-2) and (4).)

In the proof of the step, a choice of Cartier L is crucial. The divisor L is defined up to $\sim$ on Z. Canonical divisors K, K_Z are also defined up to $\sim$ on X, Z respectively. Moreover, the linear equivalence $K \sim K'$ for K, that is, $K - K'$ should be vertical principal. Thus in $K + D \sim_I f^*(L/I)$ we can suppose that $\sim_I$, that is, $K + D - f^*(L/I)$, is fixed. This gives $\mathbb{Q}$-divisors L/I and D_{mod} with Cartier index I which are defined up to $\sim_I$. Thus the adjunction (7.1.1) become (1) in 7.3 with $\sim_I$ instead of general $\sim_{\mathbb{R}}$ (cf. [PSh08, Conjecture 7.13 and (7.13.4)]).

Step 3. (Addenda.) Addendum 32 follows from 7.1, 7.3, (3) and Theorem 8.

In Addendum 33, we can take

$$\Gamma^{\mathrm{h}}(d) = \Gamma \cap \frac{\mathbb{Z}}{I} \subseteq \mathfrak{R}'' = [0,1] \cap \frac{\mathbb{Z}}{I}$$

by (2) and (4) of 7.3 (cf. (4) above).

Addendum 34 follows from (7.2.1), Addendum 33 and 6.10.

Similarly, Addendum 35 follows from (7.2.1) and 6.8 with $\mathfrak{R}'$ defined by (6.8.1). Notice that $\mathfrak{R}'$ depends on d, Γ and $\mathfrak{R}$ because $\mathfrak{R}''$ depends on d, Γ.

Finally, all the same works for 0-contractions of bd-pairs of Addendum 36 with the additional new parameter m and the new assumption $m|I$.

$\square$

7.4 Generically crepant adjunctions

Let $f\colon (X, D) \to Z, f'\colon (X', D') \to Z'$ be two 0-contractions as in 7.1. We say that they are *birationally equivalent* or *crepant generically over* Z or Z' if there exist nonempty open subsets U, U' in X, X' respectively such that $f_{|U}\colon (X_U, D_U) \to U, f'_{|U'}\colon (X'_{U'}, D'_{U'}) \to U'$ are crepant, where $X_U = f^{-1}U, D_U = D_{|X_U}$ and $X'_{U'} = f'^{-1}U', D_{U'} = D_{|X_{U'}}$ respectively. Equivalently, $X' \to Z'$ is a model of $X \to Z$ and $\mathbb{D}^{\mathrm{h}} = \mathbb{D}'^{\mathrm{h}}$ under the birational equivalence, where $\mathbb{D}' = \mathbb{D}(X', D')$. We say also that $(X', D') \to Z'$ is a (*crepant*) *model* of $(X, D) \to Z$. In this situation, $X' \to Z'$ is a model of $X \to Z$ with given proper birational isomorphisms $X' \dashrightarrow X, g'\colon Z' \dashrightarrow Z$. Thus we can compare certain (birational) invariants of adjunction, e.g., the moduli part of adjunction $\mathcal{D}_{\mathrm{mod}}$ on Z with that of $\mathcal{D}'_{\mathrm{mod}}$ on Z', where $\mathcal{D}'_{\mathrm{mod}}$ is the moduli part of adjunction of $(X', D') \to Z'$. So, $\mathcal{D}_{\mathrm{mod}} = \mathcal{D}'_{\mathrm{mod}}$ means that $\mathcal{D}'_{\mathrm{mod}} = g'^*\mathcal{D}_{\mathrm{mod}}$. (In our applications g' is identical on a nonempty open subset U' in X'.) Notice that we can omit (2) in 7.1, the effective property of $D^{\mathrm{h}}, D'^{\mathrm{h}}$. In some applications we omit also the assumption that $X \to Z, X' \to Z'$ are regular. But we still assume that they are proper rational contractions and $f\colon (X, D) \dashrightarrow Z, f'\colon (X', D') \dashrightarrow Z'$ are proper rational 0-contractions. Moreover, the latter ones are crepant respectively over Z, Z' to 0-contraction as in 7.1 (cf. Lemma 10). Thus we apply the definition and results to the latter contractions.

The same applies to bd-pairs.

Some of important invariants of adjunction in 7.1 depend on $(X, D) \to Z$ only generically over Z. The same applies to bd-pairs.

Proposition 13. *Let $f\colon (X, D) \to Z, f'\colon (X', D') \to Z'$ be two birationally equivalent generically over Z 0-contractions as in 7.1. Then they have the same following invariants under the birational equivalence:*

(1) *the moduli part of adjunction:*

$$\mathcal{D}_{\mathrm{mod}} = \mathcal{D}'_{\mathrm{mod}} \text{ and } D'_{\mathrm{mod}} = (\mathcal{D}_{\mathrm{mod}})_{Z'};$$

(2) *if*

$$K + D \sim_{\mathbb{Q}} f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) \text{ and } \mathbb{K} + \mathbb{D} \sim_{\mathbb{Q}} f^*(\mathbb{K}_Z + \mathbb{D}_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$$

then respectively

$$K_{X'} + D' \sim_{\mathbb{Q}} f^*(K_{Z'} + D'_{\mathrm{div}} + D'_{\mathrm{mod}}) \text{ and } \mathbb{K}_{X'} + \mathbb{D}' \sim_{\mathbb{Q}} f^*(\mathbb{K}_{Z'} + \mathbb{D}'_{\mathrm{div}} + \mathcal{D}'_{\mathrm{mod}});$$

the same holds for $\sim_I$ instead of $\sim_{\mathbb{Q}}$, where I is a positive integer;

(3) *the adjunction correspondence of (b-)$\mathbb{R}$-divisors;*

(4) *the adjunction constants l_P, r_P at every vertical over Z prime b-divisor P of X; actually, l_P depend only on $f\colon X \to Z$ generically over Z;*

(5) *the adjunction index I if such one exists for either of models; and*

(6) *the horizontal b-codiscrepancies: $\mathbb{D}^{\mathrm{h}} = \mathbb{D}'^{\mathrm{h}}$.*

The same holds for bd-pairs with same $\mathcal{P}$. In (2) and (5) the bd-pairs have index $m|I$.

Proof. Up to a birational equivalence as in 7.1, we can suppose that $(X/Z, D^{\mathrm{h}})$ is equal to $(X'/Z', D'^{\mathrm{h}})$ (cf. (1) in 7.5). In this situation we can use [PSh08, Remark 7.5.1] for (1-2). Of course, we suppose that $\mathbb{K}_{X'} = \mathbb{K}$ and $\sim_{\mathbb{R}}$ (respectively $\sim_{\mathbb{Q}}, \sim_I$) is the same under the birational equivalence.

(3) holds by definition (see 7.5 below).

For (4) we can use reduction to a 1-dimensional base Z [PSh08, Remark 7.3, (i)]. In this case l_P depends only on $X \to Z$ and r does so additionally on D^{h}. Indeed, c_Q depend on D but $D + c_Q f^*Q$ depend only on D^{h}.

(5-6) hold by definition and already established facts about other invariants.

The same works for bd-pairs. □

Corollary 12. *Let $(X, D) \to Z$ be a 0-contraction as in 7.1 and additionally D^{h} be a $\mathbb{Q}$-divisor. Then (7.1.1) holds with $\sim_{\mathbb{Q}}$ instead of $\sim_{\mathbb{R}}$, $\mathcal{D}_{\mathrm{mod}}$ is a b-$\mathbb{Q}$-divisor and every adjunction constant r_P is rational.*

The same holds for bd-pairs with a b-$\mathbb{Q}$-divisor $\mathcal{P}$.

Proof. We can suppose that D' is a $\mathbb{Q}$-divisor. Thus (7.1.1) holds with $\sim_{\mathbb{Q}}$, $\mathcal{D}_{\mathrm{mod}}$ is a b-$\mathbb{Q}$-divisor and all c_Q, r_P are rational by construction.

Similarly we can treat bd-pairs.

□

7.5 Adjunction correspondence of divisors

Let $f\colon (X, D) \to Z$ be a 0-contraction as in 7.1. We say now that b-$\mathbb{R}$-divisor $\mathbb{D}_{\mathrm{div}}$ *corresponds under adjunction* to the b-$\mathbb{R}$-divisor $\mathbb{D} = \mathbb{D}(X, D)$, the b-codiscrepancy of (X, D). We can formally extend this correspondence to all b-$\mathbb{R}$-divisors $\mathbb{D}_Z$ of Z: the *adjunction correspondent* to $\mathbb{D}_Z$ on X is the b-$\mathbb{R}$-divisor $\mathbb{D}'$ of X such that

(1) $\mathbb{D}'^{\mathrm{h}} = \mathbb{D}^{\mathrm{h}}$; and

(2) $\mathbb{D}'^{\mathrm{v}}$ can be determined by the adjunction correspondence of multiplicities in 7.2: for every vertical over Z prime b-divisor P,

$$\mathrm{mult}_P \mathbb{D}' = d'_P = r_P - l_P + l_P d_Q \text{ (cf. (7.2.2))}$$

where r_P, l_P are the adjunction constants of $(X, D) \to Z$ at P, $Q = f(P)$ and $d_Q = \operatorname{mult}_Q \mathbb{D}_Z$.

$\mathbb{D}'$ is a b-$\mathbb{R}$-divisor but it is not necessarily satisfies BP as the b-discrepancy $\mathbb{D}$ (cf. (3) below). If we consider the correspondence for all b-$\mathbb{R}$-divisors $\mathbb{D}_Z$ of Z then their image consists of b-$\mathbb{R}$-divisors $\mathbb{D}'$ of X which satisfy (1) of the definition and $\mathbb{D}' - \mathbb{D} = f^*\mathbb{D}'_Z$ for some b-$\mathbb{R}$-divisor $\mathbb{D}'_Z$ of Z. The pull back f^* here is birational, that is, for every model $f' \colon X' \to Z'$ of $X \to Z$,

$$\mathbb{D}'_{X'} - \mathbb{D}_{X'} = f^*(\mathbb{D}'_{Z,Z'}) \tag{7.5.1}$$

generically over divisorial points of Z. Such a divisor $\mathbb{D}'_Z$ is unique and $= \mathbb{D}_Z - \mathbb{D}_{\text{div}}$ by (7.2.1-7.2.2).

Equivalently, we can define the inductive limit for the correspondence between $\mathbb{R}$-divisors in the opposite directions. For every model $X' \to X$ of $X \to Z$, which is isomorphic to $X \to Z$ generically over Z, and every b-$\mathbb{R}$-divisor $\mathbb{D}'$ under above assumptions (in the image), $(X', D') \to Z'$ is a 0-contraction as in 7.1 generically over divisorial points of Z', where $D' = \mathbb{D}'_{X'}$. Thus the $\mathbb{R}$-divisor D'_{div} is well-defined and *adjunction corresponds* to the $\mathbb{R}$-divisor D'. Since every prime b-divisor Q is a divisor on Z' for an appropriate model $X' \to Z'$, $\mathbb{D}'_{\text{div}}$ is well-defined and adjunction corresponds to $\mathbb{D}'$: $\mathbb{D}'_{\text{div}} = \mathbb{D}_Z$. We usually denote by $\mathbb{D}'_{\text{div}}$ the adjunction correspondent b-$\mathbb{R}$-divisor to $\mathbb{D}'$, that is, the *divisorial part of adjunction* for $f \colon (X, \mathbb{D}') \to Z$. According to the above (cf. (7.5.1))

$$\mathbb{D}' = \mathbb{D} + f^*(\mathbb{D}'_{\text{div}} - \mathbb{D}_{\text{div}}). \tag{7.5.2}$$

Equivalently, the adjunction correspondence can be defined and determined by the *actual* adjunction formula (cf. (7.1.2)):

$$\mathbb{K} + \mathbb{D}' \sim_{\mathbb{R}} f^*(\mathbb{K}_Z + \mathbb{D}'_{\text{div}} + \mathcal{D}_{\text{mod}}), \tag{7.5.3}$$

where $\mathbb{K}_Z$ denotes a canonical b-divisor of Z. The moduli part of adjunction independent of $\mathbb{D}'$: $\mathcal{D}'_{\text{mod}} = \mathcal{D}_{\text{mod}}$. By definition, Corollary 12 and 7.3, (1), in the adjunction formula $\sim_{\mathbb{R}}$ can be replaced by $\sim_{\mathbb{Q}}$, if $D^{\text{h}} \in \mathbb{Q}$, and even by $\sim_I$, if $(X, D) \to Z$ has an adjunction index I (cf. Proposition 13, (2)).

A similar construction works a 0-contraction $(X, D + \mathcal{P}) \to Z$ with a bd-pair $(X, D + \mathcal{P})$ as in 7.1. We denote by $(\mathbb{D}' + \mathcal{P})_{\text{div}}$ the b-$\mathbb{R}$-divisor adjoint correspondent to $\mathbb{D}'$. However, in this situation $\mathbb{D}'$ should satisfy the following assumption: for every model $X' \to Z'$ of $X \to Z$, which is isomorphic to

$X \to Z$ generically over Z, $(X', D' + \mathcal{P}) \to Z'$ is a 0-contraction as in 7.1 generically over divisorial points of Z', where $D' = \mathbb{D}'_{X'}$ In particular, for $\mathbb{D}' = \mathbb{D}(X, D + \mathcal{P}_X) - \mathcal{P}$, $(\mathbb{D}' + \mathcal{P})_{\mathrm{div}} = (\mathbb{D}(X, D + \mathcal{P}_X))_{\mathrm{div}} = (D + \mathcal{P})_{\mathrm{div}}$ holds, that is, the b-divisorial part of $(X, D + \mathcal{P})$ adjunction corresponds to the b-divisorial part of adjunction of $(X, D + \mathcal{P}) \to Z$. In general

$$\mathbb{K} + \mathbb{D}' + \mathcal{P} \sim_{\mathbb{R}} f^*(\mathbb{K}_Z + (\mathbb{D}' + \mathcal{P})_{\mathrm{div}} + (D + \mathcal{P})_{\mathrm{mod}})$$

with the moduli part $(\mathbb{D}' + \mathcal{P})_{\mathrm{mod}} = (\mathbb{D}(X, D + \mathcal{P}_X))_{\mathrm{mod}} = (D + \mathcal{P})_{\mathrm{mod}}$ independent of $\mathbb{D}'$.

The correspondence for $\mathbb{R}$-divisors is considered for any fixed model $X' \to Z'$ of $X \to Z$ and generically over divisorial points. In this situation $\mathbb{D}'$ satisfies BP (generically over divisorial points), e.g., by (3) below because every $\mathbb{R}$-divisor in divisorial points satisfies BP and additionally is $\mathbb{R}$-Cartier.

The correspondence has the following properties.

(1) *Injectivity.* The adjunction correspondence of b-$\mathbb{R}$-divisors and $\mathbb{R}$-divisors of Z' is 1-to-1 on its image in b-$\mathbb{R}$-divisors and $\mathbb{R}$-divisors of X' respectively.

Proof. Immediate by (7.2.1-7.2.2). □

(2) *Linearity.* The correspondence is a affine $\mathbb{R}$-linear isomorphisms of affine $\mathbb{R}$-spaces.

Proof. Immediate by (7.5.2). □

(3) *BP.*

$$\mathbb{D}' \text{ satisfies BP} \ \Leftrightarrow \ \text{so does } \mathbb{D}'_{\mathrm{div}}.$$

This gives a 1-to-1 affine $\mathbb{R}$-linear isomorphism of affine $\mathbb{R}$-spaces of b-$\mathbb{R}$-divisors under BP. Recall that BP (boundary property) of a b-$\mathbb{R}$-divisor $\mathbb{D}$ of X means that there exists a model X' of X such that $\mathbb{D} = \mathbb{D}(X', \mathbb{D}_{X'})$ [Sh03, p. 125]. Equivalently, $\mathbb{K} + \mathbb{D}$ is b-$\mathbb{R}$-Cartier. In particular, if $\mathbb{K} + \mathbb{D} \sim_{\mathbb{R}} 0$, b-$\mathbb{R}$-principal, or $\sim_{\mathbb{R},Z} 0$, relative b-$\mathbb{R}$-principal, then $\mathbb{D}$ satisfies BP.

If $\mathbb{D}'$ satisfies BP than there exists a 0-contraction $(X', D') \to Z'$ as in 7.1, with $D' = \mathbb{D}'_{X'}$ and $\mathbb{D}' = \mathbb{D}(X', D')$, which generically over Z' is isomorphic to $(X, D) \to Z$ and hence is generically crepant to the latter. The new contraction $(X', D') \to Z'$ gives the same adjunction correspondence (see Proposition 13, (3)). In particular, all facts that are stated for $\mathbb{D}$ can be applied to $\mathbb{D}'$ under BP with (X', D') as in this paragraph.

Proof. On X this is the affine $\mathbb{R}$-space of b-$\mathbb{R}$-divisors $\mathbb{D}+\mathcal{C}$, where $\mathcal{C}$ is vertical b-$\mathbb{R}$-Cartier over Z, that is, $\mathcal{C}=f^*\mathcal{C}_Z$ for some b-$\mathbb{R}$-Cartier $\mathcal{C}_Z$ of Z. So, on Z this is the affine $\mathbb{R}$-space of b-$\mathbb{R}$-divisors $\mathbb{D}_{\mathrm{div}}+\mathcal{C}_Z$, where $\mathcal{C}_Z$ is b-$\mathbb{R}$-Cartier. In both cases BP follows from BP for (X,D) and $(Z,\mathbb{D}_{\mathrm{div}})$ respectively.

The last statement about the correspondence means the change of the origin in both affine R-spaces: $\mathbb{D}',\mathbb{D}'_{\mathrm{div}}$ instead of $\mathbb{D},\mathbb{D}_{\mathrm{div}}$ respectively (cf. (7.5.2)):

$$\mathbb{D}'-\mathbb{D}=f^*(\mathbb{D}'_{\mathrm{div}}-\mathbb{D}_{\mathrm{div}}),$$

where both differences are b-$\mathbb{R}$-Cartier.

$\square$

(4) *Monotonicity.*

$$\mathbb{D}''\ge\mathbb{D}'\Leftrightarrow\mathbb{D}''_{\mathrm{div}}\ge\mathbb{D}'_{\mathrm{div}};$$

and

$$\mathbb{D}''_{X'}=D''_{X'}\ge D'\Leftrightarrow D''_{\mathrm{div},Z'}\ge D'_{\mathrm{div},Z'}$$

Proof. Immediate by (7.2.2) and the equation (1) in the definition of the correspondence. $\square$

(5) *Rationality.* If D^{h} is a $\mathbb{Q}$-divisor then

$$\mathbb{D}' \text{ is a } \mathbb{Q}\text{-divisor } \Leftrightarrow \mathbb{D}'_{\mathrm{div}} \text{ is a } \mathbb{Q}\text{-divisor};$$

and

$$D' \text{ is a } \mathbb{Q}\text{-divisor } \Leftrightarrow D'_{\mathrm{div},Z'} \text{ is a } \mathbb{Q}\text{-divisor}.$$

That is the correspondence is $\mathbb{Q}$-linear.

Proof. Immediate by (7.2.1). Indeed, by definition every constant $r=r_P\in\mathbb{Q}$ in this case (cf. Addendum 33, Corollary 12 and [PSh08, Lemma 7.4, (iv)]). $\square$

(6) *Klt, lc and nonlc.* (Cf. the strict δ-lc property in Definition 10.)

$$(X,\mathbb{D}') \text{ is lc (respectively nonlc) } \Leftrightarrow (Z,\mathbb{D}'_{\mathrm{div}}) \text{ is lc (respectively nonlc)};$$

$$(X',D') \text{ is lc (respectively nonlc) } \Leftrightarrow (Z',D'_{\mathrm{div},Z'}) \text{ is lc (respectively nonlc)};$$

$$(X,\mathbb{D}') \text{ is klt over } Z \Leftrightarrow (Z,\mathbb{D}'_{\mathrm{div}}) \text{ is klt};$$

$$(X', D') \text{ is klt over } Z' \Leftrightarrow (Z', D'_{\mathrm{div},Z'}) \text{ is klt.}$$

If $(X, \mathbb{D}')$ is klt generically over Z, that is, $\mathbb{D}^{\mathrm{h}}$ is klt (horizontally), then

$$(X, \mathbb{D}') \text{ is klt } \Leftrightarrow (Z, \mathbb{D}'_{\mathrm{div}}) \text{ is klt;}$$

$$(X', D') \text{ is klt } \Leftrightarrow (Z', D'_{\mathrm{div},Z'}) \text{ is klt.}$$

The lc (klt) property here is formal: for all prime b-divisors P of X, $\mathrm{mult}_P \mathbb{D}' \leq 1$ (respectively < 1) (cf. [Sh96, Example 1.1.2]).

Proof. Immediate by 6.4-6.5 and (7.2.1-7.2.2). By the equation (1) and 7.1, (1), it is enough to consider only vertical prime b-divisors P of X. In this case, by 7.2, $b = d_P \leq r = r_P$ ($< r = r_P$) means that $(X', D'_{X'})$ is lc (respectively klt over Z').

□

(7) $\sim_{\mathbb{R},S} 0$. For a proper morphism $Z \to S$ to a scheme S,

$$\mathbb{K} + \mathbb{D} \sim_{\mathbb{R},S} 0 \Leftrightarrow \mathbb{K}_Z + \mathbb{D}_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}} \sim_{\mathbb{R},S} 0;$$

or equivalently,

$$K + D \sim_{\mathbb{R},S} 0 \Leftrightarrow K_Z + D_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}} \sim_{\mathbb{R},S} 0;$$

where we can replace $X \to Z$ by any its model $X' \to Z'/S$ and $\sim_{\mathbb{R},S}$ by $\equiv$ over S ($\sim_{\mathbb{Q},S}, \sim_{I,S}$ if respectively $D^{\mathrm{h}} \in \mathbb{Q}$, the 0-contraction $(X, D) \to Z$ has the adjunction index I).

We can replace $\sim_{\mathbb{R},S} 0$ by the $\mathbb{R}$-free or -antifree property over S. Respectively, we can replace the $\mathbb{R}$-free or -antifree property over S by the nef or antinef property over S, by $\mathbb{Q}(I)$-free or -antifree property over S if $D^{\mathrm{h}} \in \mathbb{Q}$ (respectively, $(X, D) \to Z$ has the adjunction index I).

Proof. Immediate by the adjunction formula (7.5.3) and a relative over S $\mathbb{R}$-version of [Sh19, Proposition 3]. Notice also that $(X', D_{X'}) \to Z'$ with $D_{X'} = \mathbb{D}_{X'}$ is a 0-contraction as in 7.1, except for (2) in 7.1 but it is crepant to $(X, D) \to Z$ and satisfies the same adjunction properties by Proposition 13.

□

(8) $\mathbb{R}$-*complements*. For a proper morphism $Z \to S$ to a scheme S,

$$\begin{gathered}(X/S, \mathbb{D}^+) \text{ is a (b-)}\mathbb{R}\text{-complement of } (X/S, \mathbb{D}) \\ \Updownarrow \\ (Z/S, \mathbb{D}^+_{\text{div}} + \mathcal{D}_{\text{mod}}) \text{ is a (b-)}\mathbb{R}\text{-complement of } (Z/S, \mathbb{D}_{\text{div}} + \mathcal{D}_{\text{mod}}).\end{gathered}$$

So,

$(X/S, \mathbb{D})$ has a (b-)$\mathbb{R}$-complement $\Leftrightarrow$ $(Z/S, \mathbb{D}_{\text{div}}+\mathcal{D}_{\text{mod}})$ has a (b-)$\mathbb{R}$-complement.

Equivalently, for $\mathbb{R}$-divisors,

$$\begin{gathered}(X/S, D^+) \text{ is an } \mathbb{R}\text{-complement of } (X/S, D) \\ \Updownarrow \\ (Z/S, D^+_{\text{div}} + \mathcal{D}_{\text{mod}}) \text{ is an } \mathbb{R}\text{-complement of } (Z/S, D_{\text{div}} + \mathcal{D}_{\text{mod}});\end{gathered}$$

$(X/S, D)$ has an $\mathbb{R}$-complement $\Leftrightarrow$ $(Z/S, D_{\text{div}}+\mathcal{D}_{\text{mod}})$ has an $\mathbb{R}$-complement.

Proof. Immediate by definition (see also Remark 4, (2)), (4) and (6-7). Notice that (X, D) is a log pair by 7.1, (3) and $(Z, D_{\text{div}} + \mathcal{D}_{\text{mod}})$ is a log bd-pair by construction.

$\square$

For n-complements, we have only inverse results (10) below for b-n-complements.

(9) *Inverse inequality* (1-b) *of Definition* 3. Suppose additionally that $(X, D) \to Z$ has an adjunction index I. Let n be a positive integer and $I|n$. Then

$$\begin{gathered}\mathbb{D}^+ \text{ is lc and satisfies (1-b) of Definition 3 with respect to } \mathbb{D} \\ \Uparrow \\ \mathbb{D}^+_{\text{div}} \text{ satisfies (1-b-2) of Definition 5 with respect to } \mathbb{D}_{\text{div}}.\end{gathered}$$

Equivalently,

$$\begin{gathered}(X, D^+) \text{ satisfies (1-b-2) of Definition 3 with respect to } (X, D) \\ \Uparrow \\ (Z, D^+_{\text{div}} + \mathcal{D}_{\text{mod}}) \text{ satisfies (1-b-2) of Definition 5 with respect to } (Z, D_{\text{div}} + \mathcal{D}_{\text{mod}}).\end{gathered}$$

Additionally,

$$\begin{gathered}\mathbb{D}^+ \geq \mathbb{D} \\ \Updownarrow \\ \mathbb{D}^+_{\mathrm{div}} \geq \mathbb{D}_{\mathrm{div}}.\end{gathered}$$

Proof. Immediate by 6.14 and 7.2. Indeed, for every vertical prime b-divisor P, $Ir = Ir_P \in \mathbb{Z}$ by (4) in 7.3 and $r \leq 1$ by 7.2. For corresponding $Q = f(P)$, $d^+ = d^+_Q = \mathrm{mult}_Q \mathbb{D}^+_{\mathrm{div}} \leq 1$ by (2) of Definition 5. This and 7.2 prove (1-b) of Definition 3 and that $\mathbb{D}^+$ is lc for vertical P. For a horizontal prime b-divisor P, we can use the equation (1): $\mathbb{D}^{+\mathrm{h}} = \mathbb{D}^{\mathrm{h}}$. Then $\mathbb{D}^{+\mathrm{h}}$ is lc by (1) in 7.1 and satisfies (1-b) of Definition 3 with respect to $\mathbb{D}^{\mathrm{h}} = \mathbb{D}^{+\mathrm{h}}$ by the equation (1), Example 6, (1) and (2) in 7.3. The additional equivalence of the monotonicity of b-n-complements follows from (4).

Notice also that (X, D) is a log pair by 7.1, (3) and $(Z, D_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$ is a log bd-pair by construction.

∎

(10) *Inverse* b-*n-complements*. Under the assumptions of (9), for a proper morphism $Z \to S$ to a scheme S,

$$\begin{gathered}(X/S, \mathbb{D}^+) \text{ is a (b-)}n\text{-complement of } (X/S, \mathbb{D}) \\ \Uparrow \\ (Z/S, \mathbb{D}^+_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}}) \text{ is a (b-)}n\text{-complement of } (Z/S, \mathbb{D}_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}}).\end{gathered}$$

So,

$(X/S, \mathbb{D})$ has a (b-)n-complement $\Leftarrow$ $(Z/S, \mathbb{D}_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$ has a (b-)n-complement.

Equivalently,

$$\begin{gathered}(X/S, D^+) \text{ is a b-}n\text{-complement of } (X/S, D) \\ \Uparrow \\ (Z/S, D^+_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}}) \text{ is a b-}n\text{-complement of } (Z/S, D_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}});\end{gathered}$$

$(X/S, D)$ has a b-n-complement $\Leftarrow$ $(Z/S, D_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$ has a b-n-complement.

Moreover, if the b-n-complement $(Z/S, D^+_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$ is monotonic, equivalently, $(Z/S, \mathbb{D}^+_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$ is monotonic, then so does $(X/S, D^+)$, equivalently, $(X/S, \mathbb{D}^+)$.

Proof. Immediate by (4),(6) and (9). As a definition in (8-10) for b-$\mathbb{R}$-divisors we use Definitions 3 and 5 with $\mathbb{D}, \mathbb{D}^+, \mathbb{D}_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}}, \mathbb{D}^+_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}}$ instead of $D, D^+, D_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}}, D^+_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}}$. □

However, in applications we use a more technical result, Theorem 10 below and its addenda.

(11) *Direct effectivity.* The vertical effective (b-)support of $\mathbb{D}'$ goes to the effective (b-)support of $\mathbb{D}'_{\mathrm{div}}$, that is, for every vertical prime b-divisor P of X,

$$\mathrm{mult}_P \mathbb{D}' \geq 0 \Rightarrow \mathrm{mult}_Q \mathbb{D}'_{\mathrm{div}} \geq 0,$$

where $Q = f(P)$. In particular,

$$D'_{X'} \geq 0 \Rightarrow D'_{\mathrm{div},Z'} \geq 0.$$

Proof. Immediate by 6.6 and (7.2.1). Indeed, every $r = r_P \leq 1$ by 7.2. □

(12) *Direct boundary property.* Suppose additionally that $(X, \mathbb{D}')$ is lc or has an $\mathbb{R}$-complement over some scheme S. Then the vertical boundary (b-)support of $\mathbb{D}'$ goes to the boundary (b-)support of $\mathbb{D}'_{\mathrm{div}}$, that is, for every vertical prime b-divisor P of X,

$$\mathrm{mult}_P \mathbb{D}' \in [0,1] \Rightarrow \mathrm{mult}_Q \mathbb{D}'_{\mathrm{div}} \in [0,1],$$

where $Q = f(P)$. In particular,

$$D'_{X'} \in [0,1] \Rightarrow D'_{\mathrm{div},Z'} \in [0,1].$$

Proof. Immediate by (6) and (11) (cf. 6.7). Note also that the existence of an $\mathbb{R}$-complement implies that $(X, \mathbb{D}')$ is lc (cf. Remark 1, (1)). □

Converses of the directed properties do not hold in general.

(13) *bd-Pairs.* All the above properties of the adjunction correspondence hold for a 0-contraction with a bd-pair $(X, D + \mathcal{P})$ as in 7.1. Notice that the correspondence is only between (b-)-divisorial parts, $\mathcal{P}$ and $\mathcal{D}_{\mathrm{mod}}$ are fixed (modulo $\sim_{\mathbb{R}}$ or $\sim_{\mathbb{Q}}, \sim_I$ if $\mathcal{P}$ is b-$\mathbb{Q}$-Cartier,$I\mathcal{P}$ is b-Cartier respectively).

For a bd-pair $(X, \mathbb{D} + \mathcal{P})$ in (3), (bd-)BP means that there exists a model $(X', \mathbb{D}_{X'} + \mathcal{P})$ which is a log bd-pair and $\mathbb{D} = \mathbb{D}(X', \mathbb{D}_{X'} + \mathcal{P}_{X'}) - \mathcal{P}$, the divisorial part of the latter bd-pair.

In (5) to add that $\mathcal{P}$ is b-$\mathbb{Q}$-Cartier.

In (6-7) $\mathbb{D} + \mathcal{P}, D + \mathcal{P}, \mathbb{D}' + \mathcal{P}, D' + \mathcal{P}$ should be instead of $\mathbb{D}, D, \mathbb{D}', D'$ respectively. Similarly, in other properties: sometimes $+\mathcal{P}$ is needed, sometimes not.

Theorem 10. *Let I, n be two positive integers such that $I|n$ and Φ, Φ' be two hyperstandard sets as in 6.8 with $\mathfrak{R}'' = [0,1] \cap (\mathbb{Z}/I)$. Let $(X, B) \to Z/S$ be a 0-contraction as in 7.1 and additionally of the adjunction index I, with a boundary B and a proper morphism $Z \to S$ to a scheme S. Suppose also that (X, B) is lc or has an $\mathbb{R}$-complement over S, and Z/S has a model Z'/S and an $\mathbb{R}$-divisor D'_{div} on Z' such that*

(1) *for every vertical over Z prime divisor P on X, the image $Q = f(P)$ of P on Z' is a divisor and*

$$d'_{\mathrm{div},Q} \geq b_{Q,n\_\Phi'}$$

where $d'_{\mathrm{div},Q} = \mathrm{mult}_Q D'_{\mathrm{div}}$ and $b_Q = \mathrm{mult}_Q \mathbb{B}_{\mathrm{div}}$;

(2) *$K_{Z'} + D'_{\mathrm{div}} + B_{\mathrm{mod},Z'}$ is antinef over S; and*

(3) *$(Z'/S, D'_{\mathrm{div}} + \mathcal{B}_{\mathrm{mod}})$ has a b-n-complement $(Z'/S, D^+_{\mathrm{div}} + \mathcal{B}_{\mathrm{mod}})$.*

Then $(X/S, D^+)$ is respectively a b-n-complement of $(X^\sharp/S, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ and an n-complement of $(X/S, B)$, where $D^+ = \mathbb{D}^+_X$ and $\mathbb{D}^+$ adjunction corresponds to $\mathbb{D}^+_{\mathrm{div}}$.

Addendum 37. *If additionally*

(4)

$$d^+_{\mathrm{div},Q} \geq b_{Q,n\_\Phi'}$$

for every Q as in (1) of the theorem, where $d^+_{\mathrm{div},Q} = \mathrm{mult}_Q D^+_{\mathrm{div}}$.

Then we can take $D'_{\mathrm{div}} = D^+_{\mathrm{div}}$ and the monotonicity

(5) $D^+ \geq B_{n\_\Phi}{}^\sharp \geq B_{n\_\Phi}$

holds. For this addendum, it is enough to be an n-complement in (3).

Addendum 38. *The same holds for every contraction $(X, B+\mathcal{P}) \to Z/S$ as in 7.1 and additionally of the adjunction index I with a bd-pair $(X/S, B+\mathcal{P})$ of index $m|I$, with a boundary B and proper $Z \to S$.*

Proof. Step 1. $b_Q \in [0,1]$. Immediate by (12). Thus the assumption (1) of the theorem is meaningful.

Step 2. $(X/S, \mathbb{D}^+)$ is a b-n-complement of $(X/S, \mathbb{D}')$, where $\mathbb{D}'$ adjunction corresponds to $\mathbb{D}'_{\mathrm{div}} = \mathbb{D}(Z', D'_{\mathrm{div}} + B_{\mathrm{mod},Z'}) - \mathcal{B}_{\mathrm{mod}}$. Immediate by (3) and the property (10), Inverse b-n-complements. However to apply (10) note that by definition, the BP property (3) of the adjunction correspondence and the assumption (2) of the theorem $\mathbb{D}', \mathbb{D}'_{\mathrm{div}}$ satisfy BP and there exists generically crepant model $(X', D') \to Z'/S$ of $(X, B) \to Z/S$ with $D' = \mathbb{D}'_{X'}, \mathbb{D}' = \mathbb{D}(X', D')$ and of the same adjunction index I by Proposition 13, (5). But D' and D'_{div} are not necessarily boundaries.

Step 3. $\mathbb{B}_{n\_\Phi}{}^\sharp \le \mathbb{D}'$ by (1-2) and Corollary 8. Indeed, the assumption (1) of the corollary follows from the assumption (2) of the theorem by the property (7) of the adjunction correspondence with b-antinef $\mathbb{K}_{Z'} + \mathbb{D}'_{\mathrm{div}} + B_{\mathrm{mod},Z'}$ over S. The BP b-$\mathbb{R}$-divisor $\mathbb{D}'$ corresponds to $\mathbb{D}'_{\mathrm{div}}$.

We take a maximal model $(X^\sharp/S, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ of Construction 2 as a log pair $(X/S, D)$ of Corollary 8. We suppose also that $X \dashrightarrow X^\sharp$ is a birational 1-contraction, that is, does not blow up divisors, and $B_{n\_\Phi}{}^\sharp{}_{X^\sharp} = B_{n\_\Phi,X^\sharp}$ has the same divisorial multiplicities as B_{n_Φ}. By construction $(X^\sharp/S, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ is a log pair and $\mathbb{B}_{n\_\Phi}{}^\sharp = \mathbb{D}(X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$. The inequality (2) of the corollary

$$B_{n\_\Phi}{}^\sharp{}_{X^\sharp} \le \mathbb{D}'_{X^\sharp} \tag{7.5.4}$$

follows from the equation (1) in the definition of the adjunction correspondence and the assumption (1) of the theorem by 6.9. Indeed, by construction and definition $\mathbb{D}'_{X^\sharp}{}^{\mathrm{h}} = B_{X^\sharp}{}^{\mathrm{h}}$, where $B_{X^\sharp} = \mathbb{B}_{X^\sharp}$. Thus, for the horizontal part of (7.5.4) over Z,

$$B_{n\_\Phi}{}^\sharp{}_{X^\sharp}{}^{\mathrm{h}} = B_{n\_\Phi,X^\sharp}{}^{\mathrm{h}} \le B_{X^\sharp}{}^{\mathrm{h}} = \mathbb{D}'_{X^\sharp}{}^{\mathrm{h}}.$$

The vertical part of (7.5.4) means that for every vertical over Z prime divisor P of $X^\sharp$,

$$b_{P,n\_\Phi} = b_{P,n\_\Phi}{}^\sharp \le d'_P, \tag{7.5.5}$$

where $b_P = \mathrm{mult}_P\, B = \mathrm{mult}_P\, B_{X^\sharp}, b_{P,n\_\Phi}{}^\sharp = \mathrm{mult}_P\, B_{n\_\Phi}{}^\sharp{}_{X^\sharp}$ and $d'_P = \mathrm{mult}_P\, \mathbb{D}'$. Note for this that by construction P is also a divisor on X and $B_{n\_\Phi}{}^\sharp{}_{X^\sharp} = B_{n\_\Phi,X^\sharp}$. Since B is a boundary and (X, B) is lc or has an $\mathbb{R}$-complement over S, $0 \le b_P \le r_p \le 1$ by 7.2. Thus (7.5.5) is meaningful. To verify (7.5.5) we apply 6.9 to $b_1 = b_P, r = r_P, l = l_P$ and $d' = \mathrm{mult}_Q\, D'_{\mathrm{div}}$, where $Q = f(P)$. The image Q is a prime divisor on Z' by (1) of the theorem. In this situation (6.9.1) of 6.9 means the inequality in (1) of the theorem. Indeed, by construction d_1 corresponds to $b_1 = b_P$ by (6.0.10) with $r = r_P, l = l_P$ and $r \in \mathfrak{R}''$ by 7.3, (4) and assumptions of the theorem. Thus by 7.2 and construction $d_1 = b_Q$. The other assumptions and notation of 6.9 hold by assumptions and notation of the theorem. By construction b' corresponds to $d' = \mathrm{mult}_Q D'_{\mathrm{div}}$ by (6.0.11) with $r = r_P, l = l_P$. Thus $d' = d'_P$ again by 7.2. Now since $b_1 = b_P$ and $b' = d'_P$, (6.9.2) means exactly (7.5.5).

Step 4. Steps 2-3 and Proposition 1 imply that $(X/S, \mathbb{D}^+)$ is a (b-)n-complement of $(X/S, \mathbb{B}_{n\_\Phi}{}^\sharp)$. Or, equivalently, $(X/S, D^+)$ is a b-n-complement of $(X^\sharp/S, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$, actually, of any maximal model of $(X/S, B_{n\_\Phi})$ in Construction 2. (In particular, we can replace X'/S by $X^\sharp/S$ for those b-n-complements.) Additionally, $(X/S, D^+)$ is an n-complement of $(X/S, B)$ by Example 6, (1) and Proposition 7.

Step 5. (Addenda.) In Addendum 37 we can replace D'_{div} by D^+_{div}. Indeed, (1) holds by (4). In (2) $K_{Z'} + D^+_{\mathrm{div}} + B_{\mathrm{mod},Z'} \sim_n 0/S$ and is antinef over S by Definition 5, (3). And, finally, (3) holds by (3) and Example 6, (2). In this situation it is enough to be an n-complement in (3).

The monotonicity (5) follows from Proposition 8 and Step 3. Indeed, in the addenda $\mathbb{D}' = \mathbb{D}^+$ adjunction corresponds to $\mathbb{D}'_{\mathrm{div}} = \mathbb{D}^+$.

The proof of the addendum for bd-pairs, is similar to the above proof for usual pairs.

□

7.6 Adjunction on divisor

Under the divisorial adjunction $\Gamma(\mathcal{N}, \Phi)$ goes to $\Gamma(\mathcal{N}, \widetilde{\Phi})$: for every (finite) set of positive integers $\mathcal{N}$ and hyperstandard sets $\Phi, \widetilde{\Phi}$ as in 6.11,

$$B \in \Gamma(\mathcal{N}, \Phi) \Rightarrow B_{S^\nu} \in \Gamma(\mathcal{N}, \widetilde{\Phi}), \qquad (7.6.1)$$

where B is a boundary of a pair $(X, B + S)$, lc in codimension ≤ 2, S is a reduced prime divisor of $B + S$, S^ν is a normalization of S and (S^ν, B_{S^ν}) is the adjoint pair of (X, B), that is, $B_{S^\nu} = \mathrm{Diff}\, B$, (see [Sh92, 3.1]). In particular, $S^\nu = S$ when S is normal. The property (7.6.1) is immediate by [Sh92, Corollary 3.10] and 6.11.

Theorem 11. *Let n be a positive integer and $(X/Z \ni o, B)$ be a pair with a boundary B such that*

(1) *$X/Z \ni o$ is proper with connected X_o;*

(2) *(X, B) is plt with a reduced divisor S of B over $Z \ni o$;*

(3) *$-(K + B)$ is nef and big over $Z \ni o$; and*

(4) *the adjoint pair* $(S/Z \ni o, B_S)$ *has a b-n-complement* $(S/Z \ni o, B_S^+)$.

Then there exists a b-n-complement $(X/Z \ni o, B^+)$ *of* $(X/Z \ni o, B)$ *with an adjunction extension* B^+ *of* B_S^+*, that is,* $\operatorname{mult}_S B^+ = 1$ *and*

(5)
$$\operatorname{Diff}(B^+ - S) = B_S^+.$$

The same holds for a bd-pair $(X/Z \ni o, B + \mathcal{P})$ *of index* $m|n$.

Note that by (3) S is actually unique and irreducible over $Z \ni o$ by its connectedness [Sh92, Connectedness Lemma 5.7] [K, Theorem 17.4] [A14, Theorem 6.3] etc. Moreover, the central fiber S_o is also connected.

Remark: 1. B is assumed to be a boundary, in particular, effective.

2. The b-n-complement on $X/Z \ni o$ is not necessarily monotonic even if the b-n-complement in (4) is monotonic. However, for special boundary multiplicities we can get the monotonic property a posteriori [PSh08, Lemma 3.3] [B, Theorem 1.7] [HLSh, Theorem 1.6].

3. The theorem is the first main theorem (from global to local), that is, similar to [PSh01, Proposition 6.2] [P, Proposition 4.4] where the condition 4),(iv) respectively imply our (4). However the proof is similar to the proof of [Sh92, Theorem 5.12].

Proof. Step 1. For construction of an n-complement we replace (X, B) by a crepant pair (Y, D). We can suppose that $\operatorname{Supp} D$ has only normal crossings. For this we can take a log resolution of $(X/Z \ni o, D)$. Then one can verify all required assumptions (1-4), except for, the boundary property of D. In particular, (4) holds by definition (cf. Remark 4, (1)). This implies the existence of a required complement for $(Y/Z \ni o, D)$ and also $(X/Z \ni o, B)$ with (5) (cf. [Sh92, Lemma 5.4]). For this we can use the adjunction formula [Sh92, 3.1] because birational modifications of pairs and their complements are crepant and b-n-complements agree with those modifications by Remark 4, (1).

Note now that every n-complement in the normal crossing case is actually a b-n-complement by the same arguments with sufficiently high crepant modifications or by Proposition 2.

Step 2. Construction of D^+. The complete linear system $|-nK_Y - \lfloor (n+1)D \rfloor + S| = |-n(K_Y + S) - \lfloor (n+1)(D - S) \rfloor|$ on Y cuts out the complete linear system

$|-nK_S - \lfloor (n+1)D_S \rfloor|$ on S over a suitable neighborhood of o in Z. (Tacit we use the same notation S for its birational transform on Y.) Indeed,

$$\lfloor (n+1)D_S \rfloor = \left\lfloor (n+1)(D-S)_{|S} \right\rfloor = \lfloor (n+1)(D-S) \rfloor_{|S}$$

by normal crossings. By [Sh92, Corollary 3.10] $D_S = (D-S)_{|S}$ because S is nonsingular. The required surjectivity follows from the Kawamata-Viehweg vanishing theorem

$$R^1\varphi_*\mathcal{O}_Y(-nK_Y - \lfloor (n+1)D \rfloor) = R^1\varphi_*\mathcal{O}_Y(K_Y + \lceil -(n+1)(K+D) \rceil) = 0$$

by (3), where $\varphi\colon Y \to Z \ni o$. So, for every effective divisor $E_S \in |-nK_S - \lfloor (n+1)D_S \rfloor|$, there exists an effective divisor $E \in |-nK_Y - \lfloor (n+1)D \rfloor + S|$ with $E_{|S} = E_S$.

By (2) the pair $(S/Z \ni o, D_S)$ is klt and by (4) it has an n-complement $(S/Z \ni o, D_S^+)$. So, by (1) and (3) of Definition 2 (cf. also [Sh92, Definition 5.1]

$$D_S^+ = \frac{E_S}{n} + \frac{\lfloor (n+1)D_S \rfloor}{n},$$

where $E_S \in |-nK_S - \lfloor (n+1)D_S \rfloor|$. Put

$$D^+ = \frac{E}{n} + \frac{\lfloor (n+1)D \rfloor}{n} - \frac{S}{n},$$

where $E \in |-nK_Y - \lfloor (n+1)D \rfloor + S|$ with $E_{|S} = E_S$.

Actually, $(Y/Z \ni o, D^+)$ is an n-complement of $(Y/Z \ni o, D)$ and induces the n-complement $(X/Z \ni o, B^+)$ of $(X/Z \ni o, B)$ with crepant $B^+ = \psi(D^+)$, where $\psi\colon Y \to X$. The n-complement properties (2-3) of Definition 2 are equivalent for both pairs (see Step 1) and one of them we verify on X (see Step 6 below).

Step 3. Adjunction: $\operatorname{mult}_S D^+ = 1$, and by normal crossings and construction

$$\begin{aligned}\operatorname{Diff}(D^+ - S) =& \left(\frac{E}{n} + \frac{\lfloor (n+1)(D-S) \rfloor}{n}\right)_{|S} = \frac{E_S}{n} + \frac{\left\lfloor (n+1)(D-S)_{|S} \right\rfloor}{n} = \\ & \frac{E_S}{n} + \frac{\lfloor (n+1)D_S \rfloor}{n} = D_S^+.\end{aligned}$$

Step 4. Since $E \geq 0$,

$$D^+ \geq \frac{\lfloor (n+1)D \rfloor}{n} - \frac{S}{n}.$$

This implies (1) of Definition 2.

Step 5. By construction in Step 2,

$$nK_Y + nD^+ = nK_Y + E + \lfloor (n+1)D \rfloor - S \sim 0.$$

This implies (3) of Definition 2.

Step 6. It is enough to verify (2) of Definition 2 on X. Notice that (5) holds by construction and Step 3 (see also Step 1). Thus by inversion of adjunction [Sh92, 3.3-4] [K, Theorem 17.7] [A14, Theorem 6.3] (X, B^+) is lc near S. The connectedness of the lc locus (if (X, B^+) is not lc) and its lc [A14, Theorem 6.3] imply lc of (X, B^+) over $Z \ni o$, that is, (2) of Definition 2. Note that in the local case for the lc connectedness we need the connectedness of $\varphi^{-1}o$ that holds by (1).

Step 7. For a bd-pair $(X/Z \ni o, B + \mathcal{P})$, we can take in Step 1 a log resolution over which $\mathcal{P}$ is stable and has normal crossings together with D. (Actually, according to the Kawamata-Viehweg vanishing theorem, we do not need the last normal crossings because $n\mathcal{P}$ is integral: $n\mathcal{P} \in \mathbb{Z}$.) We can assume also that $\mathcal{P}_Y$ is in general position (modulo $\sim_n$) with respect to S on Y and $\mathcal{P}_Y{}_{|S}$ is well-defined. Then $\mathcal{P}_S = \mathcal{P}_{\,\shortmid S} = \overline{\mathcal{P}_X{}_{|S}}$ is stable over S (a definition $\shortmid$ see in [Sh03, Mixed restriction 7.3]) and is nef Cartier on S over $Z \ni o$ because $m|n$. Then we can use above arguments with K_Y, K_S replaced by $K_Y + \mathcal{P}_Y, K_S + (\mathcal{P}_S)_S$ respectively (cf. general ideology in Crepant bd-models in Section 3). To extend a complement as in Step 2 we need only the aggregated bd-version of (3): $-(K_Y + B + \mathcal{P}_Y)$ is nef and big over $Z \ni o$ but the b-nef property of $\mathcal{P}$ is not needed here. However, to verify the lc property of Step 6 we need the lc connectedness (if $(X, B^+ + \mathcal{P})$ is not lc) which uses the pseudoeffective modulo $\sim_\mathbb{R}$ property of $\mathcal{P}_X$ over $Z \ni o$ that follows from the b-nef property of $\mathcal{P}$ over $Z \ni o$ (cf. [FS, Theorem 1.2]).

$\square$

Corollary 13. *Let n be a positive integer and $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a (finite) set of (rational) numbers $\mathfrak{R}$ in $[0, 1]$. Let $(X/Z \ni o, B)$ be a pair such that*

(1) *$X/Z \ni o$ is proper with connected X_o;*

(2) (X, B) *is plt with a boundary* B *and a reduced divisor* S *of* B *over* $Z \ni o$;

(3) $-(K + B)$ *is nef and big over* $Z \ni o$; *and*

(4) $(S'^\sharp/Z \ni o, B_{S',n\_\widetilde{\Phi}}{}^\sharp)$ *has a b-n-complement* $(S'^\sharp/Z \ni o, B^+_{S'^\sharp})$, *where* $(S'/Z \ni o, B_{S'})$ *is a highest crepant model with a boundary of the adjoint pair* $(S/Z \ni o, B_S)$ *(cf. Construction 5 and [Sh92, (3.2.3)]) and* $\widetilde{\Phi}$ *is defined in 6.11.*

Then there exists a b-n-complement $(X/Z \ni o, B^+)$ *of* $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp)$ *with an adjunction extension* B^+ *of* B^+_S, *that is,* $\operatorname{mult}_S B^+ = 1$ *and*

(5)
$$\operatorname{Diff}(B^+ - S) = B^+_S = \mathbb{B}^+_{S'^\sharp,S}.$$

The same holds for a bd-pair $(X/Z \ni o, B + \mathcal{P})$ *of index* $m|n$.

Proof. Step 1. *Construction of* $(S'/Z \ni o, B_{S'}), (S'/Z \ni o, B_{S',n\_\widetilde{\Phi}})$. By (2) the adjoint pair $(S/Z \ni o, B_S)$ is a klt pair with normal and irreducible S over $Z \ni o$. It has finitely many prime b-divisors P over $Z \ni o$ with $\operatorname{mult}_P \mathbb{B}_S \geq 0$. A *highest* crepant model $(S', B_{S'})$ of (S, B_S) with a *boundary* $B_{S'}$ (equivalently, $B_{S'} \geq 0$) blows up those exceptional P [Sh96, Theorem 3.1] [B12, Theoem 1.1]. Usually, the blowup is not unique (and is not necessarily $\mathbb{Q}$-factorial). It is defined up to a small flop over $Z \ni o$.

After that we change the boundary $B_{S'}$ and get the pair $(S'/Z \ni o, B_{S',n\_\widetilde{\Phi}})$ with a boundary $B_{S',n\_\widetilde{\Phi}}$. (By construction $(S', B_{S'})$ is a log pair. But $(S'/Z \ni o, B_{S',n\_\widetilde{\Phi}})$ is not necessarily a log pair when S' is not $\mathbb{Q}$-factorial.)

In our applications we construct a maximal model $(S'^\sharp/Z \ni o, B_{S',n\_\widetilde{\Phi}}{}^\sharp)$ using Construction 2 (see Step 5 in the proof Theorem 14). Actually in our situation we can construct such a model too even $S/Z \ni o$ has not necessarily wFt. (But $B, B_{S'}, B_{S',n\_\widetilde{\Phi}}$ are boundaries and $(S/Z \ni o, B_S), (S', B_{S'}), (S', B_{S',n\_\widetilde{\Phi}})$ have klt $\mathbb{R}$-complements by (1-3); cf. Step 2.) However, in (4) we suppose that such a model $(S'^\sharp/Z \ni o, B_{S',n\_\widetilde{\Phi}}{}^\sharp)$ exists and it has a b-n-complement $(S'^\sharp/Z \ni o, B^+_{S'^\sharp})$.

By construction $S, S', S'^\sharp$ are birationally isomorphic. So, (5) is meaningful.

Step 2. *Reduction to a plt wlF model* $(X^\sharp/Z\ni,B_{n\_\Phi}{}^\sharp)$. By Construction 2 a required maximal model exists:

$$\begin{array}{ccc}
(X,B_{n_\Phi}) & \stackrel{\psi}{\dashrightarrow} & (X^\sharp,B_{n_\Phi}{}^\sharp)\\
\searrow & & \swarrow\\
& Z\ni o &
\end{array}$$

where ψ is a birational 1-contraction and $X^\sharp/Z\ni o$ has wFt with connected $X^\sharp_o$. (For simplicity of notation we use $B_{n\_\Phi}{}^\sharp$ instead of $B_{n\_\Phi}{}^\sharp{}_{X^\sharp}$.) Indeed, by (1) and (3), $X/Z\ni o$ has wFt with connected X_o. Since $B_{n\_\Phi}\le B, \operatorname{mult}_S B_{n\_\Phi}=\operatorname{mult}_S B=1$ and by (3), (X,B_{n_Φ}) has a plt $\mathbb{R}$-complement with the same reduced divisor S of B_{n_Φ}. In particular, S is not the base locus of $-(K+B_{n\_\Phi})$. Hence we can suppose that $(X^\sharp,B_{n\_\Phi}{}^\sharp)$ is also plt with birationally the same prime reduced divisor $S^\sharp=\psi(S)\subset X^\sharp$ of $B_{n\_\Phi}{}^\sharp$. By Construction 2 and the assumption (3) respectively, $-(K_{X^\sharp}+B_{n\_\Phi}{}^\sharp)$ is $\mathbb{R}$-free and big over $Z\ni o$. Thus $(X^\sharp/Z\ni,B_{n\_\Phi}{}^\sharp)$ is wlF (a weak log Fano variety or space).

Notice that we suppose that ψ does nor blow up divisors. Hence $B_{n\_\Phi}{}^\sharp\in\Gamma(n,\Phi)$.

Step 3. *Construction of a b-n-complement* $(X/Z\ni o,B^+)$. By birational nature of b-n-complements (see Definition 3 and Remark 4, (1)), a required b-n-complement $(X/Z\ni o,B^+)$ of $(X^\sharp/Z\ni o,B_{n\_\Phi}{}^\sharp)$ can be induced from a b-n-complement $(X^\sharp/Z\ni o,B^+_{X^\sharp})$ of $(X^\sharp/Z\ni o,B_{n\_\Phi}{}^\sharp)$, that is, $B^+=\mathbb{B}^+_{X^\sharp,X}$. To construct the latter complement we apply Theorem 11 to $(X^\sharp/Z\ni,B_{n\_\Phi}{}^\sharp)$. In Step 2 we verified assumptions (1-3) of the theorem. The conclusion (5) of the theorem follows from (5) of the corollary. So, we need only to verify (4) of the theorem.

Step 4. *Construction of a b-n-complement of* $(S^\sharp/Z\ni o,B_{n\_\Phi}{}^\sharp{}_{S^\sharp})$. Actually the b-$n$-complement $(S'^\sharp/Z\ni o,B^+_{S'^\sharp})$ induces a b-n-complement $(S^\sharp,B^+_{S^\sharp})$ of $(S^\sharp/Z\ni o,B_{n\_\Phi}{}^\sharp{}_{S^\sharp})$, that is, $B^+_{S^\sharp}=B^+_{S'^\sharp,S^\sharp}$. By construction varieties $S'^\sharp$ and $S^\sharp$ are birational isomorphic over $Z\ni o$. The (2-3) of Definition 2 for $(S^\sharp,B^+_{S^\sharp})$ follows from that of for $(S'^\sharp/Z\ni o,B^+_{S'^\sharp})$. Thus it is enough to very (1-b) of Definition 3. This follows from Proposition 1, applied to crepant models of $(S^\sharp/Z\ni o,B_{n\_\Phi}{}^\sharp{}_{S^\sharp}),(S'^\sharp/Z\ni o,B_{S',n\_\widetilde{\Phi}}{}^\sharp)$ on a common model $S''/Z\ni o$ of $S/Z\ni o$. The required inequality for the statement is

$$\mathbb{B}_{n\_\Phi}{}^\sharp{}_{S^\sharp,S''}\le\mathbb{B}_{S',n\_\widetilde{\Phi}}{}^\sharp{}_{S''}.$$

Its b-version

$$\mathbb{B}_{n\_\Phi}{}^\sharp{}_{S^\sharp}\le\mathbb{B}_{S',n\_\widetilde{\Phi}}{}^\sharp.$$

By Corollary 8, the b-version follows from its divisorial version on $S^\sharp$:

$$B_{n\_\Phi}{}^\sharp{}_{S^\sharp} \le \mathbb{B}_{S',n\_\widetilde{\Phi}}{}^\sharp{}_{S^\sharp}. \tag{7.6.2}$$

Indeed, by construction $-(\mathbb{K}_{S^\sharp}+\mathbb{B}_{S',n\_\widetilde{\Phi}}{}^\sharp) = -(\mathbb{K}_{S'^\sharp}+\mathbb{B}_{S',n\_\widetilde{\Phi}}{}^\sharp)$ is nef. (For any rational differential form $\omega \ne 0$ on $S/Z \ni o$, $\mathbb{K}_{S^\sharp} = \mathbb{K}_{S'^\sharp} = (\omega)$, where the last canonical divisor is treated as a b-divisor.) This implies (1) of Corollary 8. The inequality (7.6.2) is (2) of Corollary 8. By Step 2 $(S^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{S^\sharp})$ is a log pair with $\mathbb{D}(S^\sharp, B_{n\_\Phi}{}^\sharp{}_{S^\sharp}) = \mathbb{B}_{n\_\Phi}{}^\sharp{}_{S^\sharp}$.

Again Corollary 8, but now applied to $(X^\sharp/Z \ni, B_{n\_\Phi}{}^\sharp)$ of Step 2, implies

$$\mathbb{B}_{n\_\Phi}{}^\sharp \le \mathbb{B}. \tag{7.6.3}$$

Indeed,$-(\mathbb{K}+\mathbb{B})$ is nef by (3). This gives (1) of Corollary 8. Construction in Step 2 gives (2) of Corollary 8: $\mathbb{B}_{X^\sharp} = B_{X^\sharp} \ge B_{n\_\Phi}{}^\sharp$.

The semiadditivity [Sh92, 3.2.1] and (7.6.3) implies

$$\mathbb{B}_{n\_\Phi}{}^\sharp{}_{S^\sharp} \le \mathbb{B}_S. \tag{7.6.4}$$

Finally, we verify (7.6.2) in every prime divisor P on $S^\sharp$. Note P is also a divisor on S' by the maximal property of the crepant model $(S'/Z \ni o, B_{S'})$. Indeed, if P is a divisor of $S^\sharp$ then by (7.6.4)

$$0 \le b_{n\_\Phi}{}^\sharp{}_{S^\sharp,P} = \operatorname{mult}_P \mathbb{B}_{n\_\Phi}{}^\sharp{}_{S^\sharp} \le \operatorname{mult}_P \mathbb{B}_S = b_{S,P}.$$

Hence P is a divisor on S'. On the other hand, $b_{n\_\Phi}{}^\sharp{}_{S^\sharp,P} \in \Gamma(n, \widetilde{\Phi})$ by (7.6.1). Hence by definition and Proposition 8

$$b_{n\_\Phi}{}^\sharp{}_{S^\sharp,P} \le b_{S,P,n\_\widetilde{\Phi}} = \operatorname{mult}_P \mathbb{B}_{S',n\_\widetilde{\Phi}} \le \operatorname{mult}_P \mathbb{B}_{S',n\_\widetilde{\Phi}}{}^\sharp,$$

that is, (7.6.2) in P.

bd-Pairs can be treated similarly. In this case, a maximal crepant model with boundary means that of only for the divisorial part B_S. So, we take $(S'/Z \ni o, B_{S'} + \mathcal{P}_{\mid S'})$. After that we take $(S'/Z \ni o, B_{S',n\_\widetilde{\Phi}} + \mathcal{P}_{\mid S'})$ and, finally, $(S'/Z \ni o, B_{S',n\_\widetilde{\Phi}}{}^\sharp + \mathcal{P}_{\mid S'^\sharp})$. By definition $\mathcal{P}_{\mid S} = \mathcal{P}_{\mid S'} = \mathcal{P}_{\mid S'^\sharp}$.

□

8 Semiexceptional complements: lifting exceptional

Semiexceptional pairs. They can be defined in terms of $\mathbb{R}$-complements. A pair (X,D) is called *semiexceptional* if it has an $\mathbb{R}$-complement and (X,D^+) is an $\mathbb{R}$-complement of (X,D) for every D^+ on X such that $D^+ \ge D$ and $K+D^+ \sim_{\mathbb{R}} 0$. (In this situation D^+ is always a subboundary and a boundary if so does D.)

The same definition works for pairs $(X, D+\mathcal{P})$ with an arbitrary b-divisor $\mathcal{P}$, in particular, for bd-pairs $(X, D+\mathcal{P})$ (of index m).

Construction 3 (b-Contraction associated to lc singularities)**.** Let (X,B) be a pair with a nonklt $\mathbb{R}$-complement (X,B^+). Then there exists a prime b-divisor P such that $a(X,B^+;P)=0$, equivalently, $\operatorname{mult}_P \mathbb{B}^+ = 1$, where $\mathbb{B}^+ = \mathbb{D}(X,B)$ denotes the codiscrepancy b-divisor of (X,B). Suppose also that X has wFt and B is a boundary. Then there exists a b-*contraction associated* to (X,B^+). Such a b-contraction is a diagram

$$(X,B^+) \overset{\varphi}{\dashleftarrow} (Y,B_Y^+) \xrightarrow{\psi} Z,$$

where φ is a crepant birational modification and ψ is a contraction such that

(1) (Y,B_Y^+) is lc but nonklt;

(2) φ blows up only prime b-divisors P with $a(X,B^+;P)=0$, equivalently, for every exceptional prime divisor P of φ, $\operatorname{mult}_P B_Y^+ = 1$; and

(3)
$$B^+ = B + \psi^* H \text{ and } B_Y^+ = B^{+,\log} = B^{\log} + \psi^* H,$$

where $B, B^{\log}, B^+, B^{+,\log}$ are respectively birational and log birational transforms of B, B^+ from X to Y and H is an effective ample $\mathbb{R}$-divisor on Z.

For φ we can take a dlt crepant blowup of (X,B^+) [KK, Theorem 3.1]. This gives (1-2) by construction. We can suppose that Y is $\mathbb{Q}$-factorial and has Ft, in particular, projective. (Z is also Ft by [PSh08, Lemma 2.8,(i)].) To satisfy (3) consider the effective $\mathbb{R}$-divisor $E = B^+ - B = B^{+,\log} - B^{\log}$ on Y. By construction E is supported outside $\operatorname{LCS}(Y,B_Y^+)$, in particular, does not contain prime divisors P on Y with $\operatorname{mult}_P B_Y^+ = 1$. Moreover, the dlt property implies that $(Y, B_Y^+ + \varepsilon E)$ is dlt for a sufficiently small positive real number ε. If E is nef then E is semiample [ShCh, Corollary 4.5] and gives the required contraction ψ because $K_Y + B_Y^+ \sim_{\mathbb{R}} 0$ and $\varepsilon E \sim_{\mathbb{R}} K_Y + B_Y^+ + \varepsilon E$.

If E is not nef we apply the LMMP to $(Y, B_Y^+ + \varepsilon E)$ or E-MMP to Y. A resulting model is not a Mori fibration or a fibration negative with respect to E, and E is nef on this model. The model gives a required pair (Y, B_Y^+) and φ is a birational modification of (Y, B_Y^+) to (X, B^+). Note that the LMMP contracts only divisors supported on $\operatorname{Supp} E$ and is crepant with respect to B_Y^+. Thus (1-2) hold for constructed φ. By construction

(4) $(Y/Z, B^{\log})$ is a 0-pair, in particular, $(Y, B^{\log})$ is a log pair.

Indeed, $\psi^* H$ is vertical and $\sim_{\mathbb{R}} 0$ over Z.

We can use also Construction 2 to construct $(Y, B^{\log})$ from $(Y, B^{\log})$ of the dlt blowup. Thus the model $(Y/Z, B^{\log})$ up to a crepant modification is defined by the dlt blowup and even by its exceptional divisors.

Since $K_Y + B_Y^+ \sim_{\mathbb{R}} 0$, we can associate to Z an adjoint boundary $B_Z^+ = B_{\mathrm{div}}^+ + B_{\mathrm{mod}}^+$ on Z [PSh08, Constructions 7.2 and 7.5, Remark 7.7]. If we treat B_Z^+ as a usual boundary then we suppose that B_{mod}^+ is the trace on Z of a sufficiently general effective b-divisor $\mathcal{B}_{\mathrm{mod}}^+$ of Z as in Conjecture 3. In this case the pair (Z, B_Z^+) is a usual lc pair with a boundary (cf. Corollary 34 below). However, to avoid Conjecture 3 we usually treat $B_{\mathrm{mod}}^+ = \mathcal{B}_{\mathrm{mod},Z}^+$ as the trace of a nef b-divisor $\mathcal{B}_{\mathrm{mod}}^+$ on Z. In this case the pair $(Z, B_{\mathrm{div}}^+ + \mathcal{B}_{\mathrm{mod}}^+)$ is an adjoint bd-pair with b-nef $\mathcal{P} = \mathcal{B}_{\mathrm{mod}}^+$ (see bd-Pairs in Section 3 and cf. Warning below). The pair is a log bd-pair and actually a 0-bd-pair. Note also that by (4) we can also associate to the pair $(Y, B^{\log})$ an adjoint boundary $B_Z^{\log} = B_{\mathrm{div}}^{\log} + B_{\mathrm{mod}}^{\log}$. The corresponding adjoint bd-pair is $(Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ with the same moduli b-part $\mathcal{B}_{\mathrm{mod}}^{\log} = \mathcal{B}_{\mathrm{mod}}^+$. Indeed, by [PSh08, Lemma 7.4(ii), Construction 7.5] or by Proposition 13 and 7.5

$$B_{\mathrm{div}}^+ = B_{\mathrm{div}}^{\log} + H, B_{\mathrm{mod}}^+ = B_{\mathrm{mod}}^{\log} \text{ and } B_Z^+ = B_Z^{\log} + H.$$

So, (Z, B_Z^+) is an $\mathbb{R}$-complement of $(Z, B_Z^{\log})$. Respectively, for bd-pairs, $(Z, B_{\mathrm{div}}^+ + \mathcal{B}_{\mathrm{mod}}^+)$ is an $\mathbb{R}$-complement of $(Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$. By construction $(Z, B_Z^{\log})$ is a log Fano pair with a log Fano bd-pair $(Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$.

Warning: bd-Pairs $(Z, B_{\mathrm{div}}^+ + \mathcal{B}_{\mathrm{mod}}^+), (Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ have lc singularities by definition. However, corresponding pairs $(Z, B_Z^+), (B, B_Z^{\log})$ have lc singularities only for an appropriate choice of the moduli part $\mathcal{B}_{\mathrm{mod}}^+ = \mathcal{B}_{\mathrm{mod}}^{\log}$ (cf. [PSh08, Corollary 7.18(ii)]).

The bd-pair $(Z, B_{\mathrm{div}}^{\log}+\mathcal{B}_{\mathrm{mod}}^{\log})$ actually depend on the complement (X, B^+) and on its dlt blowup (Y, B_Y^+); actually on the exceptional divisors of the blowup. If we replace in our construction (Y, B_Y^+) by any crepant model (V, B_V^+) of (X, B), possibly with a subboundary B_V^+, with a contraction to Y and to Z. Then the difference $B^+ - B = B^{+,\log} - B^{\log}$ (of birational and log birational transforms) usually is not semiample and even with fixed components. Moreover, the horizontal difference over Z has only those fixed components and is exceptional on Y, in particular, on the generic fiber of ψ.

The same construction works for a bd-pair $(X, B + \mathcal{P})$ with a nonklt $\mathbb{R}$-complement $(X, B^+ + \mathcal{P})$ and such that

X has wFt;

$\mathcal{P}$ is pseudoeffective modulo $\sim_{\mathbb{R}}$;

B is a boundary and

$(X, B + \mathcal{P})$ has a nonklt $\mathbb{R}$-complement $(X, B^+ + \mathcal{P})$.

In this case and in the case for usual pairs it is better to apply E-MMP. To construct a dlt Ft blowup we need $\mathcal{P}$ to be pseudoeffective modulo $\sim_{\mathbb{R}}$. For adjoint boundaries see 7.1 and (12-13) in 7.5.

By construction and Proposition 3, the b-contraction is invariant of small birational modifications of pairs (X, B) or bd-pairs $(X, B + \mathcal{P})$.

Proposition 14. *Let (X, B) be a semiexceptional but not exceptional pair, with wFt X and a boundary B. Then Construction 3 is applicable to (X, B).*

Every b-contraction of (X, B) is not birational, that is, $\dim Z < \dim Y = \dim X$. *Every lc centers and prime b-divisors P with $a(P; X, B^+) = 0$ are horizontal with respect to the b-contraction, that is, for every b-divisor P such that* $\operatorname{mult}_P \mathbb{B}^+ = 1$, $\psi \operatorname{center} P = Z$.

Pairs $(Z, B_{\mathrm{div}}^+ + \mathcal{B}_{\mathrm{mod}}^+), (Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ are a klt 0-bd-pair and a klt log Fano bd-pair respectively. The bd-pair $(Z, B_{\mathrm{div}}^+ + \mathcal{B}_{\mathrm{mod}}^+)$ is exceptional and $(Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ is semiexceptional.

Addendum 39. *The same holds for semiexceptional but not exceptional bd-pairs $(X, B + \mathcal{P})$ with wFt X, a boundary B and a pseudoeffective modulo $\sim_{\mathbb{R}}$ b-divisor $\mathcal{P}$.*

Proof. The construction is applicable because (X, B) has an $\mathbb{R}$-complement.

According to Construction 3, if ψ is birational then $B^+ - B$ is big on any dlt blowup $\varphi\colon (Y, B_Y^+) \to (X, B^+)$. So, there exists an effective divisor $E \sim_{\mathbb{R}} B^+ - B$ on Y such that (Y, B'_Y) is nonlc, where $B'_Y = B_Y^+ - B^+ + B + E$. Since $B_Y^+ \sim_{\mathbb{R}} B_Y^+ - B^+ + B + E$ and $\varphi(B'_Y) = B' = B + \varphi(E)$ on X, (X, B') is a noncl $\mathbb{R}$-complement of (X, B). This contradicts to the semiexceptional property of (X, B).

Similarly, if P is an lc prime b-divisor of (X, B^+) such that ψ center P is proper in Z, then there exists an effective divisor $E \sim_{\mathbb{R}} B^+ - B$ passing through P. Moreover, by construction and our assumptions Z is complete and proper over $S = \mathrm{pt}$. This leads again to a contradiction by (6) and (8) of 7.5. Hence by [PSh08, Lemma 7.4(iii)] and construction $(Z, B_{\mathrm{div}}^+ + \mathcal{B}_{\mathrm{mod}}^+))$ is a klt 0-bd-pair. Since H is effective and ample, $(Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ is a klt log Fano bd-pair. Since $(Z, B_{\mathrm{div}}^+ + \mathcal{B}_{\mathrm{mod}}^+))$ is a klt 0-bd-pair, it is exceptional. The bd-pair $(Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ is semiexceptional again by (6) and (8) of 7.5 (cf. the proof of Corollary 15).

Similarly we treat bd-pairs. In this case $(B^{\log} + \mathcal{P})_{\mathrm{div},Z}^+$ is a usual $\mathbb{R}$-divisor on Z and $(B^{\log} + \mathcal{P})_{\mathrm{mod}}^+$ is a nef b-divisor of Z (see 7.1).

$\square$

Since $\dim Z < \dim X$ and $(Z, B_{\mathrm{div}} + \mathcal{B}_{\mathrm{mod}}^{\log})$ is semiexceptional we can use the dimensional induction to construct n-complements of semiexceptional pairs (X, B) and bd-pairs $(X, B + \mathcal{P})$. Finally, this reduces construction of semiexceptional n-complements to exceptional ones. Any 0-dimensional pair is exceptional. We prefer a direct reduction to the exceptional case.

Semiexceptional type. Let (X, B) be a semiexceptional pair under the assumptions of Construction 3 and $(X, B^+) \dashrightarrow Z$ be a b-contraction associated a nonklt complement (X, B^+). The b-contraction has a pair (r, f) of invariants, where $r = \mathrm{reg}(X, B^+) = \dim \mathrm{R}(X, B)$ [Sh95, Proposition-Definition 7.9] is the *regularity* of (X, B^+), characterising topological depth [difficulty] of lc singularities of (X, B^+), and f is the dimension of Z, of the base of b-contraction. We order these pairs lexicographically:

$$(r_1, f_1) \geq (r_2, f_2) \text{ if } \begin{cases} r_1 > r_2; \text{ or} \\ r_1 = r_2 \text{ and } f_1 \geq f_2. \end{cases}$$

A *maximal* b-contraction is a largest one with respect to (r, f), that is, of the largest regularity and the largest dimension of the base for such a regularity. The *semiexceptional* type of (X, B) is such a largest pair (r, f). [If (X, B) is exceptional then r is not defined [or $= -\infty$] and $f = \dim X$ is possible (see Semiexceptional filtration below).]

For pairs of dimension d not all invariants $0 \le r \le d-1, 0 \le f \le d$ are possible. E.g., the top type in this situation is $(d-1, 0)$ according to Addendum 40 below.

Similar notion can be applied to a semiexceptional bd-pair $(X, B + \mathcal{P})$ under the assumptions of Construction 3.

Corollary 14. *Under the assumptions and notation of Construction 3,* $\mathrm{reg}(X, B^+)$ *is the same topological invariant as of a sufficiently general or generic fiber of* (Y, B_Y^+) *over* Z*: for the generic point* η *of* Z*,*

$$\mathrm{reg}(X, B) \ge \mathrm{reg}(X, B^+) = \mathrm{reg}(Y_\eta, B_\eta^+).$$

Addendum 40. $\mathrm{reg}(X, B^+) \le \dim Y_\eta - 1 = \dim X - \dim Z - 1$, *or* $\dim Z \le \dim X - \mathrm{reg}(X, B^+) - 1$.

Addendum 41. *The same holds for bd-pairs.*

Proof. Immediate by Proposition 14 and definition because all lc centers are horizontal.

Addendum 40 follows from the general fact that, for every lc pair (X, B), $\mathrm{reg}(X, B) \le \dim X - 1$.

For bd-pairs notice only that $\mathrm{reg}(X, B^+ + \mathcal{P})$ can be defined as for usual pairs. $\square$

Corollary 15. *Let* (X, B) *be a semiexceptional pair under the assumptions of Construction 3 and* $(X, B^+) \dashrightarrow Z$ *be a maximal b-contraction with respect to* r *associated to a nonklt complement* (X, B^+)*. Then* $(Z, B_Z^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ *is exceptional.*

The same hold for semiexceptional bd-pairs.

Proof. Otherwise there exists an effective $\mathbb{R}$-divisor $E \sim_{\mathbb{R}} H$ on Z such that $(Z, B_{\mathrm{div}}^{\log} + E + \mathcal{B}_{\mathrm{mod}}^{\log})$ is nonklt. Hence $(Y, B_Y^{\log} + \psi^* E)$ has a vertical lc center [PSh08, Lemma 7.4(iii)] (cf. the proof of Proposition 14). Since $\psi^* E \sim_{\mathbb{R}} \psi^* H$ and is effective this contradicts to the maximal property of ψ with respect to r.

$\square$

Construction 4. Let d be a nonnegative integer and $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0,1]$. By Theorem 9 there exists a positive integer J such that every contraction $\psi\colon (Y, B^{\log}) \to Z$ of Construction 3, (4) has the adjunction index J if

(1) $\dim X = d$ and

(2) $B^{\mathrm{h}} \in \Phi$.

Note that if $1 \notin \Phi$ then we can add 0 to $\mathfrak{R}$ and so 1 to Φ. Moreover, there exists a finite set of rational numbers $\mathfrak{R}'$ in $[0,1]$ such that $\Phi' = \Phi(\mathfrak{R}')$ satisfies Addendum 35. More precisely, $\mathfrak{R}'$ is defined by (6.8.1), where

$$\mathfrak{R}'' = [0,1] \cap \frac{\mathbb{Z}}{J}.$$

By Addendum 36, the same adjunction index J has every contraction $\psi\colon (Y, B^{\log} + \mathcal{P}) \to Z$ of Construction 3 if we apply the construction to a bd-pair $(X, B + \mathcal{P})$ and suppose additionally to (1-2) that $(X, B + \mathcal{P})$ is a bd-pair of index m, or equivalently, $(Y, B^{\log} + \mathcal{P})$ is a log bd-pair of index m.

Let I, ε, v, e be the data as in Restrictions on complementary indices in Section 1 and f be a nonnegative integer such that $f \le d-1$. By Theorem 7 or by dimensional induction there exists a finite set of positive integers $\mathcal{N} = \mathcal{N}(f, I, \varepsilon, v, e, \Phi', J)$ such that

Restrictions: every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data, in particular, $J|n$ (see Remark 5);

Existence of n-complement: if $(Z, B_Z + \mathcal{Q})$ a bd-pair of dimension f and of index J with wFt Z, with a boundary B_Z, with an $\mathbb{R}$-complement and with exceptional $(Z, B_{Z,\Phi'} + \mathcal{Q})$, then $(Z, B_Z + \mathcal{Q})$ has an n-complement $(Z, B_Z^+ + \mathcal{Q})$ for some $n \in \mathcal{N}$. We can apply dimensional induction by Addendum 30 [or by the assumption (1) Theorem 4, bd-version of Theorem 3] because $(Z, B_Z + \mathcal{Q})$ is also exceptional and has a klt $\mathbb{R}$-complement. Moreover,

(3) $B_Z^+ \ge B_{Z,n\_\Phi'}$.

The following result applies to the construction but not only.

Corollary 16. *For any given finite set $\mathcal{N}'$ of positive integers, we can suppose that $\mathcal{N}(I) = \mathcal{N}(\dots, I, \dots)$ under Restrictions on complementary indices with the given data is disjoint from $\mathcal{N}'$.*

Proof. Use $\mathcal{N}(I) = \mathcal{N}(I')$ with the same data except for I replaced by sufficiently divisible I'. $\square$

Theorem 12. *Let $d, \mathfrak{R}, \Phi, J, I, \mathfrak{R}', \Phi', \varepsilon, v, e, f, \mathcal{N}$ be the data of Construction 4 and r be a nonnegative integer. Let (X, B) be a pair with a boundary B such that*

(1) *X has wFt;*

(2) $\dim X = d$*; and*

(3) *both pairs*
$$(X, B_\Phi),\ (X, B_{\mathcal{N}\_\Phi})$$
are semiexceptional of the same type (r, f).

Then there exists $n \in \mathcal{N}$ such that (X, B) has an n-complement (X, B^+) with

(4) *$B^+ \geq B_{n\_\Phi}$.*

Notice that we do not assume that (X, B) has an $\mathbb{R}$-complement.

Addendum 42. $B^+ \geq B_{n\_\Phi}{}^\sharp \geq B_{n\_\Phi}$.

Addendum 43. *(X, B^+) is a monotonic n-complement of itself and of $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$, and is a monotonic b-$n$-complement of itself and of $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp), (X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$, if $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$ are log pairs respectively.*

Addendum 44. *The same holds for the bd-pairs $(X, B + \mathcal{P})$ of index m and with $\mathcal{N}$ as in Construction 4. That is,*

Existence of n-complement: *if $(X, B + \mathcal{P})$ is a bd-pair of index m with a boundary B, under (1-2) and such that*

both bd-pairs
$$(X, B_\Phi + \mathcal{P}),\ (X, B_{\mathcal{N}\_\Phi} + \mathcal{P})$$
are semiexceptional of type (r, f),

then $(X, B + \mathcal{P})$ has an n-complement $(X, B^+ + \mathcal{P})$ for some $n \in \mathcal{N}$.

Addenda 42 holds literally. In Addenda 43 $(X, B^+ + \mathcal{P})$ is a monotonic n-complement of itself and of $(X, B_{n\_\Phi} + \mathcal{P}), (X, B_{n\_\Phi}{}^\sharp + \mathcal{P})$, and is a monotonic b-$n$-complement of itself and of $(X, B_{n\_\Phi} + \mathcal{P}), (X, B_{n\_\Phi}{}^\sharp + \mathcal{P}), (X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp} + \mathcal{P})$, if $(X, B_{n\_\Phi} + \mathcal{P}_X), (X, B_{n\_\Phi}{}^\sharp + \mathcal{P}_X)$ are log bd-pairs respectively.

Lemma 10. *Let (X, B) be a semiexceptional pair with a boundary B of type (r, f), with wFt X and $D \geq B$ be an effective $\mathbb{R}$-divisor of X such that $-(K + D)$ is (pseudo)effective modulo $\sim_{\mathbb{R}}$. Then the pair (X, D) is also semiexceptional of type $\leq (r, f)$ with a boundary D. If (X, D) has also type (r, f) then every maximal b-contraction $(X, D^+) \dashrightarrow Z'$ is birationally isomorphic to such a b-contraction for $(X, B^+) \dashrightarrow Z$. Moreover, the isomorphism is crepant generically over Z', equivalently, over Z.*

The same holds for bd-pairs $(X, B+\mathcal{P}), (X, D+\mathcal{P})$ with $-(K+D+\mathcal{P}_X), \mathcal{P}$, pseudoeffective modulo $\sim_{\mathbb{R}}$.

Proof. By our assumptions there exists $D^+ \geq D$ such that $K + D^+ \sim_{\mathbb{R}} 0$. Since $D^+ \geq D \geq B \geq 0$ and (X, B) is semiexceptional, (X, D) is semiexceptional and D is a boundary too. In other words, (X, D^+) is lc and (X, D^+) is an $\mathbb{R}$-complement of (X, D). Hence by Addendum 4 (X, D^+) is also an $\mathbb{R}$-complement of (X, B).

Let (r', f') be the semiexceptional type of (X, D) and

$$(X, D^+) \dashleftarrow (Y', D^+_{Y'}) \xrightarrow{\psi'} Z',$$

be a maximal b-contraction associated to a nonklt $\mathbb{R}$-complement (X, D^+), that is, $\dim Z' = f'$ and $r' = \operatorname{reg}(X, D^+)$.

The complement (X, D^+) is also an $\mathbb{R}$-complement of (X, B). Hence $r \geq r'$. If $r > r'$ then $(r, f) > (r', f')$. Otherwise $r = r'$. However to verify that $f' \leq f$ we need to return back to a dlt crepant blowup of (X, D^+) of Construction 3. The same blowup can be used for (X, B) with $B^+ = D^+$. But a maximal b-contraction for (X, B) should be constructed by E-MMP with

$$E = B^+ - B = D^+ - B = D^+ - D + D - B = E' + E'',$$

where $E' = D^+ - D, E'' = D - B \geq 0$. Hence $f \geq f'$ and $(r, f) \geq (r', f')$.

If $(r', f) = (r, f)$ then in the last construction $f = f'$. In this case rational contractions $(X, D^+) \dashrightarrow Z', (X, B^+) \dashrightarrow Z$ are birationally isomorphic, or equivalently, ψ', ψ are birationally isomorphic. Generically over Z, E'' is exceptional or 0. Otherwise we can construct a b-contraction for (X, B) with the same r and $f > f'$.

By construction of the proof, Construction 3 and in its notation $\mathbb{D}^+ = \mathbb{B}^+ = \mathbb{B}^{\log}$ over the generic point of Z, where $\mathbb{B}^{\log} = \mathbb{B}(Y, B^{\log})$. Similarly, $\mathbb{D}^+ = \mathbb{D}^{\log}$ over the generic point of Z' with $\mathbb{D}^{\log} = \mathbb{D}(Y', D^{\log})$. By the above birational isomorphism, $\mathbb{D}^{\log} = \mathbb{B}^{\log}$ over the generic point of Z' which is the same as over the generic point of Z. Thus the birational isomorphism of $(Y', D^+_{Y'}) = (Y', D^{\log}), (Y, B^+_Y) = (Y, B^{\log})$ is crepant over the generic point of Z' which is the same as over the generic point of Z.

Similarly we can treat bd-pairs.

$\square$

Proof of Theorem 12. Let (X, B) be a pair satisfying (1-3) of the theorem. For simplicity of notation suppose that $B = B_{\mathcal{N}\_\Phi}$, in particular, $B \in \Gamma(\mathcal{N}, \Phi)$ (see Construction 1; but this is not important for the following). So, instead of (3) we have

(3′) both pairs

$$(X, B_\Phi),\ (X, B)$$

are semiexceptional of the same type (r, f).

Indeed, by definition and Proposition 6

$$B_{\mathcal{N}\_\Phi} = B_{\mathcal{N}\_\Phi,\mathcal{N}\_\Phi}, B_\Phi = B_{\mathcal{N}\_\Phi,\Phi} \text{ and } B_{\mathcal{N}\_\Phi,n\_\Phi} = B_{n\_\Phi}$$

for every $n \in \mathcal{N}$. By Corollary 6 an n-complement of $(X, B_{\mathcal{N}\_\Phi})$ is also an n-complement of (X, B).

Step 1. Choice of a b-contraction. Let $(X, B^{\mathbb{R}\text{-}+})$ be an $\mathbb{R}$-complement of (X, B) such that its associated b-contraction

$$(X, B^{\mathbb{R}\text{-}+}) \overset{\varphi}{\dashleftarrow} (Y, B_Y^{\mathbb{R}\text{-}+}) \xrightarrow{\psi} Z \tag{8.0.5}$$

has (maximal) type (r, f). Such an $\mathbb{R}$-complement and a b-contraction exist by (3′). By construction $(X, B^{\mathbb{R}\text{-}+})$ is lc, $B^{\mathbb{R}\text{-}+}$ is a boundary, $\dim Y = \dim X = d, \dim Z = f$ and $r = \operatorname{reg}(X, B^{\mathbb{R}\text{-}+})$.

Notice that ψ is actually a fibration, that is, $f < d$, because (X, B) is semiexceptional, $r \geq 0$ by our assumptions and the complement $(X, B^{\mathbb{R}\text{-}+})$ is nonklt. By Construction 3 we get an adjoint bd-pair $(Z, B^{\log}_{\text{div}} + \mathcal{B}^{\log}_{\text{mod}})$ of $(Y, B^{\log}) \to Z$. The bd-pair $(Z, B^{\log}_{\text{div}} + \mathcal{B}^{\log}_{\text{mod}})$ is exceptional by Corollary 15. We need a stronger fact.

Step 2. bd-Pair $(Z, B^{\log}_{\text{div},\Phi'} + \mathcal{B}^{\log}_{\text{mod}})$ is also exceptional. To verify this consider a crepant model $(Y^\sharp, B^{\log,\sharp}{}_{Y^\sharp})$ of $(Y, B^{\log})$ which blows up exactly prime b-divisors of Y which are divisors on X or, equivalently, on the dlt blowup of (X, B^+) of Construction 3. Thus if we use the same complement (X, B^+) and the same dlt blowup of (X, B^+) to construct a b-contraction for

(X, B_Φ) then we can apply E-MMP to $(Y^\sharp, B^{\log}_{Y^\sharp,\Phi})$ with $E = B^+_{Y^\sharp} - B^{\log}_{Y^\sharp,\Phi}$ or, equivalently, we construct a maximal model using antiflips of $(Y^\sharp, B^{\log}_{Y^\sharp,\Phi})$. This time we apply E-MMP only relatively over Z, that is, Construction 2 to $(Y^\sharp/Z, B^{\log}_{Y^\sharp,\Phi})$. Denote by $(Y^\sharp/Z, B^{\log}_{Y^\sharp,\Phi}{}^\sharp)$ the result. We can use the same notation $Y^\sharp$ even some divisors can be contracted because by construction $(Y^\sharp/Z, B^{\log,\sharp}{}_{Y^\sharp})$ is a 0-pair and all birational transformations over Z are flops of this pair. By 7.1, the adjoint bd-pair of $(Y^\sharp/Z, B^{\log,\sharp}{}_{Y^\sharp})$ is $(Z, B^{\log}_{\mathrm{div}} + \mathcal{B}^{\log}_{\mathrm{mod}})$, the same as of $(Y, B^{\log}) \to Z$. Equivalently,

$$B^{\log,\sharp}{}_{Y^\sharp,\mathrm{div}} = (B^{\log,\sharp}{}_{Y^\sharp})_{\mathrm{div},Z} = B^{\log}_{\mathrm{div}} \text{ and } (B^{\log,\sharp}{}_{Y^\sharp})_{\mathrm{mod}} = \mathcal{B}^{\log}_{\mathrm{mod}}.$$

On the other hand, by our assumption (3) and Lemma 10, the mobile part of E is vertical. Thus, generically over Z, the resulting model $(Y^\sharp/Z, B^{\log}_{Y^\sharp,\Phi}{}^\sharp)$ is a 0-pair. It is 0-pair over Z everywhere because it is maximal. During the construction of this model we contract all prime divisor P of $Y^\sharp$ for which

$$\mathrm{mult}_P\, B^{\log}_{Y^\sharp,\Phi}{}^\sharp > \mathrm{mult}_P\, B^{\log}_{Y^\sharp,\Phi}.$$

Thus on the resulting model $(Y^\sharp/Z, B^{\log}_{Y^\sharp,\Phi}{}^\sharp)$, for every prime divisor P on $Y^\sharp$,

$$\mathrm{mult}_P\, B^{\log}_{Y^\sharp,\Phi}{}^\sharp = \mathrm{mult}_P\, B^{\log}_{Y^\sharp,\Phi} \text{ and } (Y^\sharp/Z, B^{\log}_{Y^\sharp,\Phi}{}^\sharp) = (Y^\sharp/Z, B^{\log}_{Y^\sharp,\Phi}).$$

We can take adjunction for the last pair too. Denote its adjoint bd-pair by $(Z, B^{\log}_{Y^\sharp,\Phi,\mathrm{div}} + \mathcal{B}^{\log}_{Y^\sharp,\Phi,\mathrm{mod}})$, where $B^{\log}_{Y^\sharp,\Phi,\mathrm{div}} = (\mathbb{B}^{\log}_{Y^\sharp,\Phi})_{\mathrm{div},Z}$ and $\mathcal{B}^{\log}_{Y^\sharp,\Phi,\mathrm{mod}} = (\mathcal{B}^{\log}_{Y^\sharp,\Phi})_{\mathrm{mod}} = \mathcal{B}^{\log}_{\mathrm{mod}}$. The last equality follows again from Lemma 10 and Proposition 13, (1). Indeed, by the lemma b-contractions $(X, B^{\mathbb{R}\text{-}+}) \dashrightarrow Z, (X, B^{\mathbb{R}\text{-}+}) \dashrightarrow Z_\Phi$ for (X, B) and (X, B_Φ) are crepant generically over Z. Generically over Z these b-contractions are crepant respectively to $(Y, B^{\log}) \to Z$ and $(Y^\sharp, B^{\log}_{Y^\sharp,\Phi}) \to Z$. But by construction $(Y, B^{\log}) \to Z$ and $(Y^\sharp, B^{\log,\sharp}{}_{Y^\sharp}) \to Z$ are also crepant over Z. Thus $(Y^\sharp, B^{\log,\sharp}{}_{Y^\sharp}) \to Z$ and $(Y^\sharp, B^{\log}_{Y^\sharp,\Phi}) \to Z$ are crepant generically over Z, moreover, they are equal generically over Z (i.e., horizontally):

$$B^{\log,\sharp}{}_{Y^\sharp}{}^{\mathrm{h}} = B^{\log}_{Y^\sharp,\Phi}{}^{\mathrm{h}} = B^{\mathbb{R}\text{-}+,\mathrm{h}}_{Y^\sharp}.$$

Since crepant modifications preserve the semiexceptional property and by Proposition 8, the pair $(Y^\sharp, B^{\log}_{Y^\sharp,\Phi})$ is semiexceptional. Otherwise, by the statement and Construction 2, $(Y^{\sharp,\sharp}, B^{\log}_{Y^\sharp,\Phi}{}^\sharp{}_{Y^{\sharp,\sharp}})$ and (X, B_Φ) are not semiexceptional, a contradiction.

The adjoint pair $(Z, B^{\log}_{Y^\sharp,\Phi,\mathrm{div}} + \mathcal{B}^{\log}_{Y^\sharp,\Phi,\mathrm{mod}})$ is exceptional according to the proof of Corollary 15. Otherwise there exists an $\mathbb{R}$-complement $(Y^\sharp, (B^{\log}_{Y^\sharp,\Phi})^+)$ of $(Y^\sharp/Z, B^{\log}_{Y^\sharp,\Phi})$ with the induced complement $(X, (B^{\log}_{Y^\sharp,\Phi})^+_X)$ of (X, B_Φ) (see Remark 4, (1-2)) and with

$$\mathrm{reg}(Y^\sharp, (B^{\log}_{Y^\sharp,\Phi})^+) = \mathrm{reg}(X, (B^{\log}_{Y^\sharp,\Phi})^+_X) > r = \mathrm{reg}(Y^\sharp, B^{\log}_{Y^\sharp,\Phi}) = \mathrm{reg}(X, B^{\mathbb{R}\text{-}+}).$$

So, $(Z, B^{\log}_{\mathrm{div},\Phi'} + \mathcal{B}^{\log}_{\mathrm{mod}})$ is exceptional because

$$B^{\log}_{Y^\sharp,\Phi,\mathrm{div}} \leq B^{\log}_{\mathrm{div},\Phi'} \text{ and } \mathcal{B}^{\log}_{Y^\sharp,\Phi,\mathrm{mod}} = \mathcal{B}^{\log}_{\mathrm{mod}}.$$

The last equality we already know. By definition the inequality follows from two facts (cf. 6.9):

$$B^{\log}_{Y^\sharp,\Phi,\mathrm{div}} \in \Phi' \text{ and } B^{\log}_{Y^\sharp,\Phi,\mathrm{div}} \leq B^{\log}_{\mathrm{div}}.$$

The inclusion follows from Construction 4 and Addendum 35. Indeed, by construction $B^{\log}_{Y^\sharp,\Phi} \in \Phi \cup \{1\}$, in particular, ${B^{\log}_{Y^\sharp,\Phi}}^{\mathrm{h}} \in \Phi \cup \{1\}$. (Or we can suppose that $1 \in \Phi$ already.) The required inequality follows from (4) of 7.5 because $B^{\log}_{Y^\sharp,\Phi} \leq {B^{\log,\sharp}}_{Y^\sharp}$ and ${B^{\log,\sharp}}_{Y^\sharp,\mathrm{div}} = B^{\log}_{\mathrm{div}}$ by construction (cf. again 6.9). By Addendum 4 the last inequality also shows that $(Z, B^{\log}_{Y^\sharp,\Phi,\mathrm{div}} + \mathcal{B}^{\log}_{Y^\sharp,\Phi,\mathrm{mod}})$ actually has an $\mathbb{R}$-complement.

Notice that the exceptional property of this step can be established by more sophisticated but quite formal methods of Section 6.

Step 3. Choice of $n \in \mathcal{N}$ and construction on an n-complement $(Z, B^{\log,+}_{\mathrm{div}} + \mathcal{B}^{\log}_{\mathrm{mod}})$ of the bd-pair $(Z, B^{\log}_{\mathrm{div}} + \mathcal{B}^{\log}_{\mathrm{mod}})$. The bd-pair is a bd-pair of index J in notation of Construction 4. Indeed, according to Step 2, the adjunctions $(Y^\sharp, {B^{\log,\sharp}}_{Y^\sharp}) \to Z, (Y^\sharp, B^{\log}_{Y^\sharp,\Phi}) \to Z$ are crepant generically over Z one to another. So, by Proposition 13, (1) and (5) they have the same moduli part and the same adjunction index if such one exists for either of them. The second adjunction $(Y^\sharp, B^{\log}_{Y^\sharp,\Phi}) \to Z$ satisfies (1-2) of Construction 4 and has the adjunction index J. (We can suppose that $1 \in \Phi$ or to consider $\Phi \cup \{1\}$ in (2) instead of Φ.) Hence $(Y^\sharp, {B^{\log,\sharp}}_{Y^\sharp}) \to Z$ has also the adjunction index J. The adjoint bd-pair $(Z, {B^{\log,\sharp}}_{Y^\sharp,\mathrm{div}} + ({B^{\log,\sharp}}_{Y^\sharp})_{\mathrm{mod}}) = (Z, B^{\log}_{\mathrm{div}} + \mathcal{B}^{\log}_{\mathrm{mod}})$ has also the index J by (3) of 7.3 and Theorem 8 (cf. Addendum 32).

By construction $\dim Z = f$ and by Step 2 the bd-pair $(Z, B^{\log}_{\mathrm{div},\Phi'} + \mathcal{B}^{\log}_{\mathrm{mod}})$ is exceptional. Notice that Z has Ft by [PSh08, Lemma 2.8,(i)] because Y has Ft by construction. Thus by Existence of n-complement in Construction 4, there exists $n \in \mathcal{N}$ and a required n-complement $(Z, B^{\log,+}_{\mathrm{div}} + \mathcal{B}^{\log}_{\mathrm{mod}})$ of

$(Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$. About this complement we suppose (3) of Construction 4. However, we need slightly more:

(3″) $b_Q^+ \geq b_{Q,n\_\Phi'}$ for every prime b-divisor Q of Z which is the image of a prime divisor P on X, where $b_Q^+ = \operatorname{mult}_Q \mathbb{D}(Z, B_{\mathrm{div}}^{\log,+} + \mathcal{B}_{\mathrm{mod}}^{\log})_{\mathrm{div}}$ and $b_Q = \operatorname{mult}_Q \mathbb{D}(Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})_{\mathrm{div}} = \operatorname{mult}_Q \mathbb{B}_{\mathrm{div}}^{\log}$.

To satisfy this inequality we need to take an n-complement on an appropriate crepant model $(Z', B_{\mathrm{div},Z'}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ of $(Z, B_{\mathrm{div}}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$, in particular, both are log bd-pairs. The model should blow up those Q which are exceptional on Z. There are only finitely many of those b-divisors Q and $b_Q \geq 0$, actually, $b_Q \in [0,1)$ by (12) of 7.5 and $b_{Q,n\_\Phi'}$ is well-defined. Indeed, every $b_P = \operatorname{mult}_P B^{\log} \in [0,1]$ by Construction 3. Thus a required crepant model exists and its divisorial part is the boundary $B_{\mathrm{div},Z'}^{\log}$. The model is an exceptional bd-pair $(Z', B_{\mathrm{div},Z'}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ of dimension f and of index J. In particular, it has an $\mathbb{R}$-complement. The log bd-pair $(Z', B_{\mathrm{div},Z',\Phi'}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ also has an $\mathbb{R}$-complement and is exceptional. The $\mathbb{R}$-complement exists by Addendum 4. The exceptional property follows from the fact the image of a nonlc $\mathbb{R}$-complement of $(Z', B_{\mathrm{div},Z',\Phi'}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ gives a nonlc $\mathbb{R}$-complement of $(Z, B_{\mathrm{div},\Phi}^{\log} + \mathcal{B}_{\mathrm{mod}}^{\log})$ that contradicts to Step 2.

Step 4. Construction on an n-complement $(Y, B^{\log,+})$ of $(Y, B^{\log})$. Take the induced n-complement $(Y, B^{\log,+})$ of $(Z, B_{\mathrm{div}}^{\log,+} + \mathcal{B}_{\mathrm{mod}}^{\log})$, that is, $B^{\log,+}$ adjunction corresponds to $B_{\mathrm{div}}^{\log,+}$ as in 7.5. Actually we need to verify that it is an n-complement with the log version of (4):

(4′) $b_P^+ \geq b_{P,n\_\Phi}^{\log}$ for every prime divisor on X or on Y, where $b_P^+ = \operatorname{mult}_P \mathbb{B}^{\log,+}$, $b_{P,n\_\Phi}^{\log} = (b_P^{\log})_{n\_\Phi}$ and

$$b_P^{\log} = \begin{cases} b_P = \operatorname{mult}_P B, & \text{if } P \text{ is nonexceptional on } X; \\ 1 & \text{otherwise.} \end{cases}$$

For this we use Addendum 37.

We apply the addendum to the 0-contraction $(Y, B^{\log}) \to Z$ over $S = \mathrm{pt}$. The contraction has the adjunction index J. Indeed, by construction in Step 2, $(Y^\sharp, B^{\log,\sharp}{}_{Y^\sharp}) \to Z$ and $(Y/Z, B^{\log}) \to Z$ are crepant generically over Z and $(Y^\sharp, B^{\log,\sharp}{}_{Y^\sharp}) \to Z$ has the adjunction index J. Thus by Proposition 13, (5) the (adjunction for) 0-contraction $(Y, B^{\log}) \to Z$ has also the adjunction index J.

By Constructions 4 and 3, $(Y/Z, B^{\log})$ is lc with a boundary $B^{\log}$ and Z is projective over $S = \mathrm{pt}$.

We can suppose also that the hyperstandard sets Φ, Φ' and the set of rational numbers $\mathfrak{R}''$ of Construction 4 are the same as in 6.8 and in Theorem 10.

Again by Construction 4 and Step 3, $J|n$.

Finally, by Step 3 there exists a required model $Z'/\mathrm{pt}.$ of $Z/\mathrm{pt}.$ with a boundary $B_{\mathrm{div}}^{\log,+}{}_{Z'}$, that is we consider the n-complement of Step 3 on Z'/pt. By ($3''$), (4) of Addendum 37 holds. Thus (1-3) of Theorem 10 also hold by the addendum. Hence $(Y, B^{\log,+})$ is an n-complement $(Y, B^{\log})$ which satisfies ($4'$) by (5) of the addendum.

Step 5. Construction on an n-complement (X, B^+) of (X, B). The complement is induced by the n-complement $(Y, B^{\log,+})$, that is, (X, B^+) is crepant to $(Y, B^{\log,+})$. The properties (2-3) of Definition 2 holds automatically. The property ($4'$) implies (4) of the theorem. In its turn, (4) implies (1) of the definition as ($4'$) in Step 4 (cf. [Sh92, Lemma 5.4]).

Step 6. The proof of the addenda and, in particular, for bd-pairs, is similar to the above proof for usual pairs and/or to the proof of Theorem 7. $\square$

Semiexceptional filtration. Let d be a nonnegative integer, $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0,1]$ and $\mathcal{N} \supseteq \mathcal{N}'$ be sets of positive integers. Let

$$\mathcal{N} \supseteq \mathcal{N}^{(0,0)} \supseteq \cdots \supseteq \mathcal{N}^{(r,f)} \supseteq \cdots \supseteq \mathcal{N}^{(d-1,0)} \supseteq \mathcal{N}',\ 0 \le r \le d-f-1, 0 \le f \le d-1, \tag{8.0.6}$$

be its (decreasing) *filtration with respect to semiexceptional types in the dimension d*. Its *associated* (decreasing) filtration of hyperstandard sets is

$$\Gamma(\mathcal{N}, \Phi) \supseteq \Gamma(\mathcal{N}^{(0,0)}, \Phi) \supseteq \cdots \supseteq \Gamma(\mathcal{N}^{(r,f)}, \Phi) \supseteq \cdots \supseteq \Gamma(\mathcal{N}^{(d-1,0)}, \Phi) \supseteq \Gamma(\mathcal{N}', \Phi).$$

Notation: For the filtration (8.0.6) and type (r, f), put

$$\mathcal{N}_{(r,f)} = \mathcal{N}^{(r,f)} \setminus \mathcal{N}^{(r',f')},$$

where (r', f') is the next type if such a type exists. Otherwise $(r, f) = (d-1, 0)$ and $\mathcal{N}_{(d-1,0)} = \mathcal{N}^{(d-1,0)} \setminus \mathcal{N}'$. That is, the next set in the last case is $\mathcal{N}'$ but without any type (cf. Generic type filtration with respect to dimension in Section 9). Respectively, it is useful to suppose that $\Gamma(\mathcal{N}', \Phi)$ is the next set for $\Gamma(\mathcal{N}^{(d-1,0)}, \Phi)$.

The whole set $\mathcal{N}$ is not necessarily coincide with $\mathcal{N}^{(0,0)}$. The discrepancy

$$\mathcal{N}_{(-1,-)} = \mathcal{N} \setminus \mathcal{N}^{(0,0)}$$

corresponds to exceptional types $(-1, f)$ with $f = d$ possible. So, the low script is relevant. Respectively, we use $\mathcal{N}^{(-1,-)} = \mathcal{N}$.

We do not filter exceptional types here.

Example 10. (1) If $\mathfrak{R} = \{1\}$ is minimal then $\Gamma(\mathcal{N}', \Phi(\{1\})) = \Gamma(\mathcal{N}')$. The filtration (8.0.6) gives the associated filtration

$$\Gamma(\mathcal{N}) \supseteq \Gamma(\mathcal{N}^{(0,0)}) \supseteq \cdots \supseteq \Gamma(\mathcal{N}^{(r,f)}) \supseteq \cdots \supseteq \Gamma(\mathcal{N}^{(d-1,0)}) \supseteq \Gamma(\mathcal{N}').$$

(2) If $\mathcal{N}' = \emptyset$ then $\mathfrak{G}(\emptyset, \mathfrak{R}) = \Phi(\mathfrak{R}) = \Phi$.

Definition 6. Let (8.0.6) be a filtration in Semiexceptional filtration. Consider a certain class of pairs (X, B) of dimension d with boundaries B which have an n-complement with $n \in \mathcal{N}$. Such a *pair* (X, B) and its *n-complement have semiexceptional type* (r, f) *with respect to the filtration* (8.0.6) if both pairs

$$(X, B_{\mathcal{N}^{(r',f')}\_\Phi}),\ (X, B_{\mathcal{N}^{(r,f)}\_\Phi})$$

are semiexceptional of the same type (r, f), where (r', f') is its next type. If the next type (r', f') does not exists, we take $B_{\mathcal{N}'\_\Phi}$ instead of $B_{\mathcal{N}^{(r',f')}\_\Phi}$. Note also that a *pair* (X, B) and its *n-complement have exceptional type* $(-1, -)$ *with respect to the filtration* (8.0.6) if the pair

$$(X, B_{\mathcal{N}^{(0,0)}\_\Phi})$$

is exceptional.

We say that the *existence* n-complements *agrees with the filtration* if every pair (X, B) in the class has an *n-complement of filtration semiexceptional type* (r, f) with $n \in \mathcal{N}_{(r,f)}$ and

$$B^+ \geq B_{n\_\mathcal{N}^{(r',f')}\_\Phi}.$$

If the next type (r', f'), does not exists, we take $B_{n\_\mathcal{N}'\_\Phi}$ instead of $B_{n\_\mathcal{N}_{(r',f')}\_\Phi}$. Notice again that $\mathcal{N}'$ does not have any semiexceptional or exceptional type.

Respectively, exceptional n-complements have type $(-1,-)$, $n \in \mathcal{N}_{(-1,-)}$, and
$$B^+ \geq B_{n\_\mathcal{N}^{(0,0)}\_\Phi}.$$

The same applies to certain classes of bd-pairs $(X, B+\mathcal{P})$ of dimension d and index $m|n$ with boundaries B, which have an n-complement with $n \in \mathcal{N}$.

Warning 2. It is possible other n-complements which are not agree with the filtration. E.g., we can have a triple or a longer chain of subsequent pairs of the same type (r, f)
$$(X, B_{\mathcal{N}^{(r',f')}\_\Phi}), \ldots, \ (X, B_{\mathcal{N}^{(r,f)}\_\Phi}).$$

So, they have many types and many complementary indices which are agree to some type but not agree to other. Additionally, we can have n-complements without any type but with $n \in \mathcal{N}$.

Theorem 13 (Semiexceptional n-complements)**.** *Let d be a nonnegative integer, I, ε, v, e be the data as in Restrictions on complementary indices, and $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0,1]$. Then there exists a finite set $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi)$ of positive integers such that*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data.*

Existence of n-complement: *if (X, B) is a pair with wFt X, $\dim X = d$, with a boundary B, with an $\mathbb{R}$-complement and with semiexceptional (X, B_Φ) then (X, B) has an n-complement (X, B^+) for some $n \in \mathcal{N}$.*

Addendum 45. $B^+ \geq B_{n\_\Phi}{}^\sharp \geq B_{n\_\Phi}$.

Addendum 46. *(X, B^+) is a monotonic n-complement of itself and of $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$, and is a monotonic b-$n$-complement of itself and of $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp), (X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$, if $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$ are log pairs respectively.*

Addendum 47. *$\mathcal{N}$ has a semiexceptional filtration (8.0.6) with $\mathcal{N}' = \emptyset$ which agrees with the existence of n-complements for the class of pairs in Existence of n-complements.*

Addendum 48. *The same holds for bd-pairs $(X, B + \mathcal{P})$ of index m with $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi, m)$. That is,*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data and $m|n$.*

Existence of n-complement: *if $(X, B + \mathcal{P})$ is a bd-pair of index m with wFt X, $\dim X = d$, with a boundary B, with an $\mathbb{R}$-complement and with semiexceptional $(X, B_\Phi + \mathcal{P})$ then $(X, B + \mathcal{P})$ has an n-complement $(X, B^+ + \mathcal{P})$ for some $n \in \mathcal{N}$.*

Addenda 45 and 47 hold literally. In Addenda 46 $(X, B^+ + \mathcal{P})$ is a monotonic n-complement of itself and of $(X, B_{n\_\Phi} + \mathcal{P}), (X, B_{n\_\Phi}{}^\sharp + \mathcal{P})$, and is a monotonic b-$n$-complement of itself and of $(X, B_{n\_\Phi} + \mathcal{P}), (X, B_{n\_\Phi}{}^\sharp + \mathcal{P}), (X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp} + \mathcal{P})$, if $(X, B_{n\_\Phi} + \mathcal{P}_X), (X, B_{n\_\Phi}{}^\sharp + \mathcal{P}_X)$ are log bd-pairs respectively.

Warning 3. Sets $\mathcal{N}(d, I, \varepsilon, v, e, \Phi), \mathcal{N}(d, I, \varepsilon, v, e, \Phi, m)$ are not unique and may be not suitable in other situations of the paper (cf. Remark 2, (1)). Notation shows the parameters on which the sets depend. We can take such minimal sets under inclusion but they can be not unique.

It would be interesting to find a more canonical way to construct such set $\mathcal{N}(d, I, \varepsilon, v, e, \Phi), \mathcal{N}(d, I, \varepsilon, v, e, \Phi, m)$ and/or to find an estimation on their largest element.

Proof. Construction of semiexceptional complements starts from the top type $(d - 1, 0)$ and goes to the bottom $(-1, -)$, the exceptional one. We use induction on types (r, f) for the class of pairs in Existence of n-complements of the theorem.

Step 1. Type $(d - 1, 0)$. Consider a set of positive integers $\mathcal{N}_{(d-1,0)} = \mathcal{N}(0, I, \varepsilon, v, e, \Phi', J)$ as in Construction 4 with $f = 0$. Put $\mathcal{N}^{(d-1,0)} = \mathcal{N}_{(d-1,0)}$. By Theorem 12 this gives the lowest set in the filtration (8.0.6) above $\mathcal{N}' = \emptyset$.

Step 2. Type (r, f). We can suppose that it has the next type (r', f') and the filtration (8.0.6) is already constructed for types $\geq (r', f')$. Consider a set of positive integers $\mathcal{N}_{(r,f)} = \mathcal{N}(f, I, \varepsilon, v, e, \Phi', J)$ as in Construction 4 with a hyperstandard set $\Phi = \mathfrak{G}(\mathcal{N}^{(r',f')}, \mathfrak{R})$ and with given f. Note that Φ, Φ', m in the step can be different from that of in Step 1 or in the statement of theorem. By Corollary 16 we can suppose that $\mathcal{N}_{(r,f)}$ is disjoint from $\mathcal{N}^{(r',f')}$. Put $\mathcal{N}^{(r,f)} = \mathcal{N}_{(r,f)} \cup \mathcal{N}^{(r',f')}$. Again by Theorem 12 this gives the filtration (8.0.6) for types $\geq (r, f)$.

This concludes construction of complements in the semiexceptional case.

Step 3. Exceptional type $(-1,-)$. In this case we need to construct n-complements for (X,B) with exceptional $(X,B_{\mathcal{N}^{(0,0)}\_\Phi})$. Consider a set of positive integers $\mathcal{N}_{(-1,-)}=\mathcal{N}(d,I,\varepsilon,v,e,\Phi)$ as in Theorem 7 with the hyperstandard set $\Phi=\mathfrak{G}(\mathcal{N}^{(0,0)},\mathfrak{R})$. By Corollary 16 we can suppose that $\mathcal{N}_{(-1,-)}$ is disjoint from $\mathcal{N}^{(0,0)}$. Put $\mathcal{N}=\mathcal{N}^{(-1,-)}=\mathcal{N}_{(-1,-)}\cup\mathcal{N}^{(0,0)}$. Now by Theorem 7 this completes the filtration (8.0.6).

Step 4. Addenda follows from the above arguments and from Theorems 12, 7, in particular, for bd-pairs. However, we take $\mathcal{N}_{(-1,-)}=\mathcal{N}(d,I,\varepsilon,v,e,\Phi,m)$ in Step 3 with m as the index of bd-pair (X,B).

Notice also that by Addendum 47, Propositions 6 and 8, in Addenda 45, 46 and 48, we have stronger results with $n\_\mathcal{N}^{(r',f')}\_\Phi$ instead of n_Φ for n-complements of type (r,f) (see Definition 6).

□

Corollary 17. *In Theorem 13 we can add addition data $\mathcal{N}'$, a finite set of positive integers such that every $n\in\mathcal{N}'$ satisfies Restrictions on complementary indices with the given data. Then in Addendum 47 we can find an extended filtration (8.0.6) starting from $\mathcal{N}'$.*

Proof. Take $\Phi:=\mathfrak{G}(\mathcal{N}',\mathfrak{R})$. It is also hyperstandard by Proposition 4.

□

[Remark:] E.g., in the induction of the next section we use a set $\mathcal{N}'=\mathcal{N}^{d-1}$ of [non[semi]exceptional] complementary indices coming from low dimensions $\leq d-1$ (see Construction 6 below).

9 Generic nonsemiexceptional complements: extension of lower dimensional complements

Definition-Proposition 1. Let $(X/Z,B)$ be a pair with proper X/Z and with a boundary B such that $(X/Z,B)$ has a klt $\mathbb{R}$-complement. We say that $(X/Z,B)$ is *generic with respect to* $\mathbb{R}$*-complements* if the pair additionally satisfies one of the following equivalent properties.

(1) There exists a klt $\mathbb{R}$-complement $(X/Z,B^+)$ of $(X/Z,B)$ such that B^+-B is big over Z.

(2) There exists an $\mathbb{R}$-complement $(X/Z, B^+)$ of $(X/Z, B)$ and an effective big $\mathbb{R}$-mobile over Z divisor A on X such that $\operatorname{Supp} A \subseteq \operatorname{Supp}(B^+ - B)$. Moreover, we can suppose that A is $\mathbb{R}$-ample over Z when X/Z is projective.

(3) For every finite set of divisors $D_1, \ldots, D_n$ on X, there exists an $\mathbb{R}$-complement $(X/Z, B^+)$ of $(X/Z, B)$ such that $\operatorname{Supp} D_1, \ldots, \operatorname{Supp} D_n \subseteq \operatorname{Supp}(B^+ - B)$.

The same works for a bd-pair $(X/Z, B + \mathcal{P})$ with a boundary B.

In this section we use the word generic only in this sense.

Remark 6. For generic $(X/Z, B)$, every klt $\mathbb{R}$-complement satisfies (1) and (2) with an effective $\mathbb{R}$-mobile divisor, but (2) with $\mathbb{R}$-ample and (3) not always.

Lemma 11. *Let $(X/Z, D)$ be a pair with proper X/Z, $(X/Z, D^+)$ be its klt $\mathbb{R}$-complement and $(X/Z, D')$ be a pair such that $K + D' \sim_{\mathbb{R},Z} 0$ and $D' \geq D$. Then for every sufficiently small real positive number ε,*

$$(X/Z, (1-\varepsilon)D^+ + \varepsilon D')$$

is a klt $\mathbb{R}$-complement of $(X/Z, D)$ with

$$\operatorname{Supp}((1-\varepsilon)D^+ + \varepsilon D') = \operatorname{Supp} D^+ \cup \operatorname{Supp} D'.$$

If D is a boundary, then $D^+, (1-\varepsilon)D^+ + \varepsilon D'$ are boundaries too. The same works for bd-pairs.

Proof. Immediate by definition and [Sh92, (1,3.2)]. Recall (see Remark 1, (5)) that (3) in Definition 1 for nonlocal X/Z means that

$$K + D^+ \sim_{\mathbb{R},Z} 0.$$

Similarly we can treat bd-pairs.

□

Corollary 18. *Every generic $(X/Z, B)$ has wFt X/Z. Moreover, X/Z has Ft when X/Z is projective.*

The same holds for a generic bd-pair $(X/Z, B + \mathcal{P})$ with pseudoeffective $\mathcal{P}$ over Z.

Proof. Immediate by definition and Definition-Proposition 1, (1). Respectively, the projective case by [PSh08, Lemma-Definition 2.6, (iii)]. By Lemma 11 we can suppose that the complement in Definition-Proposition 1, (3) is klt.

Similarly we can treat bd-pairs.

□

Proof of Definition-Proposition 1. (2)⇒(1) Immediate by definition. For projective X/Z, the ample property implies the big one. Note also that by Lemma 11 we can suppose that the complement in (2) is klt.

(3)⇒(2) Apply (3) for $n = 1$ and $D_1 = A$.

(1)⇒(3) We can suppose that D_i are effective and even prime. Let $(X/Z, B^+)$ be a complement of (1). We construct another klt $\mathbb{R}$-complement $(X/Z, B')$ of $(X/Z, B)$ such that

$$\operatorname{Supp} B^+, \operatorname{Supp} D_1, \ldots, \operatorname{Supp} D_n \subseteq \operatorname{Supp} B'.$$

Since $B^+ - B$ is big, there exists an effective divisor $E \sim_{\mathbb{R},Z} B^+ - B$ such that

$$\operatorname{Supp} D_1, \ldots, \operatorname{Supp} D_n \subseteq \operatorname{Supp} E.$$

Consider $B' = E + B$. By construction $B' \geq B$ and

$$K + B' = K + B + E \sim_{\mathbb{R},Z} K + B + B^+ - B = K + B^+ \sim_{\mathbb{R},Z} 0.$$

Now Lemma 11 implies the existence of a required $\mathbb{R}$-complement $(X/Z, B')$.

Similarly we can treat bd-pairs.

□

Basic properties of generic pairs. (1) *Decreasing of boundary preserves the generic property*: if $(X/Z, B)$ is generic and B' is boundary on X such that $B' \leq B$ then $(X/Z, B')$ is also generic.

Proof. Immediate by definition. □

(2) *Small modifications preserves the generic property*: if $(X/Z, B)$ is generic and $(X'/Z, B)$ is its small modification over Z then $(X'/Z, B)$ is also generic.

Proof. Immediate by definition. The big property over Z is preserved under small modifications. □

(3) *Crepant models with a boundary preserve the generic property*: $(X/Z, B)$ is generic log pair and $(Y/Z, B_Y)$ its crepant model with a boundary B_Y then $(Y/Z, B_Y)$ is also generic.

Proof. Immediate by definition. Indeed, $\mathbb{B}^+ - \mathbb{B} = \mathbb{B}_Y^+ - \mathbb{B}_Y$ is b-big with a big trace on Y and $B_Y = \mathbb{B}_Y$ is a boundary by definition and our assumptions. □

(4) *The same properties holds for generic bd-pairs.*

Proof. Immediate by definition. □

Lemma 12. *Let $(X/Z \ni o, B)$ be a local generic pair which is not semiexceptional. Then there exists an $\mathbb{R}$-complement $(X/Z \ni o', B^+)$ with a crepant plt model*

$$\begin{array}{ccc} (Y, B_Y^+) & \overset{\varphi}{\dashrightarrow} & (X, B^+) \\ & \searrow & \downarrow \\ & & Z \ni o' \end{array}, \tag{9.0.7}$$

where φ is a crepant birational 1*-contraction φ over $Z \ni o'$ and o' is a specialization of o, such that*

Y is projective ($\mathbb{Q}$-factorial);

(Y, B_Y^+) is plt with a single lc center S;

the only possible exceptional divisor of φ is S; and

there exists an effective ample over $Z \ni o'$ divisor A on Y with

$$S \notin \operatorname{Supp} A \subseteq \operatorname{Supp}(B_Y^+ - B_Y^{\log}).$$

If $(X/Z \ni o, B)$ is not global, then the nonsemiexceptional assumption is redundant and S is over o', a sufficiently general closed point of the closure of o.

The same holds for a generic bd-pair $(X/Z, B + \mathcal{P})$ with pseudoeffective $\mathcal{P}$ over Z.

Warning: Every global plt model $(Y/Z \ni o', B_Y^{\log})$ is not generic because it is not klt. However, the klt property holds in the nonglobal case for the *generalization* $(Y/Z \ni o, B_Y^{\log})$ when $o' \neq o$.

Proof. After a small birational modification of X as in Lemma 1 and $\mathbb{Q}$-factorialization over $Z \ni o$, we can suppose that X is projective over $Z \ni o$ and $\mathbb{Q}$-factorial. Every small modification of $X/Z \ni o$ over $Z \ni o$ is again generic and nonsemiexceptional. Thus by Corollary 18 we can suppose that $X/Z \ni o$ has Ft. We can suppose also that $\dim X \geq 2$.

Suppose that $(X/Z \ni o, B^+)$ is a klt $\mathbb{R}$-compliment as in (3) with distinct very ample prime divisors $A_1, \ldots, A_r$ generating numerically $\operatorname{Pic}(X/Z \ni o)$. (Actually, the generation is modulo $\sim_{\mathbb{R}} /Z \ni o$ by [ShCh, Corollary 4.5].) We suppose also that $\cap \operatorname{Supp} A_i = \emptyset$. So, every component $a_i A_i$ in $B^+ = E' + \sum a_i A_i, a_i = \operatorname{mult}_{A_i} B^+ > 0, E' \geq 0$, can be replace by any general $\mathbb{R}$-linearly equivalent over Z divisor $a_i A_i'$, where $A_i \sim A_i'/Z \ni o$. (It is better to think about A_i as a divisor in the Alexeev sense.)

Since $(X/Z \ni o, B)$ is nonsemiexceptional, there exists an effective divisor $D' \geq B$ such that $(X/Z \ni o, D')$ is a log pair with $K + D' \sim_{\mathbb{R}} 0/Z \ni o$ (a nonlc $\mathbb{R}$-complement). Taking a weighted combinations $D' := aB^+ + (1-a)D', a \in (0,1)$, and similarly for B^+, we can suppose additionally that $\operatorname{Supp} B^+ = \operatorname{Supp} D'$. Note that if $(X/Z \ni o)$ is not global, that is, o is not a closed point or is not the image of X on Z, then such a complement always exists with only vertical nonklt centers of (X, D') over a sufficiently general closed point o' of the closure of o. In this case we replace $(X/Z \ni o, D')$ by the specialization $(X/Z \ni o', D')$. Recall that by definition o, o' belong to the image of $X \to Z \ni o$ and in the lemma we can replace o by such o'. In particular, we can suppose below that $o' = o$ is closed.

Now consider a crepant projective over $Z \ni o$ log resolution (V, D_V') of (X, D'). The same works for B^+ because $\operatorname{Supp} B^+ = \operatorname{Supp} D'$. Let $M_1, \ldots, M_n$ be a finite set of (very) ample effective generators of $\operatorname{Pic}(V/Z \ni o)$ which are in general position to the log birational transform ${D_V'}^{\log}$, in particular, they do not pass through the lc centers of (V, D_V') on V, and, moreover,

$$\operatorname{Supp}({D_V'}^{\log} + \sum M_i) = \operatorname{Supp}(B^{+\log} + \sum M_i)$$

is also with simple normal crossings. (The finite generation modulo $\sim /Z \ni o$ holds because $X/Z \ni 0$ has Ft.) Divisors M_i are possibly not in $\operatorname{Supp} {D_V'}^{\log}$ but we can add them preserving our assumptions. Indeed, we can add $\psi(M_i)$ to B^+ with certain positive multiplicities, where $\psi \colon V \to X$ is the log resolution. By construction every $\psi(M_j)$ is $\mathbb{R}$-linear equivalent over $Z \ni o$ to $\sum a_i^j A_i, a_i^j \in \mathbb{R}$, over $Z \ni o$. Thus for sufficiently small positive real number

ε,
$$(X/Z \ni o, B^+ - \varepsilon(\sum a_i^j A_i) + \varepsilon\psi(M_j))$$
is a klt $\mathbb{R}$-complement of $(X/Z \ni o, B)$ with $\psi(M_j)$ in the support of $B^+ := B^+ - \varepsilon(\sum a_i A_i) + \varepsilon\psi(M_j)$. Adding each $\psi(M_j)$ we got a complement with required properties. Again as above we can suppose that $\operatorname{Supp} B^+ = \operatorname{Supp} D'$. By construction ψ is also a log resolution for new perturbed B^+, D'. If $X/Z \ni o$ is not global, the nonklt centers of (X, D') are vertical over o. But in the global case we do not have such a control.

Taking a weighted combination $B' = aB^+ + (1-a)D', a \in (0,1)$, we can suppose that $(X/Z \ni o, B')$ is an lc but nonklt $\mathbb{R}$-complement of $(X/Z \ni o, B)$. Again $\psi\colon (V, B'_V) \to (X, B')$ is a log resolution of (X, B'). In particular, (V, B'_V) is dlt. If $X/Z \ni o$ is not global, the nonklt centers of (X, B') are vertical over o.

We would like to find another lc $\mathbb{R}$-complement $(X/Z \ni o, B'')$ with a single lc center. Let
$$E = \sum E_i$$
be the sum of prime components of B'_V with multiplicity 1 except for one E_0. (The latter component exists because (X, B'_V) is dlt but not klt.) By construction
$$E \sim \sum m_i M_i / Z \ni o, m_i \in \mathbb{Z}.$$
So, for every sufficiently small positive real number ε,
$$(V, B''_V) \text{ with } B''_V = B'_V - \varepsilon E + \varepsilon(\sum m_i M_i)$$
is plt with the single nonklt center E_0 and $\sim_{\mathbb{R}} 0/Z \ni o$. If $X/Z \ni o$ is not global E_0 is vertical over o.

Since X is $\mathbb{Q}$-factorial, the pair
$$(X/Z \ni o, B'') \text{ with } B'' = \psi(B'_V - \varepsilon E + \varepsilon(\sum m_i, M_i)) = B' - \varepsilon\psi(E) + \varepsilon(\sum m_i \psi(M_i))$$
with a single lc prime b-divisor $S = E_0$, that is, with $\operatorname{mult}_S \mathbb{B}'' = 1$. Actually, B'' is effective and a boundary for sufficiently small ε because $\psi(E), \psi(M_i)$ are supported in $\operatorname{Supp} B'$. By construction
$$E - \sum m_i M_i \sim 0/X$$

too. Hence

$$\psi^*(\psi(E - \sum m_i M_i)) = E - \sum m_i M_i \sim 0/X$$

because ψ is a birational contraction (e.g., by [Sh19, Proposition 3]). For every curve C on X over Z, there exists a curve C' on V and by the last relation

$$(C.\psi(E-\sum m_i M_i)) = (\psi(C').\psi(E-\sum m_i M_i)) = (C'.\psi^*(\psi(E-\sum m_i M_i))) = (C'.0) = 0$$

by the projection formula. The pair $(X/Z \ni o, B'')$ is a 0-pair too: by the last vanishing

$$\begin{aligned}(C.K + B'') &= (C.K + B' - \varepsilon\psi(E) + \varepsilon(\sum m_i \psi(M_i))) \\ &= (C.K + B') - \varepsilon(C.\psi(E - \sum m_i M_i)) = 0.\end{aligned}$$

So, at least the complement is numerical. But since $X/Z \ni o$ has Ft it is actually an $\mathbb{R}$-complement with a single lc prime b-divisor S.

The divisor $B^+ = B''$ gives a required plt complement. Take a dlt resolution $\varphi\colon (Y, B_Y^+) \to (X, B^+)$. It has an exceptional divisor if S is exceptional. Moreover, in this case we can suppose that φ is extremal: the last contraction in the LMMP for (Y, B_Y) over X. Otherwise it is identical (or, more precisely, the above $\mathbb{Q}$-factorialization). In the last case the property with an ample divisor A holds by construction. Otherwise S is exceptional on X and we need to construct an ample divisor A on Y. By construction, for every divisor M_j, its birational image $M_{j,Y}$ on Y is big on S over Z and does not contain S. So, $M_{j,Y}$ is ample on Y over X. Recall for this that φ is extremal. A required A we can construct as linear combination $\varphi^*(A') + \varepsilon M_{j,Y}$ for some $0 < \varepsilon \ll 1$, where A' is an ample over Z divisor on X supported in B^+ and not passing through $\varphi(S)$. Take $A' = A_i$ for such A_i that $\varphi(S) \not\subset \operatorname{Supp} A_i$. Such A_i exists because $\cap \operatorname{Supp} A_i = \emptyset$.

If $X/Z \ni o$ is not global, S is vertical over o.

Similarly we can treat bd-pairs.

□

Remark: it looks that similarly we can construct two disjoint lc center S_1, S_2 if we have at least two summands in E. However, this is impossible because in our construction every two components of E intersects each other.

Construction 5 (Adjoint pair)**.** Consider a plt model (9.0.7) of Lemma 12 and assume that Y is $\mathbb{Q}$-factorial. The model gives an *adjoint* projective log pair (S, B_S) for $(Y, B_Y^{\log})$ or $(Y, B_Y + (1 - \operatorname{mult}_S B)S)$, if S is not exceptional on X, with respectively

$$B_S = \operatorname{Diff}(B_Y^{\log} - S) \text{ or } B_S = \operatorname{Diff}(B - (\operatorname{mult}_S B)S).$$

The adjoint pair is always global by Lemma 12. By divisorial adjunction and its monotonicity, (S, B_S^+) with

$$B_S^+ = \operatorname{Diff}(B_Y^+ - S)$$

is an *induced* adjoint $\mathbb{R}$-complement of (S, B_S).

The same applies to (9.0.7) with a bd-pair $(X/Z \ni o, B + \mathcal{P})$ having pseudoeffective $\mathcal{P}$. The adjoint bd-pair $(S, B_S + \mathcal{P}_S)$ is a log one with $\mathcal{P}_S = \mathcal{P}_{\rceil S}$ (see the birational restriction $\rceil$ in [Sh03, Mixed restriction 7.3]. So, $\mathcal{P}_S$ is b-nef (and pseudoeffective) if $\mathcal{P}$ is b-nef over $Z \ni o$. Hence $(S, B_S + \mathcal{P}_S)$ is a bd-pair of index m if $(X/Z \ni o, B + \mathcal{P})$ is a bd-pair of index m.

Corollary 19. *The adjoint pair (S, B_S) is (global) generic with the induced klt $\mathbb{R}$-complement. In particular, S is irreducible of wFt.*

Addendum 49. *The same works for the adjoint bd-pair $(S, B_S + \mathcal{P}_S)$ of index m.*

Proof. By construction $(Y/Z \ni o, B_Y^+), (S, B_S^+)$ are log pairs, where S is normal irreducible [Sh92, Lemma 3.6]. Moreover, S is projective. It is true if $Y/Z \ni o$ is global because Y is projective. Otherwise by Lemma 12 S is projective over a closed point.

If $C \subseteq S$ is a curve then C is over the closed point $o' \in Z$ and

$$(C.K_S + B_S^+) = (C.K_Y + B_Y^+) = (\varphi(C).K + B^+) = 0.$$

Note that even in the local case C is over a closed point in Z and so is over Z. By construction and monotonicity of divisorial adjunction,

$$B_Y^+ \geq B_Y \text{ and } B_S^+ \geq B_S.$$

So, (S, B_S^+) is an $\mathbb{R}$-complement of (S, B_S).

The complement is generic because there exists an effective ample divisor A supported in $\operatorname{Supp}(B_Y^+ - {B_Y}^{\log})$. It is in general position with S. So, the restriction $A_{|S}$ is well-defined and is an effective ample divisor on S supported in $\operatorname{Supp}(B_S^+ - B_S)$.

For the bd-pair $(S, B_S + \mathcal{P})$ note that $\mathcal{P}_S$ is b-nef if $\mathcal{P}$ is b-nef over $Z \ni o$. $\square$

Construction 6. Let d be a nonnegative integer, I, ε, v, e be the data as in Restrictions on complementary indices, and $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0,1]$. Denote by $\widetilde{\Phi} = \Phi(\overline{\mathfrak{R}} \cup \{0\})$ the hyperstandard set constructed in 6.11.

By dimensional induction there exists a finite set of positive integers $\mathcal{N} = \mathcal{N}(d-1, I, \varepsilon, v, e, \widetilde{\Phi})$ such that

Restrictions: every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data;

Existence of b-n-complement: if (S, B_S) is a generic pair of dimension $d-1$ with Ft S and a boundary B_S then $(S^\sharp, B_{S,n\_\widetilde{\Phi}}{}^\sharp{}_{S^\sharp})$ has a b-n-complement (S, B_S^+) for some $n \in \mathcal{N}$.

For bd-pairs we add a positive integer m. So, $\mathcal{N} = \mathcal{N}(d-1, I, \varepsilon, v, e, \widetilde{\Phi}, m)$ and replace (S, B_S) by a bd-pair $(S, B_S + \mathcal{P}_S)$ of dimension $d-1$ and of index $m|n$.

Dimensional induction gives a more precise choice of n that has important geometrical implications (see Corollary 20 below). For this introduce the following (decreasing) filtration of $\mathcal{N}$.

Generic type filtration with respect to dimension i

$$\mathcal{N} = \mathcal{N}^d \supseteq \cdots \supseteq \mathcal{N}^i \supseteq \cdots \supseteq \mathcal{N}^0 \supseteq \mathcal{N}',\ 0 \le i \le d, \tag{9.0.8}$$

where $\mathcal{N}'$ is a subset of $\mathcal{N}$. Its *associated* filtration of hyperstandard sets is

$$\Gamma(\mathcal{N}^d, \Phi) \supseteq \cdots \supseteq \Gamma(\mathcal{N}^i, \Phi) \supseteq \cdots \supseteq \Gamma(\mathcal{N}^0, \Phi) \supseteq \Gamma(\mathcal{N}, \Phi).$$

For dimension $i > 0$, put

$$\mathcal{N}_i = \mathcal{N}^i \setminus \mathcal{N}^{i-1};$$

for $i = 0$, $\mathcal{N}_0 = \mathcal{N}^0 \setminus \mathcal{N}'$. That is, the next set in the last case is $\mathcal{N}^{-1} = \mathcal{N}'$ but without dimension. According to Theorem 14 below, b-n-complements of generic type are coming (extended) from dimension i and $n \in \mathcal{N}_i$ in this case.

For every i, between $\mathcal{N}^i \supseteq \mathcal{N}^{i-1}$, there exists an additional filtration with respect to semiexceptional types in the dimension i (see Semiexceptional filtration in Section 8).

Definition 7. Let (9.0.8) be a filtration of Construction 6. Consider a certain class of pairs $(X/Z \ni o, B)$ of dimension d with boundaries which have an (b-)n-complement with $n \in \mathcal{N}^d$. Such a *pair* $(X/Z \ni o, B)$ and its (b-)*n-complement have generic type of dimension* $0 \le i \le d$ if the complement is extended from a semiexceptional b-n-complement in dimension i. So, additionally, we can associate to the pair its (filtration) semiexceptional type $(r, f), 0 \le r \le i - f - 1, 0 \le f \le i - 1$ or the exceptional type $(-1, -)$.

We say that the *existence* n-complements *agrees with the generic type filtration* if every pair $(X/Z \ni o, B)$ in the class has an *n-complement* $(X/Z \ni o, B^+)$ *of filtration generic type i* with $n \in \mathcal{N}_i$. Additionally, $(X/Z \ni o, B^+)$ *of the type i* is an n-complement of itself, of $(X/Z \ni o, B_{n\_\mathcal{N}^{i-1}\_\Phi}), (X/Z \ni o, B_{n\_\mathcal{N}^{i-1}\_\Phi}{}^\sharp)$ and a b-n-complement of itself, of $(X/Z \ni o, B_{n\_\mathcal{N}^{i-1}\_\Phi}), (X/Z \ni o, B_{n\_\mathcal{N}^{i-1}\_\Phi}{}^\sharp), (X^\sharp/Z \ni o, B_{n\_\mathcal{N}^{i-1}\_\Phi}{}^\sharp{}_{X^\sharp})$, if $(X, B_{n\_\mathcal{N}^{i-1}\_\Phi}), (X, B_{n\_\mathcal{N}^{i-1}\_\Phi}{}^\sharp)$ are log pairs respectively.

The boundary $B_{n\_\mathcal{N}^{i-1}\_\Phi}{}^\sharp{}_{X^\sharp}$ can be slightly increased if we take into consideration the (filtration) semiexceptional type (r, f) of $(X/Z \ni o, B)$.

Possibly, the generic type is not unique. For the uniqueness we can take minimal (better than maximal) type. The same concerns the semiexceptional type.

The same applies to bd-pairs of dimension d.

Remark 7. (1) In particular, $(X/Z \ni o, B)$ has generic type d if the pair is generic and semiexceptional itself. If the pair is generic and not semiexceptional then by Lemma 12 and Construction 5 there exists an adjoint generic pair (S, B_S). By definition (S, B_S) has generic type $i \le d - 1$ and by dimensional induction this is a generic type of $(X/Z \ni o, B)$ by Step 6 in the proof of Theorem 14. By the induction it has also [some] semiexceptional type (r, f).

(2) Generic type d only possible for global pairs.

(3) In the proof of Theorem 14 we apply Construction 5 to $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$, where $\mathcal{N} = \mathcal{N}^{d-1}$. However, we can apply the same construction directly to $(X/Z \ni o, B)$ if the latter pair is generic (cf. Addendum 52).

(4) [Warning:] Possibly, there are other n-complements which are not agree with the filtration (cf. Example 13). Additionally, we can have n-complements without any type but with $n \in \mathcal{N}$.

Theorem 14 (Generic n-complements)**.** *Let d be a nonnegative integer, I, ε, v, e be the data as in Restrictions on complementary indices, and $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0,1]$. Then there exists a finite set $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi)$ of positive integers such that*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data.*

Existence of n-complement: *if $(X/Z \ni o, B)$ is a pair with $\dim X = d$, a boundary B, connected X_o and with generic $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ then $(X/Z \ni o, B)$ has an n-complement $(X/Z \ni o, B^+)$ for some $n \in \mathcal{N}$.*

Addendum 50. *$(X/Z \ni o, B^+)$ is an n-complement of itself and of $(X/Z \ni o, B_{n\_\Phi}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp)$, and is a b-$n$-complement of itself and of $(X/Z \ni o, B_{n\_\Phi}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp), (X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$, if $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$ are log pairs respectively.*

Addendum 51. *$\mathcal{N}$ has a generic type filtration (9.0.8) with $\mathcal{N}^d = \mathcal{N}$, with any finite set of positive integers $\mathcal{N}'$, satisfying Restrictions on complementary indices with the given data, and the existence n-complements agrees the filtration for the class of pairs under assumptions of Existence of n-complements in the theorem.*

Addendum 52. *In particular, the theorem and addenda applies to generic pairs $(X/Z \ni o, B)$ instead of with generic $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$.*

Addendum 53. *The same holds for bd-pairs $(X/Z \ni o, B + \mathcal{P})$ of index m with $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi, m)$. That is,*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data and $m|n$.*

Existence of n-complement: *if $(X/Z \ni o, B + \mathcal{P})$ is a bd-pair of index m with $\dim X = d$, a boundary B, connected X_o and with generic $(X/Z \ni o, B_{\mathcal{N}\_\Phi} + \mathcal{P})$ then $(X/Z \ni o, B + \mathcal{P})$ has an n-complement $(X/Z \ni o, B^+ + \mathcal{P})$ for some $n \in \mathcal{N}$.*

Addendum 51 holds literally. In Addendum 50 $(X/Z \ni o, B^+ + \mathcal{P})$ is an n-complement of itself and of $(X/Z \ni o, B_{n\_\Phi} + \mathcal{P}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp + \mathcal{P})$, and is a b-n-complement of itself and of $(X/Z \ni o, B_{n\_\Phi} + \mathcal{P}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp + \mathcal{P}), (X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp} + \mathcal{P})$, if $(X, B_{n\_\Phi} + \mathcal{P}_X), (X, B_{n\_\Phi}{}^\sharp + \mathcal{P}_X)$ are log bd-pairs respectively. In Addendum 52 $(X/Z \ni o, B + \mathcal{P}), (X/Z \ni o, B_{\mathcal{N}\_\Phi} + \mathcal{P})$ should be instead of $(X/Z \ni o, B), (X/Z \ni o, B_{\mathcal{N}\_\Phi})$ respectively.

Warning 4. Sets $\mathcal{N}(d, I, \varepsilon, v, e, \Phi), \mathcal{N}(d, I, \varepsilon, v, e, \Phi, m)$ are not unique and may be not suitable in other situations of the paper (cf. Remark 2, (1)). We can take such minimal sets under inclusion but they can be not unique.

It would be interesting to find a more canonical way to construct such set $\mathcal{N}(d, I, \varepsilon, v, e, \Phi), \mathcal{N}(d, I, \varepsilon, v, e, \Phi, m)$ and/or to find an estimation on their largest element.

Proof. Take $\mathcal{N} = \mathcal{N}_d \cup \mathcal{N}^{d-1}$, where $\mathcal{N}_d = \mathcal{N}(d, I, \varepsilon, v, e, \Gamma(\mathcal{N}^{d-1}, \Phi))$ is the finite set from Theorem 13 and $\mathcal{N}^{d-1} = \mathcal{N}(d-1, I, \varepsilon, v, e, \widetilde{\Phi})$ is from Construction 6. By Corollary 17 we can suppose that $\mathcal{N}_d \cap \mathcal{N}^{d-1} = \emptyset$. By construction $\mathcal{N}$ is a finite set of positive integers and satisfies Restrictions.

Let $(X/Z \ni o, B)$ be a pair satisfying the assumptions in Existence of n-complements. Since B is a boundary, $B_{\mathcal{N}\_\Phi}$ is well-defined.

Step 1. *It is enough to construct a b-n-complement $(X/Z \ni o, B^+)$ of $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ for some $n \in \mathcal{N}$.* Indeed, then we have all required complements in the theorem and in Addendum 50 by Propositions 1, 8 and Corollary 6. In particular, the corollary and fact that $(X/Z \ni o, B^+)$ is a b-n-complement of $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ imply that $(X/Z \ni o, B^+)$ is an n-complement of $(X/Z \ni o, B)$ (cf. the proof of Addendum 29).

Step 2. *We can suppose that $(X/Z \ni o, B_{\mathcal{N}^{d-1}\_\Phi})$ is generic and not semiexceptional*. Indeed by Basic properties of generic pairs (1) and Proposition 6, $B_{\mathcal{N}^{d-1}\_\Phi} \le B_{\mathcal{N}\_\Phi}$ and the pair $(X/Z \ni o, B_{\mathcal{N}^{d-1}\_\Phi})$ is also generic. If $(X/Z \ni o, B_{\mathcal{N}^{d-1}\_\Phi})$ is semiexceptional then by Addendum 46 $(X^\sharp/Z \ni o, B_{n\_\mathcal{N}^{d-1}\_\Phi}{}^\sharp{}_{X^\sharp})$ has a b-n-complement $(X/Z \ni o, B^+)$ for some $n \in \mathcal{N}_d$. We apply Theorem 13 and its Addendum 46 to the pair $(X, B_{\mathcal{N}\_\Phi})$ with $\dim X = d$, the boundary $B_{\mathcal{N}\_\Phi}$ and to the hyperstandard set $\Gamma(\mathcal{N}^{d-1}, \Phi)$ (see Proposition 4); the other data is the same. Notice that X/Z is global in this case (cf. Remark 7, (2)), that is , we can suppose that $Z =$ pt. and X/Z is just X. Since $(X, B_{\mathcal{N}^{d-1}\_\Phi}), (X, B_{\mathcal{N}\_\Phi})$ are generic, X has wFt and $(X, B_{\mathcal{N}\_\Phi})$ has an $\mathbb{R}$-complement by Corollary 18 and definition respectively. By definition and construction $\mathcal{N}^{d-1} \subseteq \mathcal{N}, B_{\mathcal{N}\_\Phi, \mathcal{N}^{d-1}\_\Phi} = B_{\mathcal{N}^{d-1}\_\Phi}$

and $(X, B_{\mathcal{N}\_\Phi,\mathcal{N}^{d-1}\_\Phi})$ is semiexceptional. By our assumptions $(X, B_{\mathcal{N}\_\Phi})$ is also semiexceptional and has generic type d. Indeed, $B_{\mathcal{N}^{d-1}\_\Phi} \leq B_{\mathcal{N}\_\Phi}$ and $(X, B_{\mathcal{N}\_\Phi})$ has an $\mathbb{R}$-complement. If $(X, B_{\mathcal{N}\_\Phi})$ is not semiexceptional then there exists $B' \geq B_{\mathcal{N}\_\Phi} \geq B_{\mathcal{N}^{d-1}\_\Phi}$ such that $K + B' \sim_{\mathbb{R}} 0$ and (X, B') is not lc. Hence $(X, B_{\mathcal{N}^{d-1}\_\Phi})$ is also not semiexceptional, a contradiction. Thus by Addendum 46 $(X^\sharp, B_{n\_\mathcal{N}^{d-1}\_\Phi}{}^\sharp{}_{X^\sharp})$ has a b-n-complements (X, B^+) with $n \in \mathcal{N}_d$. The constructed complement exactly agrees with the filtration. Hence Addendum 51 holds in this case too.

But we need a slightly weaker complement. Again by Proposition 6 $B_{n\_\Phi} \leq B_{n\_\mathcal{N}^{d-1}\_\Phi}$. Hence by Corollary 7 or arguments as in Step 4 below $\mathbb{B}_{n\_\Phi}{}^\sharp \leq \mathbb{B}_{n\_\mathcal{N}^{d-1}\_\Phi}{}^\sharp$. Thus by Proposition 1 (X, B^+) is also a b-n-complement of $(X^\sharp, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ ($X^\sharp$ here is not necessary the same as above and below).

This concludes the semiexceptional case. Below we suppose that $(X/Z \ni o, B_{\mathcal{N}^{d-1}\_\Phi})$ is generic and not semiexceptional. In this case we construct a b-n-complement $(X/Z \ni o, B^+)$ of $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ with $n \in \mathcal{N}^{d-1}$. We extend also the filtration of $\mathcal{N}$ starting from $\mathcal{N}^{d-1}$.

For simplicity of notation suppose now that $B = B_{\mathcal{N}^{d-1}\_\Phi}$, in particular, $B \in \Gamma(\mathcal{N}^{d-1}, \Phi)$ (but this is not important for the following). Indeed, by definition and Proposition 6 $B_{\mathcal{N}^{d-1}\_\Phi,n\_\Phi} = B_{n\_\Phi}$ and $B_{\mathcal{N}^{d-1}\_\Phi,n\_\mathcal{N}^{i-1}\_\Phi} = B_{n\_\mathcal{N}^{i-1}\_\Phi}$ for every $n \in \mathcal{N}^{d-1}, 0 \leq i \leq d$. By Step 2 $(X/Z \ni o, B)$ is generic and not semiexceptional itself. (Actually, if $(X/Z \ni o, B)$ is generic and not semiexceptional then the same holds for $(X/Z \ni o, B_{\mathcal{N}\_\Phi}), (X/Z \ni o, B_{\mathcal{N}^{d-1}\_\Phi})$.)

Step 3. *Construction of* $(Y/Z \ni o', B_Y^+)$ *as in Lemma 12 and* $(Y/Z \ni o', D)$ *as in Corollary 13.* Indeed, we can apply the lemma to $(X/Z \ni o, B)$. So, there exists an $\mathbb{R}$-complement with a crepant model $(Y/Z \ni o', B_Y^+)$ as in(9.0.7) and such that

(1) $Y \dashrightarrow X$ is a birational 1-contraction, in particular, every prime divisor of X is a divisor on Y;

(2) o' is a sufficiently general point of the closure of o;

(3) the central fiber $Y_{o'}$ is connected;

(4) (Y, B_Y^+) is plt with a complete single lc center S; and

(5) there exists an effective ample over $Z \ni o'$ divisor A on Y with

$$S \notin \operatorname{Supp} A \subseteq \operatorname{Supp}(B_Y^+ - B_Y^{\log}).$$

(1-2) and (4-5) hold by Lemma 12. The connectedness in (3) and (4) holds by the connectedness of X_o and (5) with lc connectedness respectively.

Now we take a boundary $D = B_Y^+ - aA$ on Y, where a is a sufficiently small real number. The pair $(Y/Z \ni o', D)$ satisfies the assumptions of Corollary 13. (1) of the corollary holds by construction and (2-3). (2) of the corollary follows from (4-5). (3) of the corollary follows from (5):

$$-(K_Y + D) = -(K_Y + B_Y^+) + aA \equiv aA/Z \ni o'.$$

By (1), construction and since B is a boundary, D is actually a boundary if a is sufficiently small. Additionally, we can suppose that

(6) for every prime divisor P of X,

$$\operatorname{mult}_P D = \operatorname{mult}_P \mathbb{D} \geq \operatorname{mult}_P B.$$

Indeed, by (1) P is a prime divisor on Y and

$$\operatorname{mult}_P \mathbb{D} = \operatorname{mult}_P D = (\operatorname{mult}_P B_Y^+) - a \operatorname{mult}_P A.$$

Thus by (5) $\operatorname{mult}_P D = \operatorname{mult}_P B_Y^+ \geq \operatorname{mult}_P B$ if $P \notin \operatorname{Supp} A \subseteq \operatorname{Supp}(B_Y^+ - B_Y^{\log})$. Otherwise P belongs to a finite set for which $\operatorname{mult}_P B_Y^+ > \operatorname{mult}_P B$. Notice also that $S \notin \operatorname{Supp} A$ by (5) and $\operatorname{mult}_S D = \operatorname{mult}_S B_Y^+ = 1$.

Step 4. *It is enough to construct a b-n-complement of $(Y^\sharp/Z \ni o', D_{n\_\Phi}{}^\sharp{}_{Y^\sharp})$ for some $n \in \mathcal{N}^{d-1}$.* Indeed, this complement will be also a b-n-complement of $(X^\sharp/Z \ni o', B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ and of $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ by Proposition 1 because o' is a specialization of o and $\mathbb{D}_{n\_\Phi}{}^\sharp \geq \mathbb{B}_{n\_\Phi}{}^\sharp{}_{X^\sharp}$. The last inequality follows from (6) by Corollary 8, actually, for any n and Φ. We apply the corollary to $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ with a birational 1-contraction $X \dashrightarrow X^\sharp/Z \ni o$. Such a contraction exists by Construction 2 because $B_{n\_\Phi} \leq B$ and $(X/Z \ni o', B_{n\_\Phi})$ has an $\mathbb{R}$-complement again by Proposition 1. The same arguments imply that Construction 2 is applicable to $(Y/Z \ni o', D_{n\_\Phi})$. The assumption (1) of Corollary 8 holds by the construction. On the other hand, by construction $Y \dashrightarrow X^\sharp/Z \ni o'$ is a birational 1-contraction and every prime divisor of $X^\sharp$ is a divisor on X and Y. Hence the assumption (2) of Corollary 8 follows from (6), the definition of $(-)_{n\_\Phi}$ and Proposition 8: for every prime divisor P of $X^\sharp$,

$$\operatorname{mult}_P B_{n\_\Phi}{}^\sharp{}_{X^\sharp} = \operatorname{mult}_P B_{n\_\Phi} \leq \operatorname{mult}_P D_{n\_\Phi} \leq \operatorname{mult}_P \mathbb{D}_{n\_\Phi}{}^\sharp.$$

Step 5. *Construction of a b-n-complement of* $(Y^\sharp/Z \ni o', D_{n\_\Phi}{}^\sharp{}_{Y^\sharp})$ *for some* $n \in \mathcal{N}^{d-1}$. We apply Corollary 13 to $(Y/Z \ni o', D)$. The assumptions (1-3) of the corollary we verified in Step 3. So, we need to verify (4) of Corollary 13. As in Corollary 19 but now by (5), the adjoint pair (S, D_S) in (4) of Corollary 13 is generic global of dimension $\dim X - 1 = d - 1$. Notice that S is irreducible by (3) and [Sh92, Lemma 3.6]. By (4) and construction, (S, D_S) is klt with a boundary D_S. It has a highest crepant model $(S', D_{S'})$ with a boundary $D_{S'}$. By Basic properties of generic pairs (3) $(S', D_{S'})$ is also generic. By Corollary 18 S' has also wFt. Thus by Construction 6 $(S'^\sharp, D_{S,n\_\widetilde{\Phi}}{}^\sharp{}_{S'^\sharp})$ has a b-n-complement $(S', D_{S'}^+)$ for some $n \in \mathcal{N}^{d-1}$. This is exactly (4) of Corollary 13. More accurately, we need to use the induced b-n-complement $(S'^\sharp, D_{S'^\sharp}^+)$.

So, the theorem and Addendum 50 are established.

Step 6. *Addendum 51.* We use dimensional induction to construct $\mathcal{N}^{d-1}$ with a required filtration. By 6.11 $\widetilde{\Phi}$ is already closed: $\widetilde{\widetilde{\Phi}} = \widetilde{\Phi}$. In the induction we work with $\widetilde{\Phi}$ instead of Φ. By Steps 1-5 above and especially by Lemma 12, we can suppose that X is global and every pair (X, B) in the induction has $\dim X \le d - 1$. By (4-5) of Step 3 (X, B) is generic. Essentially, we verify the induction in Construction 6.

Induction step $d = 0$. Take $\mathcal{N}_0 = \mathcal{N}(0, I, \varepsilon, v, e, \widetilde{\Phi})$, a finite set from Theorem 13 in dimension 0. By construction $\mathcal{N}_0$ is a finite set of positive integers and satisfies Restrictions on complementary indices with the given data. The existence of complements in dimension 0 is trivial. Moreover, we can add any finite set of positive integers $\mathcal{N}' = \mathcal{N}^{-1}$ satisfying Restrictions: $\mathcal{N}^0 = \mathcal{N}_0 \cup \mathcal{N}'$ and $\mathcal{N}_0 \cap \mathcal{N}' = \emptyset$. Actually [BSh, Corollary 1.3] is enough for this step.

General step of induction: construction of $\mathcal{N}^{i+1}$. Suppose that a filtration is constructed up to $\mathcal{N}^i, 0 \le i \le d - 2$, and the filtration satisfies Addendum 51 in dimension i with $\widetilde{\Phi}$ instead of Φ. The class of pairs consists of global generic pairs (X, B) with a boundary B and $\dim X = i$. Additionally we can assume that (X, B) is klt and highest with a boundary. Take $\mathcal{N}_{i+1} = \mathcal{N}(i + 1, I, \varepsilon, v, e, \Gamma(\mathcal{N}^i, \widetilde{\Phi}))$ of Theorem 13. By Corollary 17 we can suppose that $\mathcal{N}_{i+1}$ is disjoint from $\mathcal{N}^i$. This gives a required filtration up to $\mathcal{N}^{i+1} = \mathcal{N}_{i+1} \cup \mathcal{N}^i$ and by induction up to $\mathcal{N}^{d-1} = \mathcal{N}_{d-1} \cup \mathcal{N}^{d-2}$. By construction $\mathcal{N}^{i+1}$ is a finite set of positive integers and satisfies Restrictions. We need to verify Addendum 51 in dimension $i+1$ with $\widetilde{\Phi}$ instead of Φ. The class of pairs consists of global generic pairs (X, B) with a boundary B and

$\dim X = i+1$. Additionally we assume that (X, B) is klt and highest with a boundary. In particular, we assign generic type $0 \le t \le i+1$ for every such pair. For this we apply Steps 1-5 above with $d = i+1$ and $\Phi = \widetilde{\Phi}$.

By Step 1 it is enough to construct a b-n-complement (X, B^+) of $(X^\sharp, B_{n\_\mathcal{N}^{t-1}\_\widetilde{\Phi}}{}^\sharp{}_{X^\sharp})$ with $n \in \mathcal{N}_t$ for (X, B) of type t.

By our assumptions (X, B) is generic. Thus $(X, B_{\mathcal{N}^i\_\widetilde{\Phi}})$ is also generic by Basic properties of generic pairs (1). If $(X, B_{\mathcal{N}^i\_\widetilde{\Phi}})$ is additionally semiexceptional then its type is $i+1$ and the required b-n-complement of type $i+1$ with $n \in \mathcal{N}_{i+1}$ exists by Addendum 46.

So, we assume that $(X, B_{\mathcal{N}^i\_\widetilde{\Phi}})$ is not semiexceptional as in Step 2. In this case $t \le i$. By Steps 3-5 a required b-n-complement is extended from a b-n-complement of a generic pair (S, D_S) of dimension i. By induction the pair has some type $0 \le t \le i$. This is a type of (X, B) and of $(X, B_{\mathcal{N}^i\_\widetilde{\Phi}})$. Indeed, by induction $(S^\sharp, D_{S,n\_\mathcal{N}^{t-1}\_\widetilde{\Phi}}{}^\sharp{}_{X^\sharp})$ has a b-n-complement of type t with $n \in \mathcal{N}_t$. By Steps 4-5, the complement can be extended to a b-n-complement (X, B^+) of $(X^\sharp, B_{n\_\mathcal{N}^{t-1}\_\widetilde{\Phi}}{}^\sharp{}_{X^\sharp})$ with the same $n \in \mathcal{N}_t$ for (X, B) and same type t. Notice only that $B_{\mathcal{N}^i\_\widetilde{\Phi}, n\_\mathcal{N}^{t-1}\_\widetilde{\Phi}} = B_{n\_\mathcal{N}^{t-1}\_\widetilde{\Phi}}$ because $t \le i$ and $n \in \mathcal{N}^i, \mathcal{N}^{t-1} \subseteq \mathcal{N}^i$.

This completes the induction in Construction 6.

Finally, to complete the proof of Addendum 51 we assign type $t \le d-1$ to every pair $(Y/Z \ni o', D)$ of Step 3. This is the type of $(S', D_{S'})$ in Step 5. The same type we assign to $(X/Z \ni o, B)$. This is immediate for $(Y/Z \ni o', D)$ by Corollary 13 as in Step 5. By arguments of Step 4 this applies also to $(X/Z \ni o, B)$.

Warning: In the proof of the theorem we can't replace $(X/Z \ni o, B)$ by its crepant model $(X'/Z \ni o, B_{X'})$ because $'$ and $\mathcal{N}\_\Phi$ do not commute. Moreover, assumptions in Existence of n-complements do not imply that $(X'/Z \ni o, B_{X',\mathcal{N}\_\Phi})$ is generic. (This is true but in opposite direction.) However, we do not need this in the induction because we take $'$ on an induced model with the generic property (cf. Corollary 13 and Step 5).

Step 7. *Other addenda.* Addendum 52 follows from Basic properties of generic pairs (1).

Similarly we can treat bd-pairs.

$\square$

Corollary 20. *Under assumptions and in notation of Theorem 14 if $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ has generic type i then*

$$\operatorname{reg}(X) \ge \operatorname{reg}(X/Z \ni o, B^+) \ge d - i - 1.$$

More precisely, with the additional (filtration) semiexceptional type (r, f) *with:*

$$\mathrm{reg}(X/Z \ni o) \geq \mathrm{reg}(X/Z \ni o, B^+) \geq d + r - i.$$

In general, we can't replace $\mathrm{reg}(X/Z \ni o) = \mathrm{reg}(X/Z \ni o, 0)$ by $\mathrm{reg}(X/Z \ni o, B)$. However, this works if $B^+ \geq B$.

Proof. Indeed, the first right inequality hold for type d because $\mathrm{reg}(X/Z \ni o, B^+) \geq -1$ (-1 for the empty $\mathrm{R}(X/Z \ni o, B^+)$). Otherwise by induction, Theorem 11 and Corollary 13 the extension increase reg by 1:

$$\mathrm{reg}(X/Z \ni o, B^+) = \mathrm{reg}(S', B^+_{S'}) + 1 = \mathrm{reg}(S, B^+_S) + 1 \geq d-1-i-1+1 = d-i-1.$$

The second right inequality follows by the same arguments. However the induction use definition: if (filtration) semiexceptional generic (X, B) of dimension d has the (filtration) semiexceptional type (r, f) then $\mathrm{reg}(X, B^+) = r$ (cf. Corollary 14).

Both left inequalities with $\mathrm{reg}(X/Z \ni o)$ hold by [Sh95, Proposition-Definition 7.11]. $\square$

10 Klt nongeneric complements: lifting of generic

In this section we establish Theorem 3 under the assumption (1) of the theorem.

Klt type. A pair $(X/Z \ni, B)$ with a boundary B has *klt* type if there exists a klt $\mathbb{R}$-complement $(X/Z \ni o, B^+)$ of $(X/Z \ni o, B)$. In this situation a log pair $(X/Z \ni o, B)$ is klt itself.

The same definition works for bd-pairs $(X/Z \ni o, B+\mathcal{P})$ with a boundary$B$.

Construction 7. Let $(X/Z \ni o, B)$ be a pair of klt type with wFt $X/Z \ni o$. Then by Construction 2 there exists an *associated* b-0-contraction: a birational 1-contraction φ to a 0-contraction ψ over $Z \ni o$

$$\begin{array}{ccc} (X, B) & \overset{\varphi}{\dashrightarrow} & (X^\sharp, B_{X^\sharp}) \\ \downarrow & & \psi \downarrow \\ Z \ni o & \leftarrow & (Y, B^\sharp{}_{\mathrm{div}} + B^\sharp{}_{\mathrm{mod}}) \end{array},$$

where

$(X^\sharp/Z \ni o, B_{X^\sharp})$ is a maximal model of $(X/Z \ni o, B)$ with contracted fixed over $Z \ni o$ components of $-(K+B)$, in particular, $B^\sharp_{X^\sharp} = B_{X^\sharp}$;

ψ is a 0-contraction given over $Z \ni o$ by the nef over $Z \ni o$ divisor $-(K_{X^\sharp} + B_{X^\sharp})$; ψ satisfies the assumptions of 7.1;

${B^\sharp}_{\mathrm{div}}, {B^\sharp}_{\mathrm{mod}}$ are respectively the divisorial and moduli part of adjunction for ψ;

$(Y/Z \ni o, {B^\sharp}_{\mathrm{div}} + {\mathcal{B}^\sharp}_{\mathrm{mod}})$ is the *adjoint* generic klt bd-pair.

Indeed, since $(X/Z \ni o, B)$ has a klt $\mathbb{R}$-complement, $(X^\sharp, B_{X^\sharp})$ also has a klt $\mathbb{R}$-complement. Hence by 7.5, (6) $(Y/Z \ni o, {B^\sharp}_{\mathrm{div}} + {\mathcal{B}^\sharp}_{\mathrm{mod}})$ is klt too. The bd-pair is generic by construction and 7.5, (8) and (12).

Note that if X_o is connected then $X^\sharp_o$ and Y_o are connected too.

The same construction works for bd-pairs $(X/Z \ni o, B + \mathcal{P})$ of klt type with wFt $X/Z \ni o$.

A pair $(X/Z \ni o, B)$ of klt type is generic itself if and only if ψ is birational. Otherwise $(X/Z \ni o, B)$ is fibered and we can apply Theorem 14, the existence of generic n-complements.

Definition 8. A klt type pair $(X/Z \ni o, B)$ with wFt $X/Z \ni o$ has *klt type* f if in Construction 7 $\dim Y = f$. So, $f \in \mathbb{Z}$ and $0 \le f \le \dim X$.

The same definition works for bd-pairs $(X/Z \ni o, B + \mathcal{P})$ of klt type with wFt $X/Z \ni o$.

In particular, the generic type is exactly a klt one with $f = \dim X$.

Construction 8 (Cf. Construction 4)**.** Let d be a nonnegative integer and $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0,1]$. By Theorem 9 there exists a positive integer J such that every contraction $\psi\colon (X^\sharp, B_{X^\sharp}) \to Y/Z \ni o$ of Construction 7 has the adjunction index J if

(1) $\dim X = d$, and

(2) $B^{\mathrm{h}} \in \Phi$.

Moreover, there exists a finite set of rational numbers $\mathfrak{R}'$ in $[0,1]$ such that $\Phi' = \Phi(\mathfrak{R}')$ satisfies Addendum 35. More precisely, $\mathfrak{R}'$ is defined by (6.8.1), where

$$\mathfrak{R}'' = [0,1] \cap \frac{\mathbb{Z}}{J}.$$

By Addendum 36, the same adjunction index J has every contraction $\psi\colon (X^\sharp, B_{X^\sharp} + \mathcal{P}) \to Y/Z \ni o$ of Construction 7 if we apply the construction to a bd-pair $(X/Z \ni o, B+\mathcal{P})$ and suppose additionally to (1-2) that $(X/Z \ni o, B + \mathcal{P})$ is a bd-pair of index m, or equivalently, $(X^\sharp/Z \ni o, B_{X^\sharp} + \mathcal{P})$ is a log bd-pair of index m.

Let I, ε, v, e be the data as in Restrictions on complementary indices in Section 1 and f be a nonnegative integer such that $f \le d-1$. By Addenda 52-53 of Theorem 14 or by dimensional induction there exists a finite set of positive integers $\mathcal{N} = \mathcal{N}(f, I, \varepsilon, v, e, \Phi', J)$ such that

Restrictions: every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data[, in particular, $J|n$];

Existence of n-complement: if $(Y/Z \ni o, B_Y + \mathcal{Q})$ is a generic bd-pair of dimension f and of index J with wFt $Y/Z \ni o$, with a boundary B_Y and connected Y_o then $(Y^\sharp/Z \ni o, {B_{Y,n\_\Phi'}}^\sharp{}_{Y^\sharp} + \mathcal{Q})$ has a b-n-complement $(Y/Z \ni o, B_Y^+ + \mathcal{Q})$ for some $n \in \mathcal{N}$.

Theorem 15. *Let $d, \mathfrak{R}, \Phi, J, I, \mathfrak{R}', \Phi', \varepsilon, v, e, f, \mathcal{N}$ be the data of Construction 8. Let $(X/Z \ni o, B)$ be a pair with a boundary B, connected X_o such that*

(1) *$X/Z \ni o$ has wFt;*

(2) *$\dim X = d$; and*

(3) *both pairs*

$$(X/Z \ni o, B_\Phi),\ (X/Z \ni o, B_{\mathcal{N}\_\Phi})$$

have the same klt type f.

Then there exists $n \in \mathcal{N}$ such that $(X/Z \ni o, B)$ has an n-complement $(X/Z \ni o, B^+)$.

Notice that we do not assume that $(X/Z \ni o, B)$ has an $\mathbb{R}$-complement.

Addendum 54. *$(X/Z \ni o, B^+)$ is an n-complement of itself and of $(X/Z \ni o, B_{n\_\Phi}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp)$, and is a b-$n$-complement of itself and of $(X/Z \ni o, B_{n\_\Phi}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp), (X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$, if $(X/Z \ni o, B_{n\_\Phi}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp)$ are log pairs respectively.*

Addendum 55. *$\mathcal{N}$ disjoint from any finite set of positive integers $\mathcal{N}'$.*

Addendum 56. *The same holds for the bd-pairs $(X/Z \ni o, B + \mathcal{P})$ of index m and with $\mathcal{N}$ as in Construction 8. That is,*

Existence of n-complement: *if $(X/Z \ni o, B+\mathcal{P})$ is a bd-pair of index m with a boundary B, X_o connected, under (1-2) and such that*

both bd-pairs

$$(X/Z \ni o, B_\Phi + \mathcal{P}),\ (X/Z \ni o, B_{\mathcal{N}\_\Phi} + \mathcal{P})$$

have the same klt type f,

then $(X/Z \ni o, B + \mathcal{P})$ has an n-complement $(X/Z \ni o, B^+ + \mathcal{P})$ for some $n \in \mathcal{N}$.

In Addenda 54 $(X/Z \ni o, B^+ + \mathcal{P})$ is an n-complement of itself and of $(X/Z \ni o, B_{n\_\Phi} + \mathcal{P}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp + \mathcal{P})$, and is a b-$n$-complement of itself and of $(X/Z \ni o, B_{n\_\Phi} + \mathcal{P}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp + \mathcal{P}), (X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp} + \mathcal{P})$, if $(X/Z \ni o, B_{n\_\Phi} + \mathcal{P}_X), (X/Z \ni o, B_{n\_\Phi}{}^\sharp + \mathcal{P}_X)$ are log bd-pairs respectively.

The proof of the theorem is very similar to proofs of Theorems 12 and 14. So, we will be sketchy of it.

Proof. By Construction 8 $\mathcal{N}$ is a finite set of positive integers and satisfies Restrictions. So, it is enough to construct required n-complements.

Let $(X/Z \ni o, B)$ be a pair satisfying the assumptions in Existence of n-complements. Since B is a boundary, $B_{\mathcal{N}\_\Phi}$ is well-defined. It is enough to construct a b-n-complement $(X/Z \ni o, B^+)$ of $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ for some $n \in \mathcal{N}$ (cf. Step 1 in the proof of Theorem 14).

As in the proof of Theorem 12 we can suppose that $B = B_{\mathcal{N}\_\Phi}$, that is, $(X/Z \ni o, B)$ has also klt type f.

According to Construction 7 there exists 0-contraction $\psi\colon (X^\sharp, B_{X^\sharp}) \to (Y, B^\sharp{}_{\mathrm{div}} + B^\sharp{}_{\mathrm{mod}})$ over $Z \ni o$ with $\dim Y = f$.

Step 1. *We can suppose that ψ is defined on $(X/Z \ni o, B)$.* Equivalently, φ is identical. For this we use a small modification φ. Thus we use $(X^\sharp/Z \ni$

$o, B^\sharp_{X^\sharp,X^\sharp})$ crepant to $(X^\sharp/Z \ni o, B_{X^\sharp})$ of Construction 7. By Proposition 8 $B^\sharp_{X^\sharp,X^\sharp} \geq B$. Hence by definition and Corollary 7, $\mathbb{B}^\sharp_{X^\sharp,X^\sharp,n\_\Phi}{}^\sharp \geq \mathbb{B}_{n\_\Phi}{}^\sharp$. By Proposition 1 a b-$n$-complement of $(X^{\sharp,\sharp}/Z \ni o, B^\sharp_{X^\sharp,X^\sharp,n\_\Phi}{}^\sharp{}_{X^{\sharp,\sharp}})$ is a b-n-complement of $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$. On the other hand, $(X^\sharp/Z \ni o, B^\sharp_{X^\sharp,X^\sharp})$ also has klt type f because Construction 2 preserves the property by Proposition 8 and the invariance of klt singularities for crepant 0-pairs.

Additionally, $(X^\sharp/Z \ni o, B^\sharp_{X^\sharp,X^\sharp,\Phi})$ has the same klt type f. Indeed, by Proposition 8, construction and definition $B_{X^\sharp,\Phi} \leq B^\sharp_{X^\sharp,X^\sharp,\Phi} \leq B^\sharp_{X^\sharp,X^\sharp}$, where $B_{X^\sharp}$ is the birational transform of B on $X^\sharp$. Since $X^\sharp$ is a small modification of X over $Z \ni o$, $(X^\sharp/Z \ni o, B_{X^\sharp,\Phi})$ has the same klt type f as $(X/Z \ni o, B_\Phi)$ (cf. Basic properties of generic pairs (2)). Hence $(X^\sharp/Z \ni o, B^\sharp_{X^\sharp,X^\sharp,\Phi})$ also has klt type f because so does $(X^\sharp/Z \ni o, B^\sharp_{X^\sharp,X^\sharp})$ (cf. Lemma 10).

By Lemma 1 $X^\sharp/Z \ni o$ has wFt, $\dim X^\sharp = \dim X = d$. (Actually we need the wFt property only for Construction 7 and do not use it further.)

So, we can denote $(X^\sharp/Z \ni o, B^\sharp_{X^\sharp,X^\sharp})$ by $(X/Z \ni o, B)$. We have a 0-contraction ψ on X:

$$(X, B) \xrightarrow{\psi} (Y, B_{\mathrm{div}} + B_{\mathrm{mod}})/Z \ni o.$$

To construct a required complement we use Theorem 10 with the same sets Φ, Φ', the index J and some $n \in \mathcal{N}$. The sets Φ, Φ' of Construction 8 agree with 6.8 for $\mathfrak{R}'' = [0,1] \cap (\mathbb{Z}/J)$ and $J|n$ for every $n \in \mathcal{N}$. We apply the theorem to the 0-contraction $\psi\colon (X, B) \to Y/Z \ni o$ with lc (X, B). By construction the contraction ψ satisfies 7.1, B is a boundary and $Y \to Z \ni o$ is proper (projective). So, we need only to verify that ψ has the adjunction index J, to choose $n \in \mathcal{N}$ and to construct an appropriate model $Y'/Z \ni o$ of $Y/Z \ni o$ with an $\mathbb{R}$-divisor D'_{div} satisfying required properties.

Step 2. *$(X, B) \to Y$ has the adjunction index J.* By (3) $(X/Z \ni o, B_\Phi)$ has klt type f. By Construction 7, applied to $(X/Z \ni o, B_\Phi)$, there exists another 0-contraction $\psi''\colon (X''^\sharp, B_{\Phi,X''^\sharp}) \to Y''$ over $Z \ni o$ which is generically crepant to $(X, B) \to Y$ over Y. Indeed, $B_\Phi \leq B$ and pairs $(X/Z \ni o, B_\Phi), (X/Z \ni o, B)$ have the same klt type f. Thus $\mathbb{B}_\Phi{}^\sharp = \mathbb{B}^\sharp$ generically over Y. Thus by Proposition 13, (5) ψ, ψ'' have the same adjunction index.

On the hand, by definition and Construction 7 B_Φ and $B_{\Phi,X''^\sharp} \in \Phi$. Thus by Construction 8 and Theorem 9, $(X''^\sharp, B_{\Phi,X''^\sharp}) \to Y''$ has the adjunction index J. Hence $(X \ni o, B) \to Y$ also has the adjunction index J.

Step 3. *Choice of $n \in \mathcal{N}$ and construction of* $(Y'/Z \ni o, D'_{\rm div} + \mathcal{B}_{\rm mod})$. By Construction 7 the adjoint bd-pair $(Y/Z \ni o, B_{\rm div} + \mathcal{B}_{\rm mod})$ is generic klt. By construction $\dim Y = f$, $B_{\rm div}$ is a boundary and Y_o is connected. Additionally, $(Y/Z \ni o, B_{\rm div} + \mathcal{B}_{\rm mod})$ is a bd-pair of index J by 7.3, (3) (cf. Addendum 32). Thus by Existence of complement in Construction 8 $(Y^\sharp/Z \ni o, B_{{\rm div},n\_\Phi'}{}^\sharp{}_{Y^\sharp} + \mathcal{B}_{\rm mod})$ has a b-n-complements for some $n \in \mathcal{N}$. However we need a slightly stronger complement to apply Theorem 10.

For this take a crepant blowup $(\widetilde{Y}, B_{{\rm div},\widetilde{Y}} + \mathcal{B}_{\rm mod}) \to (Y, B_{\rm div} + \mathcal{B}_{\rm mod})$ such that, for every vertical over Y prime divisor P on X, its image $Q = \psi(P)$ on $\widetilde{Y}$ is a divisor. By construction $(\widetilde{Y}/Z \ni o, B_{{\rm div},\widetilde{Y}} + \mathcal{B}_{\rm mod})$ is a bd-pair of index J with $\dim \widetilde{Y} = f$ and connected $\widetilde{Y}_o$. By (12) in 7.5 $B_{{\rm div},\widetilde{Y}}$ is a boundary because (X, B) is lc. Hence by Basic properties of generic pairs (3-4) and Corollary 18, $(\widetilde{Y}/Z \ni o, B_{{\rm div},\widetilde{Y}} + \mathcal{B}_{\rm mod})$ is generic with wFt $\widetilde{Y}/Z \ni o$. By definition of bd-pairs of index J, $\mathcal{B}_{\rm mod}$ is b-nef and $B_{{\rm mod},\widetilde{Y}}$ is pseudoeffective. Thus again by Existence of complement in Construction 8 $(\widetilde{Y}^\sharp/Z \ni o, B_{{\rm div},\widetilde{Y},n\_\Phi'}{}^\sharp{}_{\widetilde{Y}^\sharp} + \mathcal{B}_{\rm mod})$ has a b-n-complement for some $n \in \mathcal{N}$. This is our choice of n and a required model is

$$(Y'/Z \ni o, D'_{\rm div} + \mathcal{B}_{\rm mod}) = (\widetilde{Y}^\sharp/Z \ni o, B_{{\rm div},\widetilde{Y},n\_\Phi'}{}^\sharp{}_{\widetilde{Y}^\sharp} + \mathcal{B}_{\rm mod})$$

with a small modification $\widetilde{Y} \dashrightarrow \widetilde{Y}^\sharp = Y'$. Such a model with a small modification exists by Construction 2. Notice that notation $D'_{\rm div}$ means that $\mathbb{D}'_{\rm div}$ adjunction corresponds to some b-divisor $\mathbb{D}'$ on X 7.5, (3). By construction and definition $(Y'/Z \ni o, D'_{\rm div} + \mathcal{B}_{\rm mod})$ satisfies (2-3) of Theorem 10. By construction, for every vertical over Y prime divisor P on X, its image $Q = \psi(P)$ on $\widetilde{Y}^\sharp$ is a divisor and inequality (1) of Theorem 10 holds by Proposition 8. Thus by Theorem 10 $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ has a b-n-complement.

Step 4. *Addenda.* Addendum 55 can be proved by the same arguments as Corollary 17.

Similarly we can treat bd-pairs.

□

Klt type filtration. Let d, Φ, Φ' be the same data as in Construction 8 and $\mathcal{N} \supseteq \mathcal{N}'$ be two sets of positive integers. A *klt type filtration with respect to dimension i* is a (decreasing) filtration

$$\mathcal{N} = \mathcal{N}^0 \supseteq \cdots \supseteq \mathcal{N}^i \supseteq \cdots \supseteq \mathcal{N}^d \supseteq \mathcal{N}',\ 0 \le i \le d. \qquad (10.0.9)$$

Its *associated* (decreasing) filtration of hyperstandard sets is

$$\Gamma(\mathcal{N}^0,\Phi')\supseteq\cdots\supseteq\Gamma(\mathcal{N}^i,\Phi')\supseteq\cdots\supseteq\Gamma(\mathcal{N}^d,\Phi')\supseteq\Gamma(\mathcal{N}',\Phi').$$

For dimension $0\le i<d$, put

$$\mathcal{N}_i=\mathcal{N}^i\setminus\mathcal{N}^{i+1};$$

for $i=d$, $\mathcal{N}_d=\mathcal{N}^d\setminus\mathcal{N}'$. That is, the next set in the last case is $\mathcal{N}^{d+1}=\mathcal{N}'$ but without dimension.

For every i, between $\mathcal{N}^i\supseteq\mathcal{N}^{i+1}$, there exists an additional filtration with respect to generic types in dimension i.

Definition 9. Let (10.0.9) be a klt type filtration. Consider a certain class of pairs $(X/Z\ni o,B)$ of dimension d with boundaries B which have an (b-)n-complement with $n\in\mathcal{N}^0$. Such a *pair* $(X/Z\ni o,B)$ and its (b-)n*-complement have klt type of dimension* $0\le i\le d$ *with respect to filtration* (10.0.9) if both pairs

$$(X/Z\ni o,B_{\mathcal{N}^{i+1}\_\Phi}),(X/Z\ni o,B_{\mathcal{N}^{i}\_\Phi})$$

are of klt type i. (We suppose that $\mathcal{N}^{d+1}=\mathcal{N}'$.)

We say that the *existence* n-complements *agrees with the klt type filtration* if every pair $(X/Z\ni,B)$ of klt type has an n*-complement* $(X/Z\ni,B^+)$ *of filtration klt type* i with $n\in\mathcal{N}_i$. Additionally, $(X/Z\ni o,B^+)$ *of the type* i is an n-complement of itself, of $(X/Z\ni o,B_{n\_\mathcal{N}^{i+1}\_\Phi}),(X/Z\ni o,B_{n\_\mathcal{N}^{i+1}\_\Phi}{}^\sharp)$ and a b-n-complement of itself, of $(X/Z\ni o,B_{n\_\mathcal{N}^{i+1}\_\Phi}),(X/Z\ni o,B_{n\_\mathcal{N}^{i+1}\_\Phi}{}^\sharp),(X^\sharp/Z\ni o,B_{n\_\mathcal{N}^{i+1}\_\Phi}{}^\sharp{}_{X^\sharp})$, if $(X,B_{n\_\mathcal{N}^{i+1}\_\Phi}),(X,B_{n\_\mathcal{N}^{i+1}\_\Phi}{}^\sharp)$ are log pairs respectively.

The same applies to bd-pairs of dimension d.

Remark: (1) In particular, $(X/Z\ni o,B)$ has (filtration) klt type d if the pair is of generic type. The converse holds for the plain klt type d. But this is not true for filtration klt type (cf. (3) below). If the pair has nongeneric filtration klt type i, that is, $i<d$ then by Theorems 15 above and 16 below a required (b-)n-complement can be lifted from a generic one in dimension i.

Note also that in the proof of Theorem 15 we apply Construction 7 to $(X/Z\ni o,B_{\mathcal{N}\_\Phi})$, where $\mathcal{N}=\mathcal{N}^i$ ($f=i$ in the theorem). However, we can use the same construction directly for $(X/Z\ni o,B)$ if the latter pair has the same klt type i. Otherwise we use the construction for the former pair.

(2) By the proof of Theorem 15, for a klt type pair $(X/Z \ni o, B)$, a complement can be lifted from a generic complement in dimension i ($f = i$ in the theorem). So, additionally, we can associate to the pair its generic type $j \le i$ and (filtration) semiexceptional type (r, f) (see Definition 7).

Boundary $B_{n\_\mathcal{N}^{i+1}\_\Phi^\sharp}{}_{X^\sharp}$ in Definition 9 can be slightly increased if we take into consideration the generic type j or better a (filtration) semiexceptional type of $(X/Z \ni o, B)$.

(3) By definition filtration klt type is bigger of equal than klt type. Filtration klt type gives better understanding of geometry than plain klt type (cf. Corollary 20). The same concerns filtration semiexceptional type vs semiexceptional type.

(4) Warning: It is possible other n-complements which are not agree with the filtration. Additionally, we can have n-complements without any type but with $n \in \mathcal{N}$.

Theorem 16 (Klt type n-complements)**.** *Let d be a nonnegative integer, I, ε, v, e be the data as in Restrictions on complementary indices, and $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0, 1]$. Then there exists a finite set $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi)$ of positive integers such that*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data.*

Existence of n-complement: *if $(X/Z \ni o, B)$ is a pair with* $\dim X = d$, *a boundary B, wFt $X/Z \ni o$, connected X_o and with klt type $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ then $(X/Z \ni o, B)$ has an n-complement $(X/Z \ni o, B^+)$ for some $n \in \mathcal{N}$.*

Addendum 57. *$(X/Z \ni o, B^+)$ is an n-complement of itself and of $(X/Z \ni o, B_{n\_\Phi}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp)$, and is a b-$n$-complement of itself and of $(X/Z \ni o, B_{n\_\Phi}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp), (X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$, if $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$ are log pairs respectively.*

Addendum 58. *$\mathcal{N}$ has a klt type filtration (10.0.9) with $\mathcal{N}^0 = \mathcal{N}$, with any finite set of positive integers $\mathcal{N}'$, satisfying Restrictions on complementary indices with the given data, and the existence n-complements agrees the filtration for the class of pairs under assumptions of Existence of n-complements in the theorem.*

Addendum 59. *In particular, the theorem and addenda applies to klt type pairs $(X/Z \ni o, B)$ instead of with klt type $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$.*

Addendum 60. *The same holds for bd-pairs $(X/Z \ni o, B + \mathcal{P})$ of index m with $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi, m)$. That is,*

Restrictions: *every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data and $m|n$.*

Existence of n-complement: *if $(X/Z \ni o, B+\mathcal{P})$ is a bd-pair of index m with* $\dim X = d$*, a boundary B, wFt $X/Z \ni o$, connected X_o and with klt type $(X/Z \ni o, B_{\mathcal{N}\_\Phi} + \mathcal{P})$ then $(X/Z \ni o, B+\mathcal{P})$ has an n-complement $(X/Z \ni o, B^+ + \mathcal{P})$ for some $n \in \mathcal{N}$.*

Addendum 58 holds literally. In Addendum 57 $(X/Z \ni o, B^+ + \mathcal{P})$ is an n-complement of itself and of $(X/Z \ni o, B_{n\_\Phi} + \mathcal{P}), (X/Z \ni o, {B_{n\_\Phi}}^\sharp + \mathcal{P})$, and is a b-$n$-complement of itself and of $(X/Z \ni o, B_{n\_\Phi} + \mathcal{P}), (X/Z \ni o, {B_{n\_\Phi}}^\sharp + \mathcal{P}), (X^\sharp/Z \ni o, {B_{n\_\Phi}}^\sharp{}_{X^\sharp} + \mathcal{P})$, if $(X, B_{n\_\Phi} + \mathcal{P}_X), (X, {B_{n\_\Phi}}^\sharp + \mathcal{P}_X)$ are log bd-pairs respectively. In Addendum 59 $(X/Z \ni o, B + \mathcal{P}), (X/Z \ni o, B_{\mathcal{N}\_\Phi} + \mathcal{P})$ should be instead of $(X/Z \ni o, B), (X/Z \ni o, B_{\mathcal{N}\_\Phi})$ respectively.

Proof. Construction of klt type complements starts from the top type d and descends to the bottom 0, where we use Corollary 31, the boundedness of lc index. We use [decreasing] induction on klt type i in dimension d. In other words, we prove the theorem and Addendum 58 simultaneously. In the induction we suppose that $\mathcal{N} = \mathcal{N}^i$ and, in Existence of n-complements, $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ has klt type $\geq i$ instead of just klt type, that is, ≥ 0. We take the same class of pairs in Addendum 58.

By Proposition 1 and Corollary 7, more subtle complements of Addendum 58 with $B_{n\_\mathcal{N}^{i+1}\_\Phi}$ give the same complements of Addendum 57 with B_{n_Φ} instead because $B_{n\_\Phi} \leq B_{n\_\mathcal{N}^{i+1}\_\Phi}$.

Step 1. *Klt type filtration for $i = d$.* Consider a set of positive integers $\mathcal{N}_d = \mathcal{N}(d, I, \varepsilon, v, e, \Phi)$ of Theorem 14. Then we are done if $\mathcal{N}' = \emptyset$: $\mathcal{N} = \mathcal{N}^d = \mathcal{N}_d$. In general we put $\mathcal{N} = \mathcal{N}^d = \mathcal{N}_d \cup \mathcal{N}'$ assuming that $\mathcal{N}_d \cap \mathcal{N}' = \emptyset$. The last condition holds for $\mathcal{N}_d = \mathcal{N} \setminus \mathcal{N}'$ of Addendum 51 (cf. Corollary 17). Note that in this case generic or filtration klt type d pairs $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ are only possible under inductive version of Existence of n-complements. Indeed, $B_\Phi \leq B_{\mathcal{N}\_\Phi}$ and $(X/Z \ni o, B_\Phi)$ is also generic and of klt type d by Basic properties of generic pairs, (1).

Step 2. *Klt type filtration for $i < d$.* We can suppose that the theorem is established for klt types $\geq i+1$ and $\mathcal{N}^{i+1}, i+1 \leq d$, the next filtration klt type set is already constructed with a filtration (10.0.9) for the class of pairs in Existence of n-complements with klt types $\geq i+1$ $(X/Z \ni o, B_{\mathcal{N}^{i+1}\_\Phi})$. Consider a set of positive integers $\mathcal{N}_i = \mathcal{N}(i, I, \varepsilon, v, e, \Phi', J)$ as in Construction 8 with a hyperstandard set $\Phi = \Gamma(\mathcal{N}^{i+1}, \mathfrak{R})$ and $f = i$. (Note that Φ, Φ', J in the step depends also on i.) By Addendum 55 we can suppose that $\mathcal{N}_i$ is disjoint from $\mathcal{N}^{i+1}$. Put $\mathcal{N} = \mathcal{N}^i = \mathcal{N}_i \cup \mathcal{N}^{i+1}$. Now by Theorem 15 this proves Theorem 16 and gives a required filtration (10.0.9) for klt types $\geq i$. Note that the complements agree with the filtration still in the class of pairs $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ of klt type $\geq i$. If $(X/Z \ni o, B_{\mathcal{N}^{i+1}\_\Phi})$ has klt type i then $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ has also klt type i and we use Theorem 15 to construct a required n-complement with $n \in \mathcal{N}_i$. Otherwise, $(X/Z \ni o, B_{\mathcal{N}^{i+1}\_\Phi})$ has klt type $\geq i+1$ and we use induction.

This concludes construction of complements for the klt type case when $i = 0$.

Step 3. *Other Addenda.* Addendum 59 follows from the fact that if $(X/Z \ni o, B)$ has klt type then so does $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ because $B_{\mathcal{N}\_\Phi} \leq B$.

Similarly we can treat bd-pairs.

□

11 Lc type complements

In this section we establish Theorem 3 under assumption (2-3) of the theorem. For this we consider complements for the pairs $(X/Z \ni o, B)$ without a klt $\mathbb{R}$-complement, equivalently, for the pairs $(X/Z \ni o, B)$ with only lc but not klt $\mathbb{R}$-complements. However, we suppose that an (lc) $\mathbb{R}$-complement exists. An *lc type* pair is a local morphism $(X/Z \ni o, B)$ of this kind. Note that, if this is the case, $(X/Z \ni o, B)$ is lc (possibly klt) itself when it is a log pair.

We start with two examples which show that n-complements of lc type are not bounded in any dimension $d \geq 3$. We construct such examples in two extremal situations when $X/Z \ni o$ is identical (local) or with $Z = o = \text{pt}$. (global).

Example 11. (1) Let $(Z \ni o, B_Z)$ be a pair such that

Z is a normal variety of the dimension $d \geq 3$ and o is its closed point;

$B_Z = \sum_{i=1}^{l+d} D_i$ be a reduced divisor near o with $l+d$ prime Weil divisorial components; integers $d, l \geq 0$; and

(Z, B_Z) is *maximally* lc (of index 1), that is, lc (of index 1) and with an lc center o.

We suppose also that there exists a $\mathbb{Q}$-factorialization $\varphi\colon X \to Z \ni o$ near o such that each divisor D_i, more precisely, its birational transform, with $i \geq d+1$ intersects D_1, respectively, its birational transform, on X, and the intersection $D_1 \cap D_i$ is exceptional on D_1. Such a factorialization is typical for toric morphisms associated with a triangulation of a polyhedral cone with the same $l+d$ edges and one common edge for all simplicial subcones of dimension d.

Now we perturb $B_X = B = \sum_{i=1}^{l+d} D_i$ on X as follows. Every divisor D_i is mobile and moreover its linear system is big. Replace each D_i with $i \geq d+1$ by

$$\sum_{j=1}^{i-d} \frac{1}{i-d} D_{i,j},$$

where $D_{i,j}$ are $i-d$ sufficiently general divisors in $|D_i|$. They are also prime Weil. So, after the perturbation

$$B = \sum_{i=1}^{d} D_i + \sum_{i=1}^{l} \sum_{j=1}^{i} \frac{1}{i} D_{i+d,j}.$$

By construction $(X/Z \ni o, B)$ is an lc 0-pair. It is an $\mathbb{R}$-complement of itself. Also by construction the intersection $D_1 \cap D_{i+d}, l \geq i \geq 1$, is exceptional on D_1 on X. Thus every $D_{i+d,j}$ passes through the intersection and (X, B) is maximally lc near the intersection.

We contend that $(X/Z \ni o, B)$ does not have n-complements for all $1 \leq n \leq l-1$. Suppose such an n-complement $(X/Z \ni o, B^+)$ exists. Then

$$\left\lfloor (n+1)\frac{1}{n+1} \right\rfloor /n = 1/n > \frac{1}{n+1}.$$

Hence, by Definition 2 (1), $B^+ > B$ near the intersection $D_1 \cap D_{n+d+1}$, a contradiction by Definition 2 (2). Indeed, it was noticed above that (X, B) is maximally lc near the intersection.

Finally note that $l = 0$ in the dimension $d \leq 2$. However, for $d \geq 3$, l can be any natural number.

We can construct a required φ as a toric morphism. We can push the example to $Z \ni o$.

(2) We construct a locally trivial $\mathbb{P}^1$-bundle $\varphi\colon X \to Y$ with projective X, $\dim X = d$ and with a reduced divisor $D = \sum_{i=1}^{l} D_i$ on X with only simple normal crossings and such that

(1) divisors D_1 and D_2 are disjoint sections of φ;

(2) (X, D) is a projective 0-pair;

(3) all other divisors $D_i, i \geq 3$, are vertical with respect to φ;

(4) every irreducible closed curve $C_j, j \in J$, of X with the generic point in the 1-dimensional strata of D is rational and has a closed point in the 0-dimensional strata;

(5) those curves generate the cone of effective curves of X, that is, every 1-dimensional effective algebraic cycle of X is numerically equivalent to $\sum r_j C_j, r_i \in \mathbb{Q}$, and $r_i \geq 0$;

(6) D_1 is base point free and big on X, ample on itself; and

(7) the restriction of $|D_1|$ on $D^1 = \sum_{i=2}^{l} D_i$ is surjective.

We start our construction from a product $X = Y \times \mathbb{P}^1$ of projective toric nonsingular varieties with the natural projection $\varphi\colon X \to Y$, $\dim X = d$ and with the invariant divisor D on X. We can suppose that D_1, D_2 are horizontal. The product satisfies the properties (1-5).

To satisfy (6-7) we transform φ as follows. Take a general hyperplane section H on Y for some projective embedding of Y. So, $\varphi^{-1}H + D$ has only simple normal crossings too. Let $\psi\colon X' \to X$ be the blowup of $D_2 \cap \varphi^{-1}H$ in X. Denote its exceptional divisor by E and by E' the proper inverse transform of the vertical divisor $\varphi^{-1}H$. Now we can blow down E' and construct a required locally trivial $\mathbb{P}^1$-bundle X'/Y with the birational transform D' of D. It also satisfies the properties (1-5). To prove (5) it is better to know that $X' \to Y$ is toric again (e.g. by the criterion in [BMSZ]; this can be done also directly by a toric construction).

For sufficiently ample H, D'_1 is very ample on D'_1. We can suppose that, for every closed curve C_j generically in the 1-dimensional strata of D in D_1,

$$(C_j, D'_1) \geq 2.$$

By construction and (5), D'_1 is nef and big on X. Since X is projective toric it is Ft and D'_1 is semiample. This concludes (6).

Denote the constructed pair $(X'/Y, D')$ by $(X/Y, D)$. The restriction of linear systems

$$|D_1| \dashrightarrow \left| D_{1|D^1} \right|$$

is surjective due to the vanishing

$$H^1(X, D_1 - D^1) = H^1(X, D_1 - (D - D_1)) = H^1(X, 2D_1 - D) = H^1(X, K + 2D_1) = 0$$

by the Grauert-Riemenschneider vanishing. This gives (7).

Actually, we can prove now that D_1 is base point free. Moreover, let $P_j \in C_j$ be arbitrary closed points on the vertical curves C_j of the 1-dimensional strata of D. Then the points P_j are the only base points of the linear system $|D_1 - \sum P_j|$. In particular, the linear system is nonempty. To construct a divisor M in the linear system we can use the dimensional induction. If the dimension of $Y = 1$, then $Y = \mathbb{P}^1$ and we have only two vertical (disjoint) curves $C_1 = D_3, C_2 = D_4$, $D^1 = D_2 + D_3 + D_4$ and $P_1 + P_2 \in \left| D_{1|D^1} \right|$. By the surjectivity in (7) there exists such an effective divisor M on X that $M_{|D^1} = P_1 + P_2$ and $M \in |D_1 - P_1 - P_2|$. Moreover, $M \cap D_2 = \emptyset$ for general M. Suppose by induction that on every vertical divisor $D_i, i \geq 3$, generically in the $(d-1)$-dimensional (divisorial) strata of D there exists an effective divisor

$$M_i \in \left| D_{1|D_i} - \sum_{P_j \in D_i} P_j \right|.$$

We can also suppose by induction that the divisors M_i are agree on the vertical varieties V generically in the smaller dimensional strata, that is, if $D_l, l \geq 3$, is another vertical divisor generically in the $(d-1)$-dimensional strata of D with the effective divisor M_l and $V \subseteq D_i, D_l$ then

$$M_{i|V} = M_{l|V}.$$

So we can glue the divisors M_i into a single Cartier divisor M_{D^1} on D^1 such that

$$M_{D^1}{}_{|D_i} = M_i.$$

Note also for this that the general divisors M_i as D_1 do not interest D_2 by construction: $D_1 \cap D_2 = \emptyset$. By construction $M_{D^1} \in \left| D_1{}_{|D^1} - \sum P_j \right|$. By (7) there exists $M \in |D_1 - \sum P_j|$. For general M, $M \cap D_2 = \emptyset$.

Actually, general M behaves as D_1, that is, if we replace D_1 by M we have the same properties (1-7). So, to verify that P_j are the only base points of $|D_1 - \sum P_j|$, it is enough to consider the case when P_j are points of the 0-dimensional strata on D_1. If $\dim Y = 1$, then there exists M such that P_1, P_2 are the only base points of $|D_1 - P_1 - P_2|$ because D_1 is a 1-dimensional strata and $D_1^2 \geq 2$. So, the base locus does not contain the 1-dimensional strata. In higher dimensions the base locus should be a closed torus invariant subset in D_1. Since it contains the points P_j but does not contain the 1-dimensional strata, the base locus have only the points P_j.

According to the above construction for any choice of $M_V \in \left| D_1{}_{|V} - P_{1,V} - P_{2,V} \right|$ there exists $M \in |D_1 - \sum P_j|$ with $M_{|V} = M_V$, where V are the vertical irreducible surfaces generically in the 2-dimensional strata and $P_{1,V}, P_{2,V}$ are the only points on V from the points P_j on D_1. Suppose that $V_1, \dots, V_m$ are the vertical irreducible surfaces generically in the 2-dimensional strata. Then we can construct $m!$ distinct divisors $M_i, i = 1, \dots, m!$, as follows. Take j divisors (curves) $C_{j,h}, h = 1, \dots, j$, on V_j in $\left| D_1{}_{|V_j} - P_{1,V_j} - P_{2,V_j} \right|$. Then take M_i with

$$M_i{}_{|V_j} = C_{j,h}$$

for some $h = 1, \dots, j$. But each curve $C_{j,h}$ on V_j is repeated $m!/j$ times in this construction. So, we use $m!$ curves $C_{j,h}$ with the multiplicity $m!/j$ on V_j.

Now we perturb D_1:

$$B = \sum_{i=1}^{m!} \frac{1}{m!} M_i + \sum_{i=2}^{l} D_i.$$

Finally, we blow up every curve $C_{j,h}$:

$$(X', B') \to (X, B)$$

with the crepant boundary B' and with exceptional divisors $E_{j,h}$. (The order of blowups is unimportant.) Note that (X', B') is again 0-pair, in particular, lc. Since, every divisor M_i intersects every surface V_j transversally in $C_{j,h}$ and $m!/j$ such divisors passing through $C_{j,h}$,

$$\operatorname{mult}_{E_{j,h}} B' = \frac{m!}{j}\frac{1}{m!} = \frac{1}{j}$$

and every $E_{j,h}$ intersects V_j transversally along the generic point of $C_{j,h}$, the proper birational transform of $C_{j,h}$ on V_j, the proper birational transform of V_j on X'.

The pair (X', B') is an $\mathbb{R}$-complement of itself. But it has only n-complements for $n \geq m$. Indeed, if $n \leq m-1$ and (X', B^+) is an n-complement of (X', B') then near V_{n+1} the pair $K_{X'} + B^+$ is not $\equiv 0$, a contradiction. For this note that

$$\left\lfloor (n+1)\frac{1}{n+1} \right\rfloor /n = 1/n > 1/(n+1).$$

On the other hand, the reduced part of $D^1 = \sum_{i=2}^{l} D_i$ already gives the lc singularity along V_{n+1}. So,

$$B^+ \geq \sum_{i=2}^{l} D_i + \sum_{i=1}^{n+1} \frac{1}{n} E_{n+1,i}$$

and B^+ does not contain other divisors than $D_i, i = 2, \dots, l$, passing through V_{n+1} (maximally lc). Take C, the birational transform of a generic fiber (vertical curve) of V_{n+1} over Y. Then by construction and adjunction,

$$0 = (C.K_{X'} + B^+) \geq (C.K_{X'} + \sum_{i=2}^{l} D_i + \sum_{i=1}^{n+1} \frac{1}{n} E_{n+1,i}) =$$

$$(C.K_{V_{n+1}} + D_2{}_{|V_{n+1}} + \sum_{i=1}^{n+1} \frac{1}{n} E_{n+1,i}{}_{|V_{n+1}}) \geq$$

$$(C.K_{V_{n+1}} + D_2{}_{|V_{n+1}} + \sum_{i=1}^{n+1} \frac{1}{n} C_{n+1,i}) \geq -2 + 1 + (n+1)\frac{1}{n} = \frac{1}{n} > 0.$$

Note that $(C.D_2{}_{|V_{n+1}}) = (C.D_2) = 1$.

Finally, if $\dim X = d \geq 3$ or, equivalently, $\dim Y \geq 2$, we can find such a pair (X, B) with $m \gg 0$. Thus in dimensions $d \geq 3$ the global n-complements of lc type are not bounded.

(3) Let d be a positive integer ≥ 3 and $\mathcal{N}$ be a finite set of positive integers. Then, for every integer $r \gg 0$, there exists a pair $(X/Z \ni o, B)$ with a $\mathbb{Q}$-boundary $B = \sum_{i=1}^{r} b_i D_i$, $\dim X = d$ and connected X_o such that $(X/Z \ni o, B)$ has an $\mathbb{R}$-complement but does not have an n-complement for every $n \in \mathcal{N}$, where $D_1, \dots, D_r$ are effective Weil divisors (cf. (2-3) of Theorem 3). Replace in Example (2) $m!$ by $\mathcal{N}! = \prod_{n \in \mathcal{N}} (n+1)$ and $1/j$ by $1/(n+1), n \in \mathcal{N}$. The divisor $\sum_{i=1}^{\mathcal{N}!} M_i$ can be replaced by a single prime one. This gives a global example. A local example can redone from Example (1).

Actually, for appropriate examples, the low bound for r depend only on d and the number of elements in $\mathcal{N}$. For this we should aggregate curves $C_{j,h}$ into at most 4 curves $C_{h,j}, h = 1, 2, 3, 4$, with 4 multiplicities $a_h \mathcal{N}!/j$ such that a_h is a positive integer $< j$ with $\sum_{h=1}^{4} a_h/j = 2$. However, the multiplicities b_i in those examples with all possible bounded $\mathcal{N}$ can't belong to a dcc set, in particular, to a finite set by Theorem 3 under the assumption (3) [HLSh, Theorem 1.6]. Thus the existence of $\mathbb{R}$-complements for every D in (3) of the theorem is essential.

This section provides a construction of bounded n-complements under the additional assumption (finiteness).

Maximal lc 0-pairs. Let $(X/Z \ni o, D)$ be a 0-pair, that is, (X, D) is lc and $K + D \sim_{\mathbb{R}} 0/Z \ni o$. We say that $(X/Z \ni o, D)$ is a *maximal lc* 0-pair if additionally $(X/Z \ni o, D)$ is the only possible $\mathbb{R}$-complement of $(X/Z \ni o, D)$. By definition $(X/Z \ni o, D)$ is an $\mathbb{R}$-complement of $(X/Z \ni o, D)$. So, the maximal lc property means that if $(X/Z \ni o, D^+)$ is another $\mathbb{R}$-complement of $(X/Z \ni o, D)$ then $D^+ = D$. By (1) of Definition 1 every global 0-pair is maximally lc (even it is klt). However, every local maximal lc 0-pair $(X/Z \ni o, D)$ should have an lc center over o as a point in every connected component of X_o (cf. Example 1, (3)). Notice that there are no such nonglobal and nonlocal 0-pairs $(X/Z, D)$.

The same applies to 0-bd-pairs $(X/Z \ni o, D + \mathcal{P})$.

Let d be a nonnegative integer and Γ be a set of boundary multiplicities: $\Gamma \subseteq [0, 1]$, including 0. The set Γ has *bounded rational maximal lc multiplicities in dimension d* if the following set of rational numbers in Γ is finite:

$$\Gamma_{\max} = \{b \in \Gamma \cap \mathbb{Q} \mid (X/Z \ni o, B) \text{ is a maximal lc 0-pair,}$$
$$\dim X = d, B \in \Gamma \cap \mathbb{Q}, \text{ and } b \text{ is a multiplicity of } B\}.$$

E.g., $0 \in \Gamma_{\max}$. Note that this assumption allows any irrational numbers of $[0,1]$ in Γ.

The same applies to 0-bd-pairs $(X/Z \ni o, B + \mathcal{P})$ of dimension d with $B \in \Gamma \cap \mathbb{Q}$ and a multiplicity b of B.

Moreover, the finiteness and Boundedness of lc index conjecture (Corollary-Conjecture 1) imply that there exists a positive integer $I = I(d, \Gamma \cap \mathbb{Q})$ depending only on d and Γ such that if $(X/Z \ni o, B)$ is a maximal lc 0-pair in dimension d with $B \in \Gamma \cap \mathbb{Q}$ then $I(K + B) \sim 0/Z \ni o$. For wFt $X/Z \ni o$ this holds without Boundedness of lc index conjecture by Corollary 31 (or [B, Theorem 1.7]). We say that I is the *rational maximal lc index of* 0*-pairs in dimension d with respect to* Γ.

The same applies to maximal lc 0-bd-pairs $(X/Z \ni o, B + \mathcal{P})$ of index m but only for wFt $X/Z \ni o$ (cf. Example 16, Conjectures 1, 3 and Corollary 34).

Example 12. Every dcc subset $\Gamma \subset [0,1]$ with 0 has bounded rational maximal lc multiplicities in dimension d under wFt. In general this is expected by Boundedness of lc index conjecture. In particular, every hyperstandard set associated to a finite set of rational numbers satisfies the property under wFt. Indeed, we can suppose that the dcc set Γ is rational. Then $\Gamma_{\max}$ is finite by [HX] or Corollary 31. Under wFt means that $X/Z \ni o$ in the definition of $\Gamma_{\max}$ is a wFt morphisms.

The same holds for bd-pairs of index m under wFt.

Construction 9. Let d be a nonnegative integer, I, ε, v, e be the data as in Restrictions on complementary indices, $\Phi = \Phi(\mathfrak{R})$ be a hyperstandard set associated with a finite set of rational numbers $\mathfrak{R}$ in $[0,1]$ and Γ be a subset of $[0,1]$ with finite $\Gamma_{\max}$. Denote by $I(d, \Gamma \cap \mathbb{Q})$ the corresponding rational maximal lc index. In particular, by Example 12 the finiteness holds and the index exists under wFt if $\Gamma \cap \mathbb{Q}$ is a dcc set as in Theorem 3. We can suppose also that $I = I(d, \Gamma \cap \mathbb{Q})$, or equivalently, that I is sufficiently divisible.

By Theorem 16 and Addendum 57 there exists a finite set of positive integers $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi)$ such that

Restrictions: every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data;

Existence of n-complement: if $(X/Z \ni o, B)$ is a pair with $\dim X = d$, a boundary B, wFt $X/Z \ni o$, connected X_o and with klt type $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ then $(X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$ is a b-n-complement $(X/Z \ni o, B^+)$ for some $n \in \mathcal{N}$.

For bd-pairs we add a positive integer m. So, $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, \Phi, m)$ and replace $(X/Z \ni o, B)$ by a bd-pair $(X/Z \ni o, B + \mathcal{P})$ of dimension d and of index $m|n$.

Theorem 17 (Lc type n-complements)**.** *Let* d,I, ε, v, e,$\Gamma, \mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e)$ *be the data as in Construction 9. Then every* $n \in \mathcal{N}$ *satisfies Restrictions on complimentary indices with the given data and*

Existence of n-complement: *if* $(X/Z \ni o, B)$ *is a pair with* $\dim X = d$*,* $B \in \Gamma$*, wFt* $X/Z \ni o$*, connected* X_o *and with lc type* $(X/Z \ni o, B)$*, or more generally, with an* $\mathbb{R}$*-complement, then* $(X/Z \ni o, B)$ *has an* n*-complement* $(X/Z \ni o, B^+)$ *for some* $n \in \mathcal{N}$*.*

Addendum 61. *The same holds for bd-pairs* $(X/Z \ni o, B + \mathcal{P})$ *of index* m *with* $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, m)$*. That is,*

Restrictions: *every* $n \in \mathcal{N}$ *satisfies Restrictions on complementary indices with the given data and* $m|n$*.*

Existence of n-complement: *if* $(X/Z \ni o, B + \mathcal{P})$ *is a bd-pair of index* m *with* $\dim X = d$*,* $B \in \Gamma$*, wFt* $X/Z \ni o$*, connected* X_o *and with lc type* $(X/Z \ni o, B + \mathcal{P})$ *then* $(X/Z \ni o, B + \mathcal{P})$ *has an* n*-complement* $(X/Z \ni o, B^+ + \mathcal{P})$ *for some* $n \in \mathcal{N}$*.*

We do not use Φ in the statement of the theorem (cf. Remark 8 below), that is, we can take any Φ, e.g., $\Phi = \{1\}$. However, Φ is hidden in the proof of Theorem 17. The proof below uses a reduction to the klt type of Theorem 16 and Addendum 59. An alternative and more right proof is sketched in Remark 8.

Proof. By construction $\mathcal{N}$ is a finite set of positive integers and satisfies Restrictions.

Let $(X/Z \ni o, B)$ be a pair satisfying the assumptions in Existence of n-complements. Since B is a boundary, $B_{\mathcal{N}\_\Phi}$ is well-defined. By Propositions 1 and 8, $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ has an $\mathbb{R}$-complement because so does $(X/Z \ni o, B)$. If the pair $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ has klt type then the result holds by Theorem 16.

Otherwise $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ has lc type. In this case we use approximation and Theorem 16 again.

Step 1. *Reduction to the wlF pair* $(X^\sharp/Z \ni o, B_{\mathcal{N}\_\Phi}{}^\sharp{}_{X^\sharp})$ *with a* 0*-contraction*

$$\psi\colon (X^\sharp, B_{\mathcal{N}\_\Phi}{}^\sharp{}_{X^\sharp}) \to Y/Z \ni o$$

such that

(1$^\sharp$) $\psi^* H \sim_{\mathbb{R}} -(K_{X^\sharp} + B_{\mathcal{N}\_\Phi}{}^\sharp{}_{X^\sharp})$ for some ample over $Z \ni o$ $\mathbb{R}$-divisor H on Y.

Use Construction 2. We can suppose that the modification $\varphi\colon X \dashrightarrow X^\sharp$ is small. So, if $n \in \mathcal{N}$ gives an n-complement $(X^\sharp/Z \ni o, B^+)$ of $(X^\sharp/Z \ni o, B_{\mathcal{N}\_\Phi}{}^\sharp{}_{X^\sharp})$, then the complement induces an n-complement $(X/Z \ni, B^+)$ of $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ by Propositions 3, 1 and 8. On its turn, $(X/Z \ni, B^+)$ is a required n-complement of $(X/Z \ni o, B)$ too by Corollary 6.

By construction $\dim X^\sharp = d$, $X^\sharp/Z \ni o$ has wFt and connected $X^\sharp_o$. Additionally, $(X^\sharp/Z \ni o, B_{\mathcal{N}\_\Phi}{}^\sharp{}_{X^\sharp})$ has an $\mathbb{R}$-complement. That is, $(X^\sharp/Z \ni o, B_{\mathcal{N}\_\Phi}{}^\sharp{}_{X^\sharp})$ satisfies the assumptions of Existence of n-complements except for $B_{\mathcal{N}\_\Phi}{}^\sharp{}_{X^\sharp} \in \Gamma$. To compensate this we verify the following property.

Step 2. $(X^\sharp, B_{\mathcal{N}\_\Phi}{}^\sharp{}_{X^\sharp} \to Y)$ *has index* I *over* $Y \ni P$ *if the* 0*-contraction is maximal lc over* $Y \ni P$, in particular, over the generic point of Y. Equivalently, by (6) 7.5 P is an lc center of the adjoint bd-pair $(Y, B_{\mathcal{N}\_\Phi}{}^\sharp{}_{\mathrm{div}} + \mathcal{B}_{\mathcal{N}\_\Phi}{}^\sharp{}_{\mathrm{mod}})$ (see 7.1). So, there exits only finitely many of those points $P \in Y/Z \ni o$. We call them *maximal lc*. Let $P \in Y$ be such a maximal lc point. By Construction 2 the 0-pair $(X^\sharp/Y, B_{\mathcal{N}\_\Phi}{}^\sharp{}_{X^\sharp})$ is crepant over $Z \ni o$ to a 0-pair $(X'/Y, B_{\mathcal{N}\_\Phi, X'})$ with a boundary $B_{\mathcal{N}\_\Phi, X'}$ being the birational transform of $B_{\mathcal{N}\_\Phi}$ on X' by a birational 1-contraction $X \dashrightarrow X'/Z \ni o$. The maximal lc property is an invariant of crepant models, e.g., since the adjoint bd-pair does so. Thus $(X'/Y \ni P, B_{\mathcal{N}\_\Phi, X'})$ is maximal lc. On the other hand, $(X/Z \ni o, B)$ has an $\mathbb{R}$-complement $(X/Z \ni o, B^{+,\mathbb{R}})$. Hence $(X'/Z \ni o, B^{+,\mathbb{R}}_{X'})$ is also an $\mathbb{R}$-complement of $(X'/Z \ni o, B_{X'})$, where $B^{+,\mathbb{R}}_{X'} = \mathbb{B}^{+,\mathbb{R}}_{X'}$ and $B_{X'}$ is the birational transform of B on X'. By construction and definition $B \geq B_{\mathcal{N}\_\Phi}$ and $B^{+,\mathbb{R}}_{X'} \geq B_{X'} \geq B_{\mathcal{N}\_\Phi, X'}$; $(X'/Z \ni o, B^{+,\mathbb{R}}_{X'})$ is also an $\mathbb{R}$-complement of $(X'/Z \ni o, B_{\mathcal{N}\_\Phi, X'})$. Since $\sim_{\mathbb{R}}$ over $Z \ni o$ gives $\sim_{\mathbb{R}}$ over $Y \ni P$ for every point $P \in Y$ over o, $(X'/Y \ni P, B^{+,\mathbb{R}}_{X'})$ is an $\mathbb{R}$-complement of $(X'/Y \ni P, B_{\mathcal{N}\_\Phi, X'})$ too. Notice also that by construction $(X'/Y \ni P, B_{\mathcal{N}\_\Phi, X'})$ is an $\mathbb{R}$-complement of itself. By the maximal lc property over $Y \ni P$, $B^{+,\mathbb{R}}_{X'} = B_{X'} = B_{\mathcal{N}\_\Phi, X'}$ over $Y \ni o$. Thus

$B_{X'} = B_{\mathcal{N}\_\Phi,X'}$ is rational and $\in \Gamma$ over $Y \ni o$. Therefore by our assumptions $(X'/Y \ni P, B_{\mathcal{N}\_\Phi,X'}) = (X'/Y \ni P, B_{,X'})$ has index I. Now the require index property follows from the invariance of index under crepant modifications.

For simplicity of notation, we denote $(X^\sharp/Z \ni o, B_{\mathcal{N}\_\Phi^\sharp X^\sharp})$ by $(X/Z \ni o, B)$. By Step 1 $(X/Z \ni o, B)$ is a wlF pair and has a 0-contraction

$$\psi\colon (X, B) \to Y/Z \ni o$$

such that

(1) $\psi^* H \sim_{\mathbb{R}} -(K + B)$ for some ample over $Z \ni o$ $\mathbb{R}$-divisor H on Y.

Additionally, $(X/Z \ni o, B)$ satisfies the assumptions of Existence of n-complements except for $B \in \Gamma$. By Step 2 the 0-contraction ψ has index I locally over the lc centers of $(Y, B_{\mathrm{div}} + \mathcal{B}_{\mathrm{mod}})$, including the generic point of Y. By (6) 7.5 the lc centers of (X, B), including the generic point of X, correspond to that of $(Y, B_{\mathrm{div}} + \mathcal{B}_{\mathrm{mod}})$ under ψ. Hence there exists an open non empty subset U in $Y/Z \ni o$ such that all lc centers of (X, B) are generically over U and (X, B) is a 0-pair over U of index I. In particular,

(2) (X, B) is klt over $Y \setminus U/Z \ni o$; and

(3) all multiplicities b of B over U belong to

$$[0, 1] \cap \frac{\mathbb{Z}}{I}.$$

In Step 4 below we construct an n-complement of $(X/Z \ni o, B)$ for some $n \in \mathcal{N}$.

Step 3. *Approximation.* There exists a perturbation B' of the boundary B such that

(4) $(X/Z \ni o, B')$ is a klt wlF;

(5) $B' > B$ over $Y \setminus U/Z \ni o$ ($>$ in every prime divisor of X over $Y \setminus U/Z \ni o$); and

(6) the multiplicities b' of B' over U are arbitrary closed to fractions $b = l/I$ with nonnegative integer $l \leq I$.

Indeed, by (1) we can find and effective $\mathbb{R}$-divisor $E \sim_{\mathbb{R}} H/Z \ni o$ on Y over $Z \ni o$ such that $Y \setminus U \subseteq \operatorname{Supp} E$ locally over $Z \ni o$. We can suppose also that E does not pass through the lc centers of $(Y, B_{\text{div}} + \mathcal{B}_{\text{mod}})$.

First, we perturb over E: take $B_1 = B + \varepsilon\psi^* E$ for sufficiently small $\varepsilon > 0$. Then we get (5). In (6) $b_1 = b$ if $b = 1$, and in (4) lc instead of klt and even klt over $Y \setminus U/Z \ni o$ by (1) and (2).

To make a second approximation we take a boundary B_2 on X such that $(X/Z \ni o, B_2)$ is a klt 0-pair. Such a boundary exists because $X/Z \ni o$ has wFt. By (1-3) and above versions of (4-6), a required perturbation is

$$B' = (1 - \delta)B_1 + \delta B_2$$

for some $0 < \delta \ll 1$.

Step 4. *Construction of an n-complement of* $(X/Z \ni o, B)$. By (4) and Theorem 16 with Addendum 59, $(X/Z \ni o, B')$ has an n-complement $(X/Z \ni o, B^+)$ for some $n \in \mathcal{N}$. We contend that it is also an n-complement of $(X/Z \ni o, B)$. For this we need to verify only Definition 2, (1). For the prime divisors of X over E it follows from (5). For the prime divisors of X over U it follows from (6) and Lemmas 2, 3. Since $\mathcal{N}$ is finite, we can find $\delta > 0$ such that $\|b' - l/I\| < \delta \leq 1/I(n + 1)$ for all $n \in \mathcal{N}$.

Step 5. The same arguments works for the bd-pairs.

□

Remark 8. However, more conceptual and precise proof of Theorem 17 should work as follows. This allows to verify that under the assumptions and notation of Existence of n-complements of the theorem

(1) $(X/Z \ni o, B^+)$ is an n-complement of itself and of $(X/Z \ni o, B_{n\_\Phi}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp)$, and is a b-$n$-complement of itself and of $(X/Z \ni o, B_{n\_\Phi}), (X/Z \ni o, B_{n\_\Phi}{}^\sharp), (X^\sharp/Z \ni o, B_{n\_\Phi}{}^\sharp{}_{X^\sharp})$, if $(X, B_{n\_\Phi}), (X, B_{n\_\Phi}{}^\sharp)$ are log pairs respectively;

(2) $\mathcal{N}$ has a filtration similar to the klt filtration (10.0.9) of Addendum 58 and the existence n-complements agrees the filtration for the class of pairs under assumptions of Existence of n-complements in the theorem; and

(3) the bd-pairs can be treated similar to Addendum 60.

(4) However, in Existence of n-complements of the theorem we can not to replace $(X/Z \ni o, B)$ by $(X/Z \ni o, B_{\mathcal{N}\_\Phi})$ (because lc type is the top of types; cf. Addendum 59).

In particular, we use Φ here.

We sketch a construction of an lc type filtration. We consider pairs $(X/Z \ni o, B)$ under the assumptions of Existence of n-complements of the theorem. We start from the top, *big lc type*: $(X/Z \ni o, B)$ has an $\mathbb{R}$-complement $(X/Z \ni o, B^+)$ and $B^+ - B$ is big over $Z \ni o$, equivalently, $-(K + B)$ is big over $Z \ni o$ (cf. (1) of Proposition-Definition 1). The big lc type is also lc type of dimension d for $d = \dim X$. Any pair $(X/Z \ni o, B)$ of big lc type has a maximal model $(X^\sharp/Z \ni o, B_{X^\sharp})$ of Construction 2 with $\mathbb{R}$-ample $-(K_{X^\sharp} + B_{X^\sharp})$ over $Z \ni o$. The model is unique up to an isomorphism over $Z \ni o$ and has a unique minimal lc center S by lc connectedness [Sh03, p. 203] [A14, Theorem 6.3]. So, the lc type d has additional parameter $s = \dim S$, a nonnegative integer $\le d$. The birational 1-contraction preserves all multiplicities of B and we denote by $B_{X^\sharp}$ the birational transform of B on $X^\sharp$. In its turn, the finite set of positive integers $\mathcal{N}^d$ for lc type d of the lc filtration in (2) has a (deceasing) subfiltration

$$\mathcal{N}^d = \mathcal{N}^{(d,0)} \supseteq \mathcal{N}^{(d,1)} \supseteq \cdots \supseteq \mathcal{N}^{(d,d-1)} \supseteq \mathcal{N}^{(d,d)}$$

with respect to s. A pair $(X/Z \ni o, B)$ has lc type (d, s) with respect to the filtration if both pairs

$$(X/Z \ni o, B_{\mathcal{N}^{(d,s+1)}\_\Phi}), (X/Z \ni o, B_{\mathcal{N}^{(d,s)}\_\Phi})$$

have lc type (d, s). The corresponding (b-)n-complement of type (d, s) for some $n \in \mathcal{N}_{(d,s)} = \mathcal{N}^{(d,s)} \setminus \mathcal{N}^{(d,s+1)}$ is extended from an (b-)n-complement of an adjoint bd-pair on $(S^\sharp/Z \ni o, B_{n\_\mathcal{N}^{(d,s+1)}}{}^\sharp{}_{\mathrm{div}} + \mathcal{B}_{n\_\mathcal{N}^{(d,s+1)}}{}^\sharp{}_{\mathrm{mod}})$ of dimension s. The adjoint bd-pair has generic klt type and a finite set of positive numbers $\mathcal{N}_{(d,s)} = \mathcal{N}(d, I, \varepsilon, v, e, \widetilde{\Phi}', I)$ exists by Addendum 60. So, it looks that we do not need the assumption $B \in \Gamma$. No, we use the assumption for the log adjunction on $S^\sharp$ for $s \le d - 2$ (in codimension ≥ 2). Indeed, such an adjunction (after a dlt or better log resolution) is a sequence of adjunctions on divisors as in 7.6 concluding by an adjunction for a 0-contraction as in 7.1. The last adjunction has index $I = I(d, \Gamma \cap \mathbb{Q})$ that can be verified as in Step 2 in the proof of Theorem 17. For this we use lc type property of $(X/Z \ni o, B)$ and the assumption that $B \in \Gamma$. Respectively, in general we change Φ into $\widetilde{\Phi}$

by (7.6.1) for adjunctions on divisors and, finally, into $\widetilde{\Phi}'$ by Addendum 35 for the adjunction of the 0-contraction. In the reverse direction, first we lift n-complements by Theorem 10. Then we extend them by induction and by Theorem 11. During the extension we glue these complements on the reduced divisors of a dlt or log resolution and, finally, extend on X.

Notice that the subfiltration starts from $\mathcal{N}^{(d,d)}$ and $\mathcal{N}^{(d,d+1)} = \emptyset$ or $\mathcal{N}'$ as in Addendum 58. Type (d,d) is generic klt type with the minimal lc center $S = X, S^\sharp = X^\sharp$ and Φ instead of $\widetilde{\Phi}'$. However, the previous type $(d, d-1)$ is lc type but it is plt with the divisorial minimal lc center $S, S^\sharp$, a reduced divisor of $B_{n\_\mathcal{N}^{(d,d)}}, B_{n\_\mathcal{N}(d,d)}{}^\sharp$ respectively and $\widetilde{\Phi}$ instead of $\widetilde{\Phi}'$. The starting type $(d,0)$ has $S^\sharp =$ pt., a closed point and $\mathcal{N}^d = \mathcal{N}^{(d,0)}$.

After that we continue with lc type of dimension $i \le d-1$, where the 0-contraction $(X^\sharp, B^\sharp{}_{X^\sharp}) \to Y/Z \ni o$ is fibered and $\dim Y = i$. By (8) of 7.5 the adjoint bd-pair has lc type but in general without index because B may have real horizontal over $Z \ni o$ multiplicities. Unfortunately, a straightforward reduction to dimension i does not work because does not preserve the assumption $B \in \Gamma$ and not preserve Γ. However, the index I is preserved for 0-contractions

$$(X^\sharp, B_{n\_\mathcal{N}^{(i,s+1)}\_\Phi}{}^\sharp{}_{X^\sharp}), \to Y/Z \ni o, n \in \mathcal{N}_{(i,s)} = \mathcal{N}^{(s,i)} \setminus \mathcal{N}^{(i,s+1)},$$

where type (i,s) is the *filtration lc type* with $i = \dim Y, s = \dim S$ and S is a minimal lc center of the adjoint bd-pair on Y. The types are ordered lexicographically. E.g., lc type $(d-1, d-1)$ precede to lc type $(d,0)$; in this case $\dim Y = d-1$, the adjoint bd-pair is generic klt and the adjunction index is $I = I(d, \Gamma \cap \mathbb{Q})$. In this case we can construct n-complements by Addendum 60 and Theorem 10. In general construction of n-complements is more involved but extend the construction for types (d,s): first we construct a (b-)n-complement on Y and then lift it to $X^\sharp$.

The bottom lc type $(0,0)$ has only global pairs with $S, S^\sharp =$ pt. and amounts the special global case of Corollary 31.

In particular, Theorem 17 can be applied to any dcc set $\Gamma \subset [0,1]$ by Example 12. Moreover, under the dcc assumption on Γ in Theorem 17 and in the remark we can suppose that

(5) $\mathcal{N}$ has a single element, or equivalently, there exists a positive integer $n = n(d, I, \varepsilon, v, e)$ such that Existence of n-complements holds for this n [HLSh, Theorem 1.6].

Indeed, all n-complements are coming from the exceptional case. By 6.10 and a similar fact for adjunction on a divisor, we can suppose that the exceptional pairs also have boundaries with dcc multiplicities. Actually in this situation (and even in general) increasing multiplicities we can suppose that the boundary multiplicities form a finite set [HLSh, Theorem 5.20]. Then we can find a single complementary index n (cf. [B, Theorem 1.7]).

The same works for bd-pairs.

Similar results expected for a-lc complements and not only under wFt (cf. Conjecture 5 below). One of crucial pieces – Corollary 31, the boundedness of lc index, - does not hold in general for maximal a-lc 0-pairs but may hold under slightly stricter assumption (cf. Addendum 90).

Affine maps to divisors. Let $\mathbb{R}^r$ be a finite dimensional $\mathbb{R}$-linear space and X be an algebraic variety or space. An affine map A into $\mathbb{R}$-divisors of X is a map

$$A\colon \mathbb{R}^r \to \mathrm{WDiv}_{\mathbb{R}} X$$

which is $\mathbb{R}$-linear for every multiplicity of $\mathbb{R}$-divisors. That is, for every prime divisor P on X, there exist real numbers $a_P, a_{P,1}, \ldots, a_{P,r}$ such that, for every point $(x_1, \ldots, x_r) \in \mathbb{R}^r$,

$$\mathrm{mult}_P A(x_1, \ldots, x_r) = a_P + \sum_{i=1}^{r} a_{P,i} x_i. \tag{11.0.10}$$

The map A is $\mathbb{Q}$-affine if all $a_P, a_{P,i} \in \mathbb{Q}$. The hight of such a $\mathbb{Q}$-affine map A is

$$h(A) = \max\{h(a_P), h(a_{P,i})\},$$

where $h(a)$ is the usual hight of $a \in \mathbb{Q}$. The boundedness of $h(A)$ does not imply the finiteness of maps A but the finiteness of their linear components (11.0.10). More generally, the finiteness holds if $A \in \mathfrak{A}$, where $\mathfrak{A}$ is finite set of real numbers and $A \in \mathfrak{A}$ means that every $a_P, a_{P,i} \in \mathfrak{A}$.

Construction 10. Let d be a nonnegative integer, I, ε, v, e be the data as in Restrictions on complementary indices, $\mathfrak{A}$ be a finite set of real numbers and Δ be a compact subset, e.g., a compact polyhedron, in a finite dimensional $\mathbb{R}$-linear space $\mathbb{R}^r$.

For every $x \in \mathbb{R}^r$, the set of real numbers

$$\Gamma(x) = \Gamma(x, \Delta, \mathfrak{A}) = \{a + \sum_{i=1}^{r} a_i x_i \mid a, a_1, \ldots, a_r \in \mathfrak{A}\} \cap [0, 1]$$

is finite. Hence $\Gamma(x)$ satisfies the assumption of Construction 9: $\Gamma(x)_{\max}$ is finite too. Let $I(x) = \Gamma(d, \Gamma \cap \mathbb{Q})$ be the corresponding rational maximal lc index. Additionally, we can suppose that $I|I(x)$.

By Theorem 17 there exists a finite set of positive integers $\mathcal{N}(x) = \mathcal{N}(d, I(x), \varepsilon, v, e)$ such that

Restrictions: every $n \in \mathcal{N}(x)$ satisfies Restrictions on complementary indices with the given data;

Existence of n-complement: if $(X/Z \ni o, B)$ is a pair with $\dim X = d$, $B \in \Gamma(x)$, wFt $X/Z \ni o$, connected X_o and with lc type $(X/Z \ni o, B)$ then $(X/Z \ni o, B)$ has an n-complement $(X/Z \ni o, B^+)$ for some $n \in \mathcal{N}(x)$.

Additionally, by Lemma 13 below with $\Gamma = \Gamma(x)$ and $\mathcal{N} = \mathcal{N}(x)$ there exists a positive real number $\delta(x)$.

By the finiteness of linear functions $L(y_1, \ldots, y_r) = a + \sum_{i=1}^r a_i y_i, a, a_1, \ldots, a_r \in \mathfrak{A}$, there exists an open neighborhood $U(x)$ of x in $\mathbb{R}^r$ such that for every those function L and $(y_1, \ldots, y_r)$ in U

$$\|L(x_1, \ldots, x_r) - L(y_1, \ldots, y_r)\| < \delta(x).$$

Finally, since Δ is compact there exists a finite covering

$$\Delta \subset \bigcup_j U(x^j), x^j = (x_1^j, \ldots, x_r^j) \in \Delta.$$

Respectively, consider

$$\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e) = \bigcup_j \mathcal{N}(x^j) = \bigcup_j \mathcal{N}(d, I(x^j), \varepsilon, v, e).$$

Thus

Restrictions: every $n \in \mathcal{N}$ satisfies Restrictions on complementary indices with the given data.

For bd-pairs we add a positive integer m. So, $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, m), \mathcal{N}(x) = \mathcal{N}(d, I(x), \varepsilon, v, e, m)$ and $m|n \in \mathcal{N}, \mathcal{N}(x)$.

Theorem 18 (Lc type n-complements with A)**.** *Let d,I, ε, v, e,Δ, $\mathfrak{A}$, $\mathcal{N}$ be the data as in Construction 10. Then every $n \in \mathcal{N}$ satisfies Restrictions on complimentary indices with the given data and*

Existence of n-complement: *if* $(X/Z \ni o, B)$ *is a pair with* $\dim X = d$, *wFt* $X/Z \ni o$, *connected* X_o *and such that there exists an affine map* $A\colon \mathbb{R}^r \to \mathrm{WDiv}_{\mathbb{R}} X$, *where* $A \in \mathfrak{A}$,

$$A(\Delta) \subseteq \mathfrak{C}_{\mathbb{R}} \cap \mathfrak{D}_{\mathbb{R}}^{+} = \{D \in \mathrm{WDiv}_{\mathbb{R}} X \mid (X/Z \ni o, D) \text{ has an } \mathbb{R}\text{−complement and } D \geq 0\}$$

and $B \in A(\Delta)$, *then* $(X/Z \ni o, B)$ *has an* n*-complement* $(X/Z \ni o, B^+)$ *for some* $n \in \mathcal{N}$.

Addendum 62. *The same holds for bd-pairs* $(X/Z \ni o, B + \mathcal{P})$ *of index* m *with* $\mathcal{N} = \mathcal{N}(d, I, \varepsilon, v, e, m)$. *That is,*

Restrictions: *every* $n \in \mathcal{N}$ *satisfies Restrictions on complementary indices with the given data and* $m|n$.

Existence of n-complement: *if* $(X/Z \ni o, B+\mathcal{P})$ *is a bd-pair of index* m *with* $\dim X = d$, *wFt* $X/Z \ni o$, *connected* X_o *and such that there exists an affine map* $A\colon \mathbb{R}^r \to \mathrm{WDiv}_{\mathbb{R}} X$, *where* $A \in \mathfrak{A}$,

$$A(\Delta) \subseteq \mathfrak{C}_{\mathbb{R}}\, \mathcal{P} \cap \mathfrak{D}_{\mathbb{R}}^{+} = \{D \in \mathrm{WDiv}_{\mathbb{R}} X \mid (X/Z \ni o, D+\mathcal{P}) \text{ has an } \mathbb{R}\text{−complement and } D \geq$$

and $B \in A(\Delta)$, *then* $(X/Z \ni o, B + \mathcal{P})$ *has an* n*-complement* $(X/Z \ni o, B^+ + \mathcal{P})$ *for some* $n \in \mathcal{N}$.

The proof uses the following.

Lemma 13 (Approximation of n-complements)**.** *Let* Γ *be a finite subset in* $[0, 1]$ *and* $\mathcal{N}$ *be a finite set of sufficiently divisible positive integers:* $nb \in \mathbb{Z}$ *for every* $n \in \mathcal{N}$ *and rational* $b \in \Gamma$. *Then there exists a positive real number* δ *with the following approximation property. If* $(X/Z, B^+)$ *is an* n*-complement of* $(X/Z, B)$ *with* $B \in \Gamma$ *with* $n \in \mathcal{N}, B \in \Gamma$ *and* D *is a subboundary on* X *such that* $\|B - D\| < \delta$ *then* $(X/Z, B^+)$ *is also an* n*-complement of* $(X/Z, D)$.

Addendum 63. *Let* $\Phi = \Phi(\mathfrak{R})$ *be a hyperstandard set associated with a finite set of rational numbers* $\mathfrak{R}$ *in* $[0, 1]$. *Suppose additionally that* D *is a boundary. Then every b-*n*-complement* $(X/Z, B^+)$ *of* $(X/Z, B_{n\_\Phi})$ *is also a b-*n*-complement of* $(X/Z, D_{n\_\Phi})$ *if* $(X, B_{n\_\Phi}), (X, D_{n\_\Phi})$ *are log pairs.*

Addendum 64. *The same holds for bd-pairs* $(X/Z, B + \mathcal{P}), (X/Z, D + \mathcal{P})$ *of index* $m|n$ *with the same b-divisor* $\mathcal{P}$.

Proof. Immediate by definition and Lemmas 2, 3. Actually, we need to verify only that (1) of Definition 2 for B^+ with respect to B implies that of with respect to D (cf. Proposition 7). For rational b, we use Lemmas 2, 3. (The case with $d \leq b = 0$ is easy to add to Lemma 2.) For irrational b, we can use the continuity of $\lfloor (n+1)d \rfloor /n$ in a neighborhood of b.

To prove Addendum 63 it is sufficient to verify that $D_{n\_\Phi} \leq B_{n\_\Phi}$ under our assumptions. This follows from definition and Corollary 4.

Similarly we can treat bd-pairs.

$\square$

Proof of Theorem 18. Restrictions on complementary indices hold by Construction 10.

Let $(X/Z \ni o, B)$ be a pair under the assumptions of Existence of n-complements in the theorem and $A\colon \mathbb{R}^r \to \mathrm{WDiv}_{\mathbb{R}} X$ be a corresponding affine map. By our assumptions $B = A(x)$ for some $x \in \Delta$. On the other hand, by Construction 10, $x \in U(x^j)$, where $x^j \in \Delta$ and

$$\left\| A(x^j) - B \right\| < \delta(x^j).$$

Again by Construction 10 $(X/Z \ni o, A(x^j))$ has an n-complement $(X/Z \ni o, B^+)$ for some $n \in \mathcal{N}(x^j)$. Indeed, $(X/Z \ni o, A(x^j))$ satisfies the assumptions of Existence of n-complements in Construction 10. In particular, $(X/Z \ni o, A(x^j))$ has an $\mathbb{R}$-complement or lc type because $A(x^j) \in \mathfrak{C}_{\mathbb{R}}$ by our assumptions. This with $A(x^j) \geq 0$ implies that $A(x^j)$ is a boundary and $\in \Gamma(x^j)$ (cf. Remark 1,(1)).

Finally, $(X/Z \ni o, B^+)$ is also an n-complement of $(X/Z \ni o, B)$ by Construction 10 and Lemma 13.

Similarly we can treat bd-pairs.

$\square$

Remark 9. (1) Actually Existence of n-complements in Theorem 18 holds for a slightly large set of B because the covering of Δ is larger than Δ itself.

(2) For an analog of Remark 8 and Theorem 18 we can use the remark and Addendum 63. In this situation under the assumptions of Existence of n-complements of Theorem 18, $(X/Z \ni o, B^+)$ is a b-n-complement of $(X/Z \ni o, B_{n\_\Phi}))$ too, if (X, B_{n_Φ}) is a log pair. Actually, in the proof it is better to take a $\mathbb{Q}$-factorialization at the begging.

The same works for bd-pairs.

Proof of Theorems 3 and 4. The theorem under the assumption (1) is immediate by Theorem 16 and Addendum 59.

For the assumption (2) we can use Theorem 18 with an appropriate set of real numbers $\mathfrak{A}$. The P-component (11.0.10) of a map A in this case has $a_P = 0, a_{P,i} = \operatorname{mult}_P D_i, i = 1, \ldots, r$, nonnegative integers. By definition, for $(d_1, \ldots, d_r) \in \mathbb{R}^r$,

$$A(d_1, \ldots, d_r) = \sum_{i=1}^{r} d_i D_i \text{ and } \operatorname{mult}_P A(d_1, \ldots, d_r) = \sum_{i=1}^{r} a_{P,i} d_i.$$

On the other hand, for $(d_1, \ldots, d_r) \in \Delta$, all $d_i \geq 0$ and $\sum_{i=1}^{r} a_{P,i} d_i \leq 1$ hold by our assumptions. Hence we can suppose that all numbers $a_{P,i}$ belong to a finite set $\mathfrak{A}$, including 0. Indeed, if $a_{P,i} \gg 0$ then every $d_i = 0$ and we can take $a_{P,i} = 0$.

Theorem 3 under the assumption (3) is immediate by Theorem 17 and Example 12.

To prove Addendum 1 we can consider in our dimensional induction simultaneously a bounded number of lc centers, e.g., in Lemma 12 with nonconnected X_o a required plt model for every connected component of X_o. Equivalently, we can prove Theorem 3 relaxing the assumption that X is irreducible but assuming that X has a bounded number of irreducible (connected) components X_i and $\dim X = \max \dim X_i$. All (b-)n-complements are coming from the exceptional one of bounded dimension. So, we add also the boundedness of components of the exceptional pairs or consider exceptional pairs with bounded number of irreducible (connected) components. Notice that in the proof of Theorem 7 we use only boundedness of exceptional pairs but not their irreducibility. Since we are working with algebraic spaces X too, it is possible to use an appropriate étale neighborhood (a branch) for every connected component of X_o. We can take the same n-complements for isomorphic neighborhoods. Thus it is to count only nonisomorphic neighborhoods.

Similarly we can treat bd-pairs.

To prove Addendum 2, for a global pair (X, B), we use the invariance of H^0 with respect to the algebraic closure. The connectedness of X_0 depends on the algebraic closure but independent modulo conjugation of the closure. Thus after taking the algebraic closure we can use Addendum 1. Note also that existence of an n-complement means existence of an element $\overline{B}$ in a linear system

$$|-nK - nS - \lfloor (n+1)(B-S)\rfloor|$$

such that (X, B^+) is lc, where $S = \lfloor B \rfloor$, the reduced part of B, and

$$B^+ = B + \frac{1}{n}(\lfloor (n+1)(B-S)\rfloor + \overline{B})$$

[Sh92, after Definition 5.1]. Thus $\overline{B}$ should be sufficiently general in the Zariski topology. If such element exists over the algebraic closure $\overline{k}$, it exists also over k because k is infinite (cf. Example 1, (5)).

The same arguments work for global G-pairs assuming that n is sufficiently divisible. Indeed, we can suppose that a canonical divisor K of X is G-semicanonical: $|G|K$ is G-invariant.

Similarly we can treat nonglobal pairs and G-pairs.

$\square$

In dimension 1 such subtleties are not need.

Example 13 (n-complements in dimension 1). Theorem 3 for $\dim X = 1$ holds for every pair $(X/Z, B)$, without the local (with unbounded number of leaves for X/Z), wFt and connectedness of X_o assumptions. Any such a local pair with a boundary B has an $\mathbb{R}$-complement and n-complement for any positive integer n satisfying Restrictions on complementary indices.

In the global case we suppose existence of an $\mathbb{R}$-complement instead of (1) in Theorem 3 (cf. Example 3).

The global case with $X = E$, a curve of genus 1, is also trivial: $B^+ = B = 0$ and n is any positive integer satisfying Restrictions as above.

So, the main interesting case as in Example 3 concerns global pairs $(\mathbb{P}^1, B)$ with $B = \sum_{i\ge 1} b_i P_i, 1 \ge b_1 \ge b_2 \ge \cdots \ge b_i \ge \cdots \ge 0$ and $\sum_{i\ge 1} b_i \le 2$. However this time to exclude points P_i with small b_i we use our general approach with low approximations but, for simplicity, without Restrictions on complementary indices.

Lc type: $b_1 = 1$ and B has at most two points P_i with $b_i = 1$. In this case, $(\mathbb{P}^1, B)$ has 1-complement, except for,

$$(\mathbb{P}^1, P_1 + \frac{1}{2}b_2 + \frac{1}{2}b_3).$$

The pair is a 2-complement of itself.

Generic pairs: every $b_i < 1$ and $\sum_{i\geq 2} b_i < 1$. In this case we have again a 1-complement by an elementary computation (cf. Theorem 11). Moreover, we can suppose that $(\mathbb{P}^1, B_1)$ has generic type where B_1 is the approximation of B with $\Gamma(1, \emptyset) = \{0, 1/2, 1\}$. (For $\Phi = \emptyset$, we take an abridged set of multiplicities, in Hyperstandard sets of Section 3, with only $l = 1$.) So, $(\mathbb{P}^1, B_1)$ is generic only for

$$B_1 = \begin{cases} 0, & \text{if } 1/2 > b_1 \\ \frac{1}{2}P_1, & \text{if } b_1 \geq 1/2 > b_2 \\ \frac{1}{2}P_1 + \frac{1}{2}P_2, & \text{if } b_2 \geq 1/2 > b_3. \end{cases}$$

In all these cases $(\mathbb{P}^1, B)$ has a 1-complement.

Semiexceptional pairs: $(\mathbb{P}^1, B_1)$ with

$$B_1 = \begin{cases} \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3, & \text{if } b_3 \geq 1/2 > b_4 \\ \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + \frac{1}{2}P_4, & \text{if } b_1 = b_2 = b_3 = b_4 = 1/2 \text{ and } b_5 = 0. \end{cases}$$

In the last case

$$(\mathbb{P}^1, \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + \frac{1}{2}P_4)$$

is a 2-complement of itself.

Thus we need to construct n-complements only in the case

$$B_1 = \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3.$$

The pair $(\mathbb{P}^1, B_1)$ has the semiexceptional type $(0, 0)$ and this is the top type with the next generic type. The previous type $(-1, -)$ is exceptional. According to our general approach we need to extend the set $\mathcal{N}' = \{1\}$ to another set of positive integers $\mathcal{N}^{(0,0)}$ (see Semiexceptional filtration in Section 8). For simplicity consider $\mathcal{N}_{(0,0)} = \{2\}, \mathcal{N}^{(0,0)} = \{1, 2\}$ and $\Phi = \emptyset$. Then $\Gamma(\{1, 2\}, \emptyset) = \{0, 1/3, 1/2, 2/3, 5/6, 1\}$. For the last set of boundary multiplicities $(\mathbb{P}^1, B_{\{1,2\}})$ is again semiexceptional but not exceptional only for

$$B_{\{1,2\}} = \begin{cases} \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3, & \text{if } 2/3 > b_1 \geq b_3 \geq 1/2 \text{ and } 1/3 > b_4 \\ \frac{2}{3}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3, & \text{if } 5/6 > b_1 \geq 2/3 > b_2 \geq b_3 \geq 1/2 \text{ and } 1/3 > b_4 \\ \frac{5}{6}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3, & \text{if } b_1 \geq 5/6, 2/3 > b_2 \geq b_3 \geq 1/2 \text{ and } 1/3 > b_4. \end{cases}$$

In all three cases $(\mathbb{P}^1, B)$ has a 2-complement by a direct computation (cf. Theorem 12).

Exceptional pairs: $(\mathbb{P}^1, B_{\{1,2\}})$ with $B_{\{1,2\}} \in \{0, 1/3, 1/2, 2/3, 5/6, 1\}$ form a bounded family. As in Example 3 we can join all small multiplicities $b_i < 1/3$ to b_1 (or to any other $b_i \geq 1/3$; cf. Step 7 of the proof of Theorem 7). The new (X, B) will have also exceptional $(\mathbb{P}^1, B_{\{1,2\}})$ (possibly with different $B_{\{1,2\}}$). Moreover, $B = b_1 P_1 + b_2 P_3 + b_3 P_4$, except for, B with

$$B_{\{1,2\}} = \frac{2}{3}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + \frac{1}{3}P_4.$$

In this case $(\mathbb{P}^1, B)$ is a 6-complement of itself (and has also a 4-complement).

In all other cases $B = b_1 P_1 + b_2 B_2 + b_3 P_3$ and $(\mathbb{P}^1, B_{\{1,2\}})$ is exceptional. In these cases we can find a finite set of small complementary indices, e.g., $\{3, 4, 6\}$ as in [Sh92, Example 5.2]. Or we can find a finite set of complementary indices satisfying Restrictions using Theorem 7 or [Sh95, Example 1.11].

12 Applications

Inverse stability for $\mathbb{R}$-complements. Let $(X/Z, D)$ be a pair and ε be a positive integer. We say that the *inverse stability for $\mathbb{R}$-complements* holds for $(X/Z, D)$ if, for every boundary B on X such that $\|B - D\| < \varepsilon$ and under certain additional assumptions, the existence of an $\mathbb{R}$-complement for $(X/Z, B)$ implies that of for $(X/Z, D)$.

The same definition works for bd-pairs $(X/Z, B + \mathcal{P}), (X/Z, D + \mathcal{P})$, that is, we compare only the divisorial parts B, D.

Theorem 19. *Let d, h, l be nonnegative integers and $v \in \mathbb{R}^l$ be a vector. Then there exists a positive real number ε such that, for every*

$\mathbb{Q}$-affine map $A \colon \mathbb{R}^l \to \mathrm{WDiv}_{\mathbb{R}} X$ of the height not exceeding h; and

pair $(X/Z \ni o, B)$ with wFt $X/Z \ni o$, $\dim X = d$, a boundary B with $\|B - A(v)\| < \varepsilon$ and under either of the (additional) assumptions (1-3) of Theorem 3,

the inverse stability for $\mathbb{R}$-complements holds for $(X/Z \ni o, A(v))$.

Notice that we do not suppose that X_o is connected.

Addendum 65. *We can assume that* B *is given locally over* $Z \ni o$ *(even in the étale topology) near every connected component of* X_o*.*

Addendum 66. $A(v)$ *is a boundary.*

Addendum 67. *There exists a neighborhood* U *of* v *in* $\langle v\rangle$ *such that, for every vector* $u \in U$*,* $A(u)$ *is a boundary and* $(X/Z \ni o, A(u))$ *has an* $\mathbb{R}$*-complement.*

Addendum 68. *The same holds for bd-pairs* $(X/Z \ni o, A(v) + \mathcal{P}), (X/Z \ni o, B + \mathcal{P})$ *of index* m *with* ε *depending also on* m*.*

Corollary 21 (Direct stability for $\mathbb{R}$-complements; cf. [N, Theorem 1.6] [HLSh, Theorems 5.6 and 5.16])**.** *Under the assumptions and notation of Theorem 19 there exists a neighborhood* U *of* v *in* $\langle v\rangle$ *such that if* $A(v)$ *is a boundary and* $(X/Z \ni o, A(v))$ *has an* $\mathbb{R}$*-complement and additionally either of the assumptions (1-3) of Theorem 3 holds for* $(X/Z \ni o, A(v))$ *then, for every vector* $u \in U$*,* $A(u)$ *is a boundary and* $(X/Z \ni o, A(u))$ *has an* $\mathbb{R}$*-complement.*

The same holds for bd-pairs of index m $(X/Z \ni o, A(v) + \mathcal{P})$*.*

Proof. Take U as in Addendum 67 and apply Theorem 19 with the addendum to $B = A(v)$.

Similarly we can treat bd-pairs.

□

Corollary 22 (n-complements vs $\mathbb{R}$-complements)**.** *Let* $(X/Z \ni o, B)$ *be a pair with wFt* $X/Z \ni o$ *and a boundary* B*. Then* $(X/Z \ni o, B)$ *has an* $\mathbb{R}$*-complement under either of the following assumptions:*

Weak version: $(X/Z \ni o, B)$ *has* n*-complements for infinitely many positive integers* n*; or*

Strong version: $(X/Z \ni o, B)$ *has an* n*-complements for one but sufficiently large positive integer* n*.*

The same holds for bd-pairs $(X/Z \ni o, B + \mathcal{P})$ *of index* m*.*

Proof. Weak version. Immediate by Theorem 1.

Strong version. Immediate by the closed rational polyhedral property of Theorem 6 and by the proof of Theorem 1.

Alternatively, we can use Theorem 19 instead of Theorem 6. Indeed, according to the proof of Theorem 1 there exists a sequence of positive integers n_i and of rational boundaries $B_i = B_{[n_i]}, i = 1, 2, \ldots,$ on X such that $B = \lim_{i\to\infty} B_i, \lim_{i\to\infty} n_i = +\infty$ and every pair $(X/Z \ni o, B_i)$ has an $\mathbb{R}$- and monotonic n_i-complement. For ε of Theorem 19 and i such that $\|B_i - B\| < \varepsilon$, $n = n_i$ works. We apply the inverse stability for $\mathbb{R}$-complements to $A(v) = B = \sum b_j D_j, v = (b_j)$, and $B = B_i$.

Similarly we can treat bd-pairs.

$\square$

Theorem 20 (Inverse stability for n-complements). *Let n be a positive integer and μ be a positive real number. There exists a positive real number δ such that if*

$(X/Z, B)$ is a pair with a boundary B, having an n-complement, and

D is a $\mathbb{Q}$-divisor on X such that nD is an integral Weil divisor, $\|D - B\| < \delta/n$ and every positive multiplicity d of D is $\geq \mu$

then $(X/Z, D)$ has an n-complement.

Addendum 69. *D is a boundary.*

Addendum 70. *If $(X/Z, B^+)$ is an n-complement of $(X/Z, B)$ then $(X/Z, B^+)$ is also an n-complement of (X, D).*

Addendum 71. *Any n-complement of $(X/Z, D)$ is monotonic.*

Addendum 72. *We can take any positive real number $\delta \leq \mu n/(n+1)]$ and 1.*

Addendum 73. *For $\mu > 1$, $D = 0$ and the theorem is trivial. For $\mu \leq 1$, we can take any positive real number $\delta \leq \mu n/(n+1)$. in particular, $0 < \delta \leq \mu/2$ independent of n.*

Addendum 74. *The same holds for bd-pairs $(X/Z \ni o, B + \mathcal{P}), (X/Z \ni o, D + \mathcal{P}), (X/Z, B^+ + \mathcal{P})$ of index $m|n$.*

Proof. It is enough to verify Addendum 70. By Definition 2, it is enough to verify (1) of the definition.

Let P be a prime divisor on X. Put $d = \operatorname{mult}_P D$ and $b^+ = \operatorname{mult}_P B^+$. We need to verify that

$$b^+ \geq \begin{cases} 1, \text{ if} & d = 1; \\ \lfloor (n+1)d \rfloor / n & \text{otherwise.} \end{cases}$$

Take δ of Addendum 72.

Step 1. *Addendum 69.* D is a boundary, that is, $d \in [0, 1]$. Moreover, $d = 0$ or $\geq \mu$. By our assumptions $d = m/n$, where $m \in \mathbb{Z}$, and

$$d < b + \frac{\delta}{n} \leq 1 + \frac{1}{n},$$

where $b = \operatorname{mult}_P B$. Hence $m < n + 1$ and, moreover, $\leq n$, that is, $d \leq 1$. Similarly, $m \geq 0$ and $d \geq 0$ because $b \geq 0$.

Thus if $d \neq 0$ then d is positive and $\geq \mu$ by our assumptions. This implies Addendum 73. Indeed, for $\mu > 1$, $D = 0 \leq B$ and the theorem follows from Proposition 1.

Step 2. Addendum 71 follows from Example 5, (2).

Step 3. *Case* $d \leq b$. For $b < 1$, follows from (1) of Definition 2 for B and the monotonicity of $\lfloor\ \rfloor$: $d < 1$ and

$$\lfloor (n+1)d \rfloor / n \leq \lfloor (n+1)b \rfloor / n \leq b^+.$$

Otherwise, $b = b^+ = 1, d \leq 1 = b^+$ and the required inequality holds by definition.

Step 4. *Case* $d > b$. In this case $d > 0$ and $\geq \mu$ because $b \geq 0$. Since $\|d - b\| < \delta/n$ holds,

$$b > d - \frac{\delta}{n} \geq d - \frac{\mu n}{(n+1)n} = d - \frac{\mu}{n+1}.$$

By Step 1, $d \leq 1$ and $\mu \leq 1$ (cf. Addendum 73). Moreover, if $\mu = 1$ then $d = 1, 1 - 1/(n+1) = n/(n+1) < b < 1$ and

$$b^+ \geq \lfloor (n+1)b \rfloor / n = 1 = d.$$

Otherwise, $\mu < 1$ and by our assumptions $d = m/n \geq \mu$, where m is integer and $1 \leq m \leq n$.

If $d < 1$ then again Example 5, (2) gives the required inequality

$$b^+ \geq \lfloor (n+1)b \rfloor /n \geq \left\lfloor (n+1)(\frac{m}{n} - \mu/(n+1)) \right\rfloor /n =$$
$$\left\lfloor m + \frac{m}{n} - \mu \right\rfloor /n = m/n + \lfloor d - \mu \rfloor /n = m/n = d = \lfloor (n+1)d \rfloor /n.$$

If $d = 1$ then

$$b^+ \geq \lfloor (n+1)b \rfloor /n = 1 = d$$

because $n/(n+1) < b < 1$ as above.

Similarly we can treat bd-pairs.

□

Proof of Theorem 19. We will chose ε below.

Step 1. *Renormalization of v and I.* We can suppose that every A has integral ($\mathbb{Z}$-matrix) linear part. For this we change the standard basis $(1,0,\ldots,0),\ldots,(0,0,\ldots,1)$ of $\mathbb{R}^l$ by $(N,0,\ldots,0),\ldots,(0,0,\ldots,N)$ for sufficiently divisible positive integer N. Then we need to increase the height h and replace the vector v by v/N.

In this step we can chose a positive integer I such that IA is integral, equivalently, for every constant c of A, Ic is an integer. Hence $nA(w)$ is integral if $w \in \mathbb{Q}^l$, $nw \in \mathbb{Z}^l$ and $I|n$.

Step 2. *Choice of μ.* Since the height of A is bounded by h, every multiplicity $a = \mathrm{mult}_P A(v)$ in prime P has the form

$$a = c + \sum_{i=1}^{l} a_i v_i,$$

where $a_i \in \mathbb{Z}, \|a_i\| \leq h, c \in \mathbb{Q}, Ic \in \mathbb{Z}$, [remark: we need bounded hight of c only to bound the denominators of c] and $v = (v_1,\ldots,v_l)$. Hence the set of those multiplicities a is finite and there exists $\mu > 0$ such that $a \geq \mu$ if $a > 0$. Indeed, the constant terms c belong to a finite set, e.g., because by our assumptions

$$\|b - a\| = \left\| b - c - \sum_{i=1}^{l} a_i v_i \right\| < \varepsilon$$

and $b \in [0,1]$. This implies also Addendum 66, if ε is sufficiently small. We suppose that μ is also sufficiently small, e.g., $\mu \leq 1$.

Note also that for any two vectors $w_1, w_2 \in \mathbb{R}^l$,

$$\|A(w_1) - A(w_2)\| \leq hl \, \|w_1 - w_2\| \, .$$

Step 3. *Rational case.* For rational v, $\langle v \rangle = v$. In this case we can suppose that A is constant, e.g., all $a_i = 0$. Then Theorem 20 implies the required stability for n sufficiently divisible and sufficiently small ε. The n-complement is monotonic for $(X/Z \ni o, A(v))$ by Addenda 71.

So, we suppose below that v is irrational. Thus $r = \dim \langle v \rangle \geq 1$ and $l \geq r \geq 1$. Notice that by definition $h \geq 1$ because otherwise there are no A.

Step 4. *Choice of directions $e_0, \dots, e_r$ in $\langle v \rangle$.* Take $r+1, r = \dim \langle v \rangle$, directions $e_i, i = 0, \dots, r$, in $\langle v \rangle$ such that 0 is inside of the simplex $[e_0, e_1, \dots, e_r]$. For rather small μ, if every $\|e_i' - e_i\| \leq \mu$, for any other directions $e_0', e_1', \dots, e_r'$ in $\langle v \rangle$, then $[e_0', e_1', \dots, e_r']$ is also a simplex with 0 inside of it.

Step 5. *Choice of complementary indices.* For $i = 0, 1, \dots, r$, there exists a finite set of positive integers

$$\mathcal{N}_i = \mathcal{N}(d, I, \mu/8hl, v, e_i).$$

By Theorem 3 under either of assumptions (1-3) of the theorem, there exists such a finite set.

Step 6. *Choice of ε.* We can take $\varepsilon \leq \mu/8n$ for every $i = 0, 1, \dots, r$ and $n \in \mathcal{N}_i$.

Indeed, let B be a boundary on X such that

(4) $\|B - A(v)\| < \varepsilon$.

Then under other assumptions of Theorem 19 and either of assumptions (1-3) of Theorem 3, for every $i = 0, 1, \dots, r$, by Theorem 3 $(X/Z \ni o, B)$ has an n_i-complement $(X/Z \ni o, B_i^+)$ with $n_i \in \mathcal{N}_i$. By Restrictions of Theorem 3 we have also approximations $w_i \in \langle v \rangle$ such that

(5) $n_i w_i \in \mathbb{Z}^l$;

(6) $\|w_i - v\| < \mu/8hln_i$; and

(7)
$$\left\| \frac{w_i - v}{\|w_i - v\|} - e_i \right\| < \mu/8hl \leq \mu.$$

By Theorem 20 there exist $\mathbb{R}$-complements $(X/Z \ni o, B_i^+)$ of $(X/Z \ni o, D_i)$, where $D_i = A(w_i)$.

Indeed, by (5) and Step 1 $n_i D_i$ is integral because $I|n_i$.

By (6) and Step 2,

(8) $\|A(v) - D_i\| < hl\mu/8hln_i = \mu/8n_i \leq \mu/8.$

In particular, this implies that if $\operatorname{mult}_P A(v) = 0$ for a prime divisor P on X, then $\operatorname{mult}_P D_i = 0$ too because $n_i \operatorname{mult}_P D_i$ is integral but $\mu/8n_i \leq 1/8n_i < 1$ for $\mu \leq 1$. Otherwise, $\operatorname{mult}_P A(v) \geq \mu$ and by (8) $\operatorname{mult}_P D_i \geq \mu - \mu/8 = 7\mu/8 \geq \mu/2$. Again by (8) and (4),

$$\|D_i - B\| \leq \|B - A(v)\| + \|A(v) - D_i\| < \mu/8n_i + \mu/8n_i = \mu/4n_i.$$

Thus the inversion of Theorem 20 holds for $(X/Z \ni o, D_i)$ with $\mu/2$ instead of μ and $\delta = \mu/4$ by Addendum 73. So, $(X/Z \ni o, D_i)$ has an n_i-complement. It is monotonic by Addendum 71. Hence every $(X/Z \ni o, D_i)$ has an $\mathbb{R}$-complement too.

Step 7. *Conclusion.* By the convexity of Theorem 6 we need to verify that $A(v)$ belongs to $[D_0, D_1, \ldots, D_r]$. In its turn, this follows from the inclusion $v \in [w_0, w_1, \ldots, w_r]$.

By Step 4 and (7)

$$0 \in [\frac{w_0 - v}{\|w_0 - v\|}, \frac{w_1 - v}{\|w_1 - v\|}, \ldots, \frac{w_r - v}{\|w_r - v\|}].$$

Since the denominators $\|w_i - v\|$ are positive, there exist positive real numbers $\nu_0, \nu_1, \ldots, \nu_r$ such that

$$\nu_0(w_0 - v) + \nu_1(w_1 - v) + \cdots + \nu_r(w_r - v) = 0 \text{ and } \sum_{i=0}^{r} \nu_i = 1.$$

Hence $v = \nu_0 w_o + \nu_1 w_1 + \cdots + \nu_r w_r$ which gives the required inclusion.

We were cheating a little bit. The previous proof works only if X_o is connected. Now we make

Step 8. *General case and Addendum 65.* (Cf. Proposition 9) Suppose that $(X/Z \ni o, A(v))$ does not have $\mathbb{R}$-complement. We can assume that X is $\mathbb{Q}$-factorial. Otherwise we replace X by its $\mathbb{Q}$-factorialization. Again $(X/Z \ni o, A(v))$ does not have $\mathbb{R}$-complement. We can suppose that $X/Z \ni o$

has Ft, in particular, is projective. Otherwise we use a small modification of Lemma 1. Still $(X/Z \ni o, A(v))$ does not have $\mathbb{R}$-complement.

The pair $(X/Z \ni o, A(v))$ is lc. Otherwise $(X/Z \ni o, A(v))$ is not lc near some connected component of X_0 and does not have an $\mathbb{R}$-complement near this component. This is impossible by the connected case.

The divisor $-(K + A(v))$ is not nef. Otherwise $(X/Z \ni o, A(v))$ has $\mathbb{R}$-complement. So, we can apply $-(K + A(v))$-MMP as in Construction 2. This preserves the $\mathbb{R}$-complements by Proposition 8. Thus after finitely may steps [ShCh, Corollary 5.5] we have an extremal contraction $X/Y/Z \ni o$ which is positive with respect to $K + A(v)$. The transformations preserve connected components of X_o. So, the initial model $(X/Z \ni o, A(v))$ does not have an $\mathbb{R}$-complement near the connected component of X_o corresponding to the connected component with the nearby fibration. This is impossible by the connected case.

Step 9. *Addendum 67.* Immediate by Step 7 and the convexity of Theorem 6: U is a neighborhood of v in $[w_0, w_1, \ldots, w_r] \subset \langle v \rangle$.

Similarly we can treat bd-pairs with the assumptions (1-3-bd) of Theorem 4.

$\square$

$\mathbb{R}$-complement thresholds. Let $(X/Z, D)$ be a pair such that it has an $\mathbb{R}$-complement and $F > 0$ be an effective $\mathbb{R}$-divisor on X. Then the following threshold

$$\mathbb{R}\text{-clct}(X/Z, D; F) = \sup\{t \in \mathbb{R} \mid (X/Z, D + tF) \text{ has an } \mathbb{R} - \text{complement}\}$$

is a nonnegative real number and well-defined. If X/Z has wFt then we can use the maximum instead of the supremum by the closed property in Theorem 6. The threshold will be called the $\mathbb{R}$-*complement thereshold* of F for $(X/Z, D)$.

The same definition works for a bd-pair $(X/Z, D + \mathcal{P})$ and gives the threshold $\mathbb{R}$-clct$(X/Z, D + \mathcal{P}; F)$.

Theorem 21. *Let d be a nonnegative integer and Γ_b, Γ_f be two dcc sets of nonnegative real numbers. Then the set of thresholds*

$$\{\mathbb{R}\text{-clct}(X/Z, B; F) \mid \dim X = d, X/Z \textit{ has wFt}, B \in \Gamma_b \textit{ and } F \in \Gamma_f\}$$

satisfies the acc.

Addendum 75. *Instead of $B \in \Gamma_b, F \in \Gamma_f$, we can suppose that $B = \sum b_i E_i, F_i = \sum f_i E_i$, where E_i are effective Weil $\mathbb{Z}$-divisors and $b_i \in \Gamma_b$ and $f_i \in \Gamma_f$.*

Addendum 76. *The same holds for bd-pairs $(X/Z, B + \mathcal{P})$ of index m and the set of corresponding thresholds depends also on m.*

Actually, the set of $\mathbb{R}$-complement thresholds is a union of two well-known sets of thresholds.

Lemma 14. *Let $(X/Z, D)$ be a pair with wFt X/Z and F be an effective and $\neq 0$ divisor on X. Then*

$$\mathbb{R}\text{-clct}(X/Z, D; F) = \begin{cases} \operatorname{lct}(X' \ni o', D'; F') = \mathbb{R}\text{-clct}(X' \ni o', D'; F') \textit{ or} \\ \operatorname{act}(X''/o'', D''; F'') = \mathbb{R}\text{-clct}(X''/o'', D''; F'') \end{cases},$$

where lct, act *are respectively* log canonical *and* anticanonical *thresholds; X', X'' are $\mathbb{Q}$-factorial of dimension $\leq \dim X$, $\rho(X'') = 1$, $X' \ni o', X''$ are Ft and o', o'' are closed points; the multiplicities of D', D'' and of F', F'' are respectively multiplicities of D and of F. In particular, if $D \in \Gamma_d, F \in \Gamma_f$ then $D', D'' \in \Gamma_d, F', F'' \in \Gamma_f$ respectively.*

Conversely, $\operatorname{lct}(X' \ni o', D'; F') = \mathbb{R}\text{-clct}(X' \ni o', D'; F')$ *and* $\operatorname{act}(X''/o'', D''; F'') = \mathbb{R}\text{-clct}(X''/o'', D''; F'') = a$ *if $(X'', D'' + aF'')$ is lc.*

The same holds for bd-pairs:

$$\mathbb{R}\text{-clct}(X/Z, D; F+\mathcal{P}) = \begin{cases} \operatorname{lct}(X' \ni o', D'; F' + \mathcal{P}') = \mathbb{R}\text{-clct}(X' \ni o', D'; F' + \mathcal{P}') \textit{ or} \\ \operatorname{act}(X''/o'', D''; F'' + \mathcal{P}'') = \mathbb{R}\text{-clct}(X''/o'', D''; F'' + \mathcal{P}''). \end{cases}$$

Additionally, the bd-pairs $(X' \ni o', D'; F' + \mathcal{P}'), (X''/o'', D''; F'' + \mathcal{P}'')$ have index m if $(X/Z, D + \mathcal{P})$ has index m.

Proof. Put $t = \mathbb{R}\text{-clct}(X/Z, D; F)$. We assume that $t \geq 0$ and is well-defined.

Step 1. *We can suppose that X is $\mathbb{Q}$-factorial and Ft.* Taking a $\mathbb{Q}$-factorialization $Y \to X$, we reduce the proof to the $\mathbb{Q}$-factorial case by Proposition 3.

Below we assume that X is $\mathbb{Q}$-factorial. By Lemma 1 we can suppose also that X/Z is Ft, in particular, projective over Z.

Step 2. *We can suppose that $(X, D + tF)$ is klt over* $\operatorname{Supp} F$, that is, the lc centers of $(X, B + tF)$ are not in $\operatorname{Supp} F$. If $p \in X$ is a point such that $(X, D+tF)$ is lc but not klt in p, and F passes through the point (i.e., $F > 0$ near p), then we have the *lc threshold* in p: $\operatorname{lct}(X \ni p, D; F) = t = \mathbb{R}\text{-clct}(X/Z, D; F)$. Taking hyperplane sections we can suppose also that p is closed (cf. Step 6).

Warning: p is not necessarily over o but contains a point over o.

So, we can assume that F only passes klt points of $(X, D+tF)$ or $(X, D+tF)$ is klt near $\operatorname{Supp} F$.

Step 3. *We can suppose that $-(K+D+tF)$ is nef over Z.* Otherwise, there exists an extremal contraction $X \to Y/Z$ which is positive with respect to $K+D+tF$. The contraction is birational because otherwise by Definition 1, (1) and (3)

$$K + B^+ - (K + D + tF) = B^+ - D - tF \geq 0$$

is numerically negative over Y, a contradiction, where $(X/Z, B^+)$ is an $\mathbb{R}$-complement of $(X/Z, D+tF)$.

If the contraction is small then we make an antiflip. The antiflip preserves the threshold t by Proposition 3. If after that F passes a nonklt point then we go to Step 2.

If the contraction is divisorial then we contract a prime divisor P. The contraction preserves the $\mathbb{R}$-complements and $t = \mathbb{R}\text{-clct}(X/Z, D; F) = \mathbb{R}\text{-clct}(Y/Z, D_Y; F_Y)$, where D_Y, F_Y are images of respectively D, F on Y. Indeed, by definition $(Y/Z, D_Y + tF_Y)$ has the $\mathbb{R}$-complement $(Y/Z, B_Y^+)$ with the image B_Y^+ of B^+ on Y. Thus $\mathbb{R}\text{-clct}(Y/Z, D_Y; F_Y) \geq t$. Actually, $= t$ holds by Proposition 8 applied to $(X/Y, D+tF)$. Indeed, for sufficiently small real number $\varepsilon > 0$, $K + D + (t+\varepsilon)F$ is negative over Y and $(X^\sharp/Y, (D+(t+\varepsilon)F)^\sharp_{X^\sharp}) = (Y/Y, D_Y + (t+\varepsilon)F_Y)$. Moreover, if $(Y/Z, D_Y + (t+\varepsilon)F_Y)$ has an $\mathbb{R}$-complement $(Y/Z, B')$. Then it induces an $\mathbb{R}$-complement $(X/Z, B'_X)$, with crepant B'_X, of $(X/Z, D+(t+\varepsilon)F)$, a contradiction. By definition it is enough to verify that

$$\operatorname{mult}_P B'_X \geq \operatorname{mult}_P(D + (t+\varepsilon)F).$$

This is equivalent to (1) of Definition 1 over Y locally near the center (image) of P. This holds by Proposition 8 or Addendum 8. In particular we prove that $\operatorname{Supp} F \neq P$ and $F_Y \neq 0$. If after the divisorial contraction F passes a nonklt point then we go again to Step 2.

So, by the termination [ShCh, Corollary 5.5], after finitely many steps we get nef $-(K+D+tF)/Z$. The last divisor $-(K+D+tF)$ gives a contraction $X \to Y/Z$.

Step 4. *$-F$ is not nef with respect to the contraction X/Y.* Since X/Z has Ft, the cone of curves $\mathrm{NE}(X/Z)$ in $\mathrm{N}(X/Z)$ is closed convex rational polyhedral with finitely many extremal rays [ShCh, Corollary 4.5]. Denote by V its face generated by the extremal rays R which are contracted on Y, that is, they have a curve C/o with $(C.K + D + tF) = 0$. Actually this is a face by Step 3. Moreover, the cone V is also closed convex rational polyhedral. If $-F$ is nef on the face V then, for every sufficiently small real number $\varepsilon > 0$, $-(K + D + (t + \varepsilon)F)$ is nef by Step 3 and semiample over Z [ShCh, Corollary 4.5]. By Step 2 we can suppose also that $(X, D + (t + \varepsilon)F)$ has lc singularities. Thus t is not $\mathbb{R}$-clct$(X/Z, D; F)$ by Addendum 8, a contradiction.

Step 5. Moreover, *we can suppose that $X/Y = Z$ itself is a fibered (extremal) contraction with o, the generic point of Z, and F is positive over Z.* To establish this we apply $(-F)$-MMP to X/Z. However, we consider only extremal rays R of the cone of curves $\mathrm{NE}(X/Z)$ which are numerically trivial with respect to $K + D + tF$, that is, $R \subseteq V$. By Step 4 there exists such an extremal ray R negative for $-F$, equivalently, F is positive for F.

If R gives a small birational contraction $X \to X'/Z$, actually, over Y then we make a flip in R and this preserves the threshold t. If R gives a divisorial contraction $X \to X'/Z$, again also over Y, then after the contraction we have the same threshold

$$\mathbb{R}\text{-clct}(X/Z, D; F) = \mathbb{R}\text{-clct}(X'/Z, D'; F'),$$

where D', F' are birational transforms of D, F respectively on X'. In particular, F is not exceptional for X/X' and $F' \neq 0$. We can argue here as in Step 3. However, F' can pass lc singularities of $(X', D' + tF')$. In this case we go again to Step 2.

By Step 4 and termination we get an extremal fibered contraction $X \to X'/Z$ for which $K+D+tF \equiv 0/X'$ and F is numerically positive over X'. By construction and since any contraction of Ft is Ft, $X/X' \ni \eta$ has Ft, where η is the generic point of X' and $\mathbb{R}\text{-clct}(X/Z, D; F) = \mathbb{R}\text{-clct}(X/\eta, D; F) = \mathrm{act}(X/\eta, D; F)$. Finally, denote X/η by X/o.

Step 6. *We can suppose that $Z = o$ is a closed point and $\rho(X/o) = 1$.* In terms of Italian understanding of the generic point we can replace X/o by X_p/p, where p is a sufficiently general closed point of Z and X_p is the fiber of X/Z, and

$$\mathbb{R}\text{-clct}(X/Z, D; F) = \mathbb{R}\text{-clct}(X/o, D; F) = \mathbb{R}\text{-clct}(X_p/p, D_p; F_p) = \mathrm{act}(X_p/p, D_p; F_p),$$

where $D_p = D_{|X_p}, F_p = F_{|X_p}$. We loose only the extremal property of contraction $X \to X'/o$ but only when $o \neq p$. By Proposition 10 and Addendum 12 and since X/o has Ft, X_p is $\mathbb{Q}$-factorial for sufficiently general $p \in Z$ (specialization). For $o \neq p$, $\dim X_p < \dim X$. So, we use dimensional induction in this case and go to Step 2. Otherwise, $o = p$ is a closed point and we are done.

Notice that our algorithm preserves multiplicities of D and of F.

The converse holds by definition. Moreover, the converse holds for constructed pairs.

Similarly we can treat bd-pairs.

$\square$

Proof of Theorem 21. We reduce the acc to two acc's: for lc and ac thresholds, local over $Z_i \ni o$.

Consider a sequence of pairs

$$(X_i/Z_i \ni o, B_i), i = 1, 2, \dots, i, \dots \tag{12.0.11}$$

with effective $\mathbb{R}$-divisors F_i on X_i, such that

(1) every $X_i/Z_i \ni o$ has wFt with $\dim X_i = d$;

(2) every $B_i \in \Gamma_b, F_i \in \Gamma_f$ and $F_i > 0$;

(3) every $(X_i/Z_i \ni o, B_i)$ has an $\mathbb{R}$-complement; and

(4) the sequence of thresholds is nonnegative and monotonic

$$\begin{aligned} 0 &\le \mathbb{R}\text{-clct}(X_1/Z_1 \ni o, B_1; F_1) \le \mathbb{R}\text{-clct}(X_2/Z_2 \ni o, B_2; F_2) \\ &\le \dots \le \mathbb{R}\text{-clct}(X_i/Z_i \ni o, B_i; F_i) \le \dots. \end{aligned}$$

For simplicity of notation we use same o everywhere instead of o_i (as same 0 for every field). We need to verify the stabilization: for every $i \gg 0$,

$$\mathbb{R}\text{-clct}(X_i/Z_i \ni o, B_i; F_i) = \mathbb{R}\text{-clct}(X_{i+1}/Z_{i+1} \ni o, B_{i+1}; F_{i+1}).$$

Step 1. *The case when every* $\operatorname{Supp} B_i, \operatorname{Supp} F_i$ *have at most* l *prime divisors.* Equivalently, $\operatorname{Supp} B_i \cup \operatorname{Supp} F_i$ has at most l prime divisors, possibly for a different natural number l. In other words, for every $i = 1, 2, \dots, i, \dots,$

there is a $\mathbb{Z}$-linear map $A_i\colon \mathbb{R}^l \to \operatorname{WDiv}_{\mathbb{R}} X_i$ of height 1 and vectors $x_i, y_i \in \mathbb{R}^l$ such that $A_i(x_i) = B_i, A_i(y_i) = F_i$.

Indeed, there exist distinct prime divisors $D_{i,j}, j = 1, \dots, l$, on X_i such that

$$B_i = \sum_{j=1}^{l} b_{i,j} D_{i,j} \text{ and } F_i = \sum_{j=1}^{l} f_{i,j} D_{i,j}.$$

We put

$$A_i(1, 0, \dots, 0) = D_{i,1}, A_i(0, 1, \dots, 0) = D_{i,2}, \dots, A_i(0, 0, \dots, 1) = D_{i,l}.$$

Hence $x_i = (b_{i,1}, b_{i,2}, \dots, b_{i,l})$ and $y_i = (f_{i,1}, f_{i,2}, \dots, f_{i,l})$, where by (2) every $b_{i,j} \in \Gamma_b$ and $f_{i,j} \in \Gamma_f$.

Since Γ_g, Γ_f satisfy the dcc, taking a subsequence of (12.0.11) we can suppose that $0 \le x_1 \le x_2 \le \dots \le x_i \le \dots$ and $0 < y_1 \le y_2 \le \dots \le y_i \le \dots$, where $\le$ for vectors as for divisors: $(v_1, \dots, v_l) \le (w_1, \dots, w_l) \in \mathbb{R}^l$ if every $v_i \le w_i$.

Put $x = \lim_{i\to\infty} x_i, y = \lim_{i\to\infty} y_i$ and

$$t = \lim_{i\to\infty} t_i, \quad t_i = \mathbb{R}\text{-clct}(X_i/Z_i \ni o, B_i; F_i).$$

The limits $x, x + ty$ and t are proper ($< +\infty$): $x, x + ty \in \mathbb{R}^l$ and $t \in \mathbb{R}$. (Thus the limit y is proper if $x + ty$ is proper and $t \ne 0$.) Indeed, for x, this follows from the subboundary property: every $x_i \le (1, 1, \dots, 1) \in \mathbb{R}^l$ or B_i is a subboundary by (3). The same works for $x + ty$ because every $x_i + t_i y_i \le (1, 1, \dots, 1)$ or $B_i + t_i F_i$ is a subboundary by the definition of $\mathbb{R}$-complements and (3) again. By (2-3) and construction every B_i and $B_i + t_i F_i$ are boundaries. On the other hand, since $F_i \ne 0$,

$$t_i \le 1/\min\{\gamma \mid \gamma \in \Gamma_f \text{ and } \gamma > 0\}.$$

The minimum exists and is positive by the dcc of Γ_f. Thus $t \in \mathbb{R}$. Note that y and every y_i are $> (0, 0, \dots, 0) \in \mathbb{R}^l$ by (2), in particular, $\ne 0 \in \mathbb{R}^l$.

Now it is enough to verify that $(X_i/Z_i \ni o, A_i(x+ty))$ has an $\mathbb{R}$-complement for every $i \gg 0$. Indeed, by construction, $A_i(x + ty) = A_i(x) + tA_i(y) \ge A_i(x_i) + t_i A_i(y_i) = B_i + t_i F_i$ by the monotonic property of every A_i. So, by definition $A_i(x + ty) = A_i(x) + tA_i(y) = A_i(x_i) + t_i A_i(y_i) = A_i(x_i + t_i y_i)$ and $x + ty = x_i + t_i y_i$ for those i because every A_i is injective. Thus $t_i = t$ for those i too because every $y \ge y_i > 0$. This is the required stabilization.

The $\mathbb{R}$-complement property follows from (1) and Theorem 19 under the assumption (3) of Theorem 2. Indeed, $\Gamma = \{b_{i,j} + t_i f_{i,j}\}$ satisfies the dcc with the only limiting points $x_1, \ldots, x_l$. On the other hand, by construction, definition and (3), every pair $(X_i/Z_i \ni o, B_i + t_i F_i)$ has an $\mathbb{R}$-complement. For every positive real number ε, the estimation

$$\|B_i + t_i F_i - A_i(x + ty)\| \leq l\,\|x_i + t_i y_i - x - ty\| < \varepsilon$$

for every $i \gg 0$ concludes the proof in Step 1.

Below we reduce the general case to the situation of Step 1. By Lemma 14 we need to consider two local over $X_i' \ni o$ or X_i''/o cases. Indeed, by the lemma every

$$t_i = \begin{cases} \operatorname{lct}(X_i' \ni o, B_i'; F_i') = \mathbb{R}\text{-clct}(X_i' \ni o, B_i'; F_i'), \text{ or} \\ \operatorname{act}(X_i''/o, B_i''; F_i'') = \mathbb{R}\text{-clct}(X_i''/o, B_i''; F_i''), \end{cases}$$

where X_i', X_i'' are $\mathbb{Q}$-factorial of dimension $\leq \dim X_i = d$, $\rho(X_i'') = 1$, $X_i' \ni o, X_i''$ are Ft and all points o are closed; $B_i', B_i'' \in \Gamma_b, F_i', F_i'' \in \Gamma_f$, and $F_i', F_i'' \neq 0$.

By dimensional induction and the converse we can suppose that every $\dim X_i' = X_i'' = d$.

Step 2. (*lct*) Suppose that there exists infinitely many lct cases with $t_i = \operatorname{lct}(X_i' \ni o, B_i'; F_i') = \mathbb{R}\text{-clct}(X_i' \ni o, B_i'; F_i')$. Taking a subsequence of (12.0.11) and changing notation we can suppose that every $t_i = \mathbb{R}\text{-clct}(X_i' \ni o, B_i'; F_i') = \operatorname{lct}(X_i' \ni o, B_i'; F_i')$, where X_i' is $\mathbb{Q}$-factorial. By [K, Theorem 18.22] $\operatorname{Supp} B_i, \operatorname{Supp} F_i$ have not more than $d/\mu_b, d/c\mu_f$ prime divisors respectively, where

$$\mu_b = \min\{\gamma \in \Gamma_b \mid \gamma > 0\}, \mu_f = \min\{\gamma \in \Gamma_f \mid \gamma > 0\} \text{ and } c = \min\{t_i\},$$

assuming that all $t_i > 0$ (see Warning below). Minima exist and are positive by the dcc of Γ_b, Γ_f and by the monotonic property of t_i (assuming that all $t_i > 0$). So, by Step 1 the required stabilization of t_i holds.

Warning: our estimations depend on c. Either we can consider a fix sequence of thresholds t_i or consider the (truncated) thresholds $\geq c > 0$. This is sufficient to prove the require acc.

Step 3. (*act*) Now we suppose that there exists infinitely many act cases with $t_i = \operatorname{act}(X_i'' \ni o, B_i''; F_i'') = \mathbb{R}\text{-clct}(X_i'' \ni o, B_i''; F_i'')$. As in Step 2, we can suppose that every $t_i = \mathbb{R}\text{-clct}(X_i'' \ni o, B_i''; F_i'') = \operatorname{act}(X_i''/o, B_i''; F_i'')$, where X_i'' is $\mathbb{Q}$-factorial Ft over o and $\rho(X_i) = 1$. By [BMSZ, Corollary 1.3] $\operatorname{Supp} B_i, \operatorname{Supp} F_i$ have not more than $(d+1)/\mu_b, (d+1)/c\mu_f$ prime divisors respectively, where μ_b, μ_f, c are same as in Step 2. Again by Step 1 the required stabilization of t_i holds.

Step 4. *Addenda.* Addendum 75 is immediate by the case with only the prime divisors E_i and the dcc property of $\{\sum \gamma_i \mid \gamma_i \in \Gamma\} \subset [0, +\infty)$ for every dcc $\Gamma \subset [0, +\infty)$.

Similarly we can treat bd-pairs.

□

Corollary 23 (Lc thresholds [HMX, Theorem 1.1])**.** *Let d be a nonnegative integer and Γ_b, Γ_f be two dcc sets of nonnegative real numbers. Then the set of lc thresholds*

$$\{\operatorname{lct}(X/X, B; F) \mid \dim X = d, B \in \Gamma_b \text{ and } F \in \Gamma_f\}$$

satisfies the acc.

Addendum 77. *Instead of $B \in \Gamma_b, F \in \Gamma_f$, we can suppose that $B = \sum b_i E_i, F_i = \sum f_i E_i$, where E_i are effective Weil $\mathbb{Z}$-divisors and $b_i \in \Gamma_b, f_i \in \Gamma_f$.*

Addendum 78. *The same holds for bd-pairs $(X/X, B + \mathcal{P})$ of index m and the set of corresponding thresholds depends also on m.*

Proof. Immediate by Theorem 21 for klt (X, B). Indeed, in this case X/X or locally X has wFt. By definition of the lc threshold we suppose that (X, B) is lc and F is > 0, $\mathbb{R}$-Cartier. Since every lc $(X/X, B + tF)$ is a 0-pair, $\operatorname{lct}(X/X, B; F) = \mathbb{R}\text{-}\operatorname{clct}(X/Z, B; F)$ by Example 1, (3).

If (X, B) is lc and F passes an lc center of (X, B) then $\operatorname{lct}(X/X, B; F) = \mathbb{R}\text{-}\operatorname{clct}(X/X, B; F) = 0$. Otherwise, (X, B) is lc and F does not pass the lc centers of (X, B) and we can replace (X, B) by its dlt resolution and F by its pull-back on the resolution. The construction preserves the lc threshold. The pull-back of F does not blow up divisors, is the birational transform of F with the same multiplicities as F. The new X is klt and we can apply Theorem 21 again.

Similarly we can treat bd-pairs.

□

Corollary 24 (Ac thresholds)**.** *Let d be a nonnegative integer and Γ_b, Γ_f be two dcc sets of nonnegative real numbers. Then the set of ac thresholds*

$$\{\mathrm{act}(X/Z, B; F) \mid \dim X = d, X/Z \text{ has wFt}, (X, B) \text{ is lc}, B \in \Gamma_b \text{ and}$$
$$F = \sum f_i F_i, \text{ where } F_i \text{ are locally free over } Z \text{ and } f_i \in \Gamma_f\}$$

satisfies the acc.

Addendum 79. *The same holds for bd-pairs $(X/Z, B + \mathcal{P})$ of index m and the set of corresponding thresholds depends also on m.*

Remark 10. In general, the *anticanonical* threshold of a log pair $(X/Z, D)$ with respect to a nef over Z divisor H on X is

$$\mathrm{act}(X/Z, D; H) = \inf\{t \in \mathbb{R} \mid K + D + tH \text{ is nef over } Z\}.$$

In [ISh, p. 47] it was used for Fano varieties as one of invariants in the Sarkisov program. In this situation Fano varieties X are $\mathbb{Q}$-factorial, have the Picard number 1, terminal singularities and H is an effective nonzero (Weil) divisor. Thus $\mathrm{act}(X, 0; H)$ is t of (12.0.12) in the proof below with $D = 0, F = H$. Moreover, such thresholds $\mathrm{act}(X, 0; H)$ form an acc set with a single accumulation point 0 if $\dim X = d$ is fixed (cf. Corollary 25). This follows from boundedness of those Fano varieties [B16, Theorem 1.1]. In general it is not true (cf. Remark 11, (1) below).

Example 14. Let S be a cone over a rational normal curve of degree n and L be its generator. Then

$$\mathrm{act}(S, 0; L) = n + 2.$$

In this situation L is not free for $n \geq 2$ and pairs (S, L) form an unbounded family. Notice also that $\mathbb{R}$-$\mathrm{clct}(S, 0; L) = 1$ and $\neq \mathrm{act}(S, 0; L)$ for $n \geq 2$. This is why we consider only act $= \mathbb{R}$-clct. But lct $= \mathbb{R}$-clct holds automatically (cf. Lemma 14).

Proof of Corollary 24. Recall that by definition of $t = \mathrm{act}(X/Z, D; F)$ we suppose that (X, D) is a log pair, $F \not\equiv 0$ generically over Z and

$$-(K + D) \equiv tF/Z. \qquad (12.0.12)$$

The proof uses Theorem 21.

Step 1. *It is enough to consider a local case* $\operatorname{act}(X/Z \ni o, B; F)$ *with a closed point* $o \in Z$. Indeed, for any closed point $o \in Z$,

$$\operatorname{act}(X/Z, B; F) = \operatorname{act}(X/Z \ni o, B; F)$$

because $F \not\equiv 0$ generically over Z. We omit locally the divisors F_i with $f_i = 0$ and suppose that every $f_i \neq 0$. Hence every $f_i \geq \mu_f > 0$, where

$$\mu_f = \min\{\gamma \in \Gamma_f \mid \gamma > 0\}.$$

We can consider truncated ac thresholds: $t \geq c$ for some positive real number c.

Step 2. *We can suppose that* t *and every* $f_i \leq 1$. Notice for this that t is bounded: $t \leq (d+1)/\mu_f$. Indeed, since $-(K+B) \equiv tF/Z \ni o$ and $t \geq c > 0$, X is covered by curves $C/Z \ni o$ with $-(C.K) \leq d+1$ and $-(C.K+B) \leq d+1$ for such a sufficiently general curve C/o. Additionally, we can assume that $-(C.K+B) > 0$. On the other hand, every $(C.F_i) \geq 0$ and by (12.0.12) some $(C.F_i) \geq 1$ because F_i are free over $Z \ni o$. Hence, for $(C.F_i) \geq 1$,

$$t\mu_f \leq tf_i \leq (C.tf_iF_i) \leq (C.tF) = -(C.K+B) \leq d+1.$$

This gives the required bound.

Similarly, we can verify that multiplicities f_i are bounded: every $f_i \leq (d+1)/c$.

We replace every f_iF_i by

$$D_i = \{f_i\}F_{i,0} + \sum_{j=1}^{\lfloor f_i \rfloor} F_{i,j},$$

where $\{f_i\}$ is the fractional part of f_i, belongs to $[0,1)$ and every $F_{i,j} \sim F_i$ over $Z \ni o$. By definition

$$t = \operatorname{act}(X/Z \ni o, B; F) = \operatorname{act}(X/Z \ni o, B; D),$$

where $D = \sum D_i$. We denote below D by F. Now F has additional multiplicities $1, \{f_i\}$ which satisfy the dcc because f_i are bounded. Thus we can add them to Γ_f and suppose that every $f_i \leq 1$.

If $t \geq 1$ we replace tF by $\{t\}F$ and B by $B' = B + \lfloor t \rfloor F$, where $\{t\}$ is the fractional part of t. The log pair (X, B') is lc with a boundary B'

by Bertini if every copy of ($\lfloor t \rfloor$ copies of) F_i in $\lfloor t \rfloor F$ is sufficiently general effective divisor in its linear system over $Z \ni o$. We need to add 1 to Γ_b if it is necessary. Here we loos the assumption that $t \geq c$. By construction

$$\{t\} = \operatorname{act}(X/Z \ni o, B'; F).$$

It is enough to verify the acc for $\{t\}$ because t is bounded. Below we denote B' by B and $\{t\}$ by t.

Step 3. *The acc for $t \leq 1$ holds.* Since every $f_i \leq 1$ and $t \leq 1$, $(X, B+tF)$ is lc again by Bertini if F_i in are sufficiently general in their linear system over $Z \ni o$. Hence by definition and since $F > 0$,

$$\operatorname{act}(X/Z \ni o, B; F) = \mathbb{R}\text{-}\operatorname{clct}(X/Z \ni o, B; F)$$

holds and the corollary holds by Theorem 21.

Similarly we can treat bd-pairs.

□

The next result (with $B = 0$) was conjectured by the author in late 80's in relation to the acc of mld's. It was suggested as a problem to V. Alexeev who was a graduate student of V. Iskovskikh in that time (cf. Remark 11, (1) below).

Corollary 25 (Acc for Fano indices)**.** *Let d be a nonnegative integer and Γ_b be a dcc set of nonnegative real numbers. Then the set of Fano indices*

$$\{0 \leq h \in \mathbb{R} \mid -(K+B) \equiv hH, \text{ where } \dim X = d, X \text{ has (w)Ft}, \\ (X, B) \text{ is lc}, B \in \Gamma_b \text{ and } H \text{ is a primitive ample divisor on } X\}$$

satisfies the acc.

Actually, X has Ft in the theorem, (X, B) in the definition of h is an *lc log Fano* variety, if $h > 0$, and h in this case is its *Fano index*.

Addendum 80. *Let Γ_h be a dcc set of nonnegative real numbers. Instead of $B \in \Gamma_b, H$, we can take $B = \sum b_i E_i, H = \sum h_i H_i \not\equiv 0$, where E_i are effective Weil $\mathbb{Z}$-divisors, H_i are nef Cartier divisors and $b_i \in \Gamma_b, h_i \in \Gamma_h$.*

Addendum 81. *The same holds for the lc Fano bd-pairs $(X, B + \mathcal{P})$ of index m with $-(K + B + \mathcal{P}_X) \equiv hH$ and the set of corresponding Fano indices depends also on m.*

Proof. Primitive (ample) means that if $H \equiv hH'$, where H' is also (ample) divisor, then $h \geq 1$.

There exists a positive integer N such that $F = NH$ is free for every (w)Ft X of dimension d [K93, Theorem 1.1] (cf. Corollary 2 above). Hence

$$h/N = \mathrm{act}(X, B; F).$$

Then Corollary 24 implies the acc for h.

In Addendum 80 we can use freeness of NH_i with N depending only on the dimension d by Corollary 2.

Similarly we can treat bd-pairs.

□

Remark 11. (1) For every positive real number ε, the Fano indices for ε-lc log Fano pairs (X, B) of the corollary with a finite set Γ_b form a finite set by BBAB [B16, Theorem 1.1], that is, there are only finitely many of those indices. This was conjectured by Alexeev for $B = 0$.

However, the union of this finite sets for all ε gives the acc set of the theorem. In general, the finiteness does not imply the acc.

(2) A Fano index is not always defined even for log Fano varieties (X, B). However, it is defined for such a pair if B is a $\mathbb{Q}$-boundary, e.g., $\Gamma_b \subset \mathbb{Q}$.

ä-Invariant. Let (X, B) be a log Fano variety. Then the ä-invariant of (X, B) is

$$\ddot{a} = \ddot{a}(X, B) = (1 - \mathrm{glct}(X, B))h,$$

where $\mathrm{glct}(X, B)$ is the global lc threshold or α-invariant of (X, B) and h is the Fano index of (X, B). Since $h > 0$, $\ddot{a} \geq 0$ if and only if $\mathrm{glct}(X, B) \leq 1$.

Notice that ä is not always defined but defined for $\mathbb{Q}$-boundaries B (cf. Remark 11, (2) and Addendum 83 below).

The same definition works for a bd-pair $(X/Z, D + \mathcal{P})$ with

$$\mathrm{glct}(X, B+\mathcal{P}) = \sup\{t \in \mathbb{R} \mid (X, B+E+\mathcal{P}) \text{ is lc for all } 0 \leq E \equiv -t(K+B+\mathcal{P}_X)\}$$

and $-(K + B + \mathcal{P}_X) = hH$, where H is a primitive ample divisor on X.

Below we discuss some properties of ä-invariant and of glct.

Corollary 26 (Acc for ä-invariant)**.** *Let d be a nonnegative integer and Γ_b be a dcc set of nonnegative real numbers. Then the set of ä-invariants*

$$\{0 \leq \ddot{a}(X, B) \mid (X, B) \textit{ is an lc log Fano variety, } \dim X = d, X \textit{ has Ft and } B \in \Gamma_b\}$$

satisfies the acc.

Addendum 82. *Let Γ_h be a dcc set of nonnegative real numbers. Instead of $B \in \Gamma_b$ and ample H in the definition of ä-invariant, we can take $B = \sum b_i E_i$ and $H = \sum h_i H_i$, where E_i are effective Weil $\mathbb{Z}$-divisors, H_i are nef Cartier divisors and $b_i \in \Gamma_b, h_i \in \Gamma_h$.*

Addendum 83. *If additionally $\Gamma_b, \Gamma_h \subset \mathbb{Q}$ then H always exists and $\ddot{a}(X,B), \mathrm{glct}(X,B) \in \mathbb{Q}$. Moreover, if Γ_b, Γ_h are closed $\mathbb{Q}$, then the sets of ä-invariants is also closed in $\mathbb{Q}$ limits.*

(Cf. other statements about limits of thresholds in Corollary 33 and its addenda.)

Addendum 84. *The same holds for the lc Fano bd-pairs $(X, B+\mathcal{P})$ of index m with $-(K+B+\mathcal{P}_X) \equiv hH$ and the set of corresponding ä-invariants ≥ 0 depends also on m.*

Proof. We can replace H by free $F = NH$, where N is a positive integer depending only on d. It is enough to verify that $t = \ddot{a}/N$ satisfies the acc. Since ä is bounded ($\leq d+1$), we can suppose that $t < 1$ for appropriate N.

The acc is enough to verify for $\ddot{a} > 0$, equivalently, $t > 0$. In this situation t is also a threshold:

$$t = \sup\{0 \leq r \in \mathbb{R} \mid K+B+rF+E \equiv 0 \text{ and } (X, B+E) \text{ is lc but nonklt pair for some effective } \mathbb{R}\text{-divisor } E \text{ on } X\}.$$

This threshold can be converted into an $\mathbb{R}$-clct one. In particular, it is attained, that is, the supremum can be replaced by the maximum (cf. Corollary 27 below). For sufficiently general effective divisor $M \sim F$, M does not pass the prime components of $\operatorname{Supp} B, \operatorname{Supp} E$, the lc centers of $(X, B+E)$ and $(X, B+rM+E)$ is lc. Take a prime b-divisor P of X such that $a(P; X, B+rM+E) = a(P; X, B+E) = 0$, log discrepancies at P. Let $(Y, (B+rM+E)_Y)$ be a crepant blowup of P. The blowup is an isomorphism $X = Y$ if P is not exceptional on X. For exceptional P, the crepant transform $(B+rM+E)_P$ of $B+rM+E$ on Y is equal to $B+rM+E+P$, where $B+rM+E$ denotes also its birational transform on Y. Then by construction in the exceptional case

$$r \leq \mathbb{R}\text{-clct}(Y, B+P; M) \leq t.$$

In $\mathbb{R}$-clct$(Y, B+P; M)$, the divisor M is the birational transform of M on Y and is free again. Note also that every Y in the construction has Ft [PSh08, Lemma-Definition 2.6, (iii)]. Replacing r by $\mathbb{R}$-clct$(Y, B + P; M)$, we can suppose that

$$r = \mathbb{R}\text{-clct}(Y, B + P; M).$$

Now we can use Theorem 19 with assumption (3) of Theorem 3 or Theorem 21 with Addendum 75 and get t in terms of an $\mathbb{R}$-complement threshold: for r sufficiently close to t,

$$t = r = \mathbb{R}\text{-clct}(Y, B + P; M).$$

Indeed, $M = 1M$ and

$$B + P, 1 \in \Gamma_b \cup \{1\}$$

and it is a dcc set. So, Theorem 21 with Addendum 75 again implies the acc for t and ä.

Similarly, if P is not exceptional then $Y = X$ and $(B + rM + E)_Y = B + rM + E$. By construction $\operatorname{mult}_P(B + rM + E) = \operatorname{mult}_P(B + E) = 1$. In this case

$$r \leq \mathbb{R}\text{-clct}(X, B + pP; M) \leq t,$$

where $p = \operatorname{mult}_P E$. (So, $\operatorname{mult}_P(B + pE) = 1, E' = E - pP \geq 0$ and $B + pP + E' = B + E$.) Replacing r by $\mathbb{R}$-clct$(X, B + pP; M)$ we can suppose that

$$r = \mathbb{R}\text{-clct}(X, B + pP; M).$$

Again we can use Theorem 19 or Theorem 21 with Addendum 75 and get t in terms of an $\mathbb{R}$-complement threshold: for r sufficiently close to t,

$$t = r = \mathbb{R}\text{-clct}(X, B + pP; M).$$

Indeed, $M = 1M$,

$$B + pP, 1 \in \Gamma_b \cup \{1\}$$

and it is a dcc set. As above Theorem 21 with Addendum 75 implies the acc for t and a.

We prove more: every ä > 0, glct < 1 invariants are attained. See explanations in Corollary 27 below. We also established that ä $\in \mathbb{Q}$ for $B \in \mathbb{Q}$ because ä $= \mathbb{R}$-clct$(Y, B+P; M)$ or $= \mathbb{R}$-clct$(X, B+pP; P)$ is rational by the Theorem 6 and $B + pP \in \mathbb{Q}$ too. This proves rationality in Addendum 83. The closed rational property follows from the similar result for $\mathbb{R}$-clct thresholds (see Corollary 33 below).

Similarly we can treat ä-invariants with H as in Addendum 82 (cf. the proof of Corollary 27) and bd-pairs.

$\square$

Corollary 27. *Let (X, B) be an lc log Fano variety with a boundary B. Then every threshold* $\mathrm{glct}(X, B) \leq 1$ *is attained, that is, there exists an effective $\mathbb{R}$-divisor E such that $(X, B+E)$ is lc but not klt and* $E \equiv -\mathrm{glct}(X, B)(K+B)$.

The same holds for $\mathrm{glct}(X, B+\mathcal{P}) \leq 1$ *of lc Fano bd-pairs $(X, B+\mathcal{P})$ of index m.*

Notice that $\ddot{a}(X, B), \ddot{a}(X, B+\mathcal{P}) \geq 0$ are also attained as it was established in the proof of Corollary 26.

Proof. We can suppose that (X, B) is a klt log Fano variety and has Ft. Otherwise, (X, B) is not klt and $\mathrm{glct}(X, B) = 0$ and $E = 0$. The cone of semiample divisors on X is rational polyhedral. In particular,

$$-(K+B) \equiv \sum_{i=1}^{l} r_i H_i,$$

where r_i are positive real numbers and H_i are very ample (Cartier) divisors. We can suppose also that every $r_i \leq 1$.

We start from the case $t = \mathrm{glct}(X, B) < 1$. Then by definition there exists a real number $r > 0$ and effective $\mathbb{R}$-divisor E such that $(X, B+E)$ is lc but not klt and

$$K + B + E + r\sum_{i=1}^{l} r_i H_i \equiv 0.$$

Those $r \leq a < 1$ and have a tendency to a, where $a = 1 - t > 0$ and

$$E \equiv (1-r)(-K-B) \equiv (1-r)\sum_{i=1}^{l} r_i H_i.$$

(Remark that this a is not the ä-invariant but it is its nonintegral and possibly irrational but more anti log canonical version.) For given r, E, take sufficiently general effective divisors F_i on X such that every $F_i \sim H_i$ does not pass the prime component of $\mathrm{Supp}\, B, \mathrm{Supp}\, E, F_j, j \neq i$, the lc centers of $(X, B+E)$ and $(X, B+E+\sum_{i=1}^{l} F_l)$ is lc. By construction $r < 1$ and $(X, B+E+r\sum_{i=1}^{l} r_i F_i)$ is an lc but not klt 0-pair. As in the proof of Corollary 26 take a prime b-divisor P such that ($P \neq F_1, \dots, F_l$ and) $a(P; X, B + E + r\sum_{i=1}^l r_i F_i) = 0$. If P is exceptional then

$$r \leq \mathbb{R}\text{-clct}(Y, B + P; \sum_{i=1}^l r_i F_i) \leq a$$

where $Y \to X$ is blowup of P and B, F_i are respectively birational transforms of B, F_i on Y. Actually, we can suppose that (for given Y and P but possibly different r, E)

$$r = \mathbb{R}\text{-clct}(Y, B + P; \sum_{i=1}^l r_i F_i).$$

We can take r arbitrary close to a. Hence by Theorem 19 with assumption (3) of Theorem 3 or Theorem 21 with Addendum 75 we get a in terms of an $\mathbb{R}$-complement threshold: for r sufficiently close to a,

$$a = r = \mathbb{R}\text{-clct}(Y, B + P; \sum_{i=1}^l r_i F_i).$$

Indeed, there exists a finite subset Γ in $[0,1]$ such that $B \in \Gamma$ and

$$B + P \text{ and every } r_i \in \Gamma \cup \{1, r_1, \dots, r_l\}$$

and it is a dcc set. So, t and a are attained because the $\mathbb{R}$-clct thresholds are attained on Ft X by the closed property in Theorem 6.

Similarly, if P is not exceptional then $Y = X$ and $\text{mult}_P(B + E + r\sum_{i=1}^l r_i F_i) = 1$, actually, $\text{mult}_P(B + E) = 1$. In this case

$$r \leq \mathbb{R}\text{-clct}(X, B + pP; \sum_{i=1}^l r_i F_i) \leq a,$$

where $p = \text{mult}_P E$. (So, $\text{mult}_P(B + pE) = 1, E' = E - pP \geq 0$ and $B + pP + E' = B + E$.) Replacing r by $\mathbb{R}\text{-clct}(X, B + pP; \sum_{i=1}^l r_i F_i)$ we can suppose that

$$r = \mathbb{R}\text{-clct}(X, B + pP; \sum_{i=1}^l r_i F_i).$$

Again we can use Theorem 19 with assumption (3) of Theorem 3 or Theorem 21 with Addendum 75 and get a in terms of an $\mathbb{R}$-complement threshold: for r sufficiently close to a,

$$a = r = \mathbb{R}\text{-clct}(X, B + pP; \sum_{i=1}^{l} r_i F_i).$$

Indeed,

$$B + pP \text{ and every } r_i \in \Gamma \cup \{1, r_1, \dots, r_l\}$$

and it is a dcc set. As above Theorem 21 with Addendum 75 implies the attainment of t and a.

The case with $t = \text{glct}(X, B) = 1$ and $a(X, B) = 0$ is more delicate because in this case $r < a = 0$ are negative. In this case, we use $r = 0$, effective $E' \equiv -(K + B)$ and a prime b-divisor P with the log discrepancy

$$a(P; X, B + E') = \varepsilon > 0$$

very close to 0. Such E' can be constructed by normalization of E in the definition: put

$$E' = \frac{1}{s} E,$$

where $(X, B + E)$ is lc but not klt and $E \equiv -s(K + B)$. By construction $s \geq 1$. By [B16, Theorem 1.1] if s goes to 1 then ε goes to 0. (In other words, if (X, B) is ε-lc then $\text{glct}(X, B) \geq \delta > 0$. This case is easier than general BBAB because here X is fixed! But B is not fixed!)

Then we apply to a sequence of ε_i, E_i with $\lim_{i\to\infty} \varepsilon_i = 0$ Theorem 21 with Addendum 75 or Theorem 19 with assumption (1) of Theorem 3 , that is, for ε sufficiently close to 0. For exceptional prime b-divisors P_i, P, in both cases we replace P_i, P by $(1 - \varepsilon_i)P, (1 - \varepsilon)P$ respectively. In the nonexceptional case we replace $B + p_i P_i, B + pP$ by $B + (p_i - \varepsilon_i)P, B + (p - \varepsilon)P$ respectively.

Similarly we can treat bd-pairs.

$\square$

Remark: In general $\text{glct}(X, B)$ behaves badly, e.g., does not satisfies the acc or dcc even if $\dim X = d$ is fixed and $B \in \Gamma_b$, a dcc set [Sh06, ??].

However, $\text{glct}(X, B)$ satisfies certain interesting properties. Some of them were conjectured by G. Tian.

Corollary 28 (glct gap)**.** *Let d be a nonnegative integer and Γ_b be a dcc set of nonnegative real numbers. There exists a positive real number g such that if (X,B) is an lc log Fano variety with $\dim X = d$, $B \in \Gamma_b$ and $\operatorname{glct}(X,B) > 1$ then $\operatorname{glct}(X,B) \geq 1 + g$.*

The same holds for $\operatorname{glct}(X, B + \mathcal{P}) > 1$ of lc Fano bd-pairs $(X, B + \mathcal{P})$ of index m with g also depending on m.

Proof. Similar to the proof of Corollary 27 in the case $t = \operatorname{glct}(X,B) = 1$. Actually, we need to prove that if $t \geq 1$ and sufficiently close to 1 then $t = 1$. However, in this situation we need BBAB of the full strength.

□

Corollary 29. *Let (X,B) be an lc log Fano variety with a rational boundary B. Then $\ddot{a}(X,B) \geq 0, \operatorname{glct}(X,B) \leq 1$ are also rational.*

The same holds for $\ddot{a}(X, B + \mathcal{P}) \geq 0, \operatorname{glct}(X, B + \mathcal{P}) \leq 1$ of lc Fano bd-pairs $(X, B + \mathcal{P})$ of index m.

Examples of nonrational $\ddot{a}(X,B) < 0, \operatorname{glct}(X,B) > 1$ with a rational boundary B are unknown.

Proof. Immediate by Addendum 83 and Addendum 84.

□

But we have a more effective statement.

Corollary 30 (Effective attainment)**.** *Let $\ddot{a} \geq 0$ be a rational number, d be a nonnegative integer and Γ_b be a rational closed dcc set in $[0,1]$. Then there exists a positive integer $n = n(d, \Gamma_b, \ddot{a})$ such that every $\ddot{a}(X,B) = \ddot{a}$, equivalently, $\operatorname{glct}(X,B)$, with $\dim X = d$, is attained by a divisor $E = D/n \equiv -\operatorname{glct}(X,B)(K+B)$, where*

$$D \in |-nK - nB - naH|$$

and H is a (primitive) ample divisor on X such that $-(K+B) \equiv hH$.

The same holds for $\ddot{a}(X, B + \mathcal{P}) = \ddot{a}, \operatorname{glct}(X, B + \mathcal{P})$ of lc Fano bd-pairs $(X, B + \mathcal{P})$ of index m with $E = D/n \equiv -\operatorname{glct}(X,B)(K + B + \mathcal{P}_X)$ and $-(K + B + \mathcal{P}) \equiv hH$, where n depends also on m and

$$D \in |-nK - n\mathcal{P}_X - nB - naH|\,.$$

.

Proof. By Corollary 27 a is attained. Actually, $t \leq 1$ is attained and $a = (1-t)h$. Thus there exists an effective $\mathbb{R}$-divisor E such that $(X, B+E)$ is lc but not klt and

$$K + B + E + aH \equiv 0.$$

There exists N depending only on d such that $NH \equiv F$, where F is a free divisor on X. In particular, we can suppose that $(X, B+E+(a/N)F)$ is lc too, that is, F does not pass the prime components of $\operatorname{Supp} B, \operatorname{Supp} E$ and the center of any prime b-divisor P with $a(P; X, B+E) = 0$. As in the proof of Corollary 27, in the exceptional case,

$$(Y, B + \frac{a}{N}F + P)$$

has an $\mathbb{R}$-complement. More precisely, in this case we suppose that every P is exceptional. The set of multiplicities $\Gamma_b \cup \{1, a/N\}$ is a rational closed dcc subset of $[0,1]$. (We can suppose that $a/N \leq 1$.) Hence there exists a monotonic n-complement (Y, B^+) of $(Y, B+(a/N)F+P)$. We suppose also that na/N is integer. Then by monotonicity the divisor (in the linear system of $\mathbb{Q}$-divisors)

$$D' = nB^+ - nB - \frac{na}{N}F - nP \in \left| -nK_Y - nB - \frac{na}{N}F - nP \right|$$

is effective. Taking the image of D' on X we get

$$D \in |-nK - nB - naH|$$

and $E = D/n$ is required effective.

Similarly, if there are nonexceptional P then $B+E = B'+E'+S$, where S is reduced part of $B+E$, that is, the sum of those P (assuming that (X,B) is klt) and $B' = B, E' = E \geq 0$ outside of $\operatorname{Supp} S$. In this situation, we have a monotonic n-complement (X, B^+) of $(X, B'+(a/N)F+S)$. Thus there exists an effective divisor

$$D' = nB^+ - nB' - \frac{na}{N}F - nS \in |-nK - nB' - naH - nS|.$$

By construction $B' = B$ outside of S and $B' = 0$ on S. Hence

$$nB' + nS \geq nB,$$

because B is a (sub)boundary and S is reduced. Hence again

$$D = D' + nB' + nS - nB = nB^+ - nB - \frac{na}{N}F \in |-nK - nB - naH|$$

is also effective and $E = D/n$ is required effective.

Similarly we can treat bd-pairs.

$\square$

Bounded affine span and index of divisor. Let $V \subseteq \mathbb{R}^n$ be a class of $\mathbb{R}$-linear spaces $\mathbb{R}^l$ with a standard basis and with their $\mathbb{Q}$-affine subspace V. Such a subspace V can be given by linear equations (possibly nonhomogeneous) with integral coefficients in the standard basis. We say that V in this class is *bounded* if, for all V of the class, the integer coefficients of equations are bounded. Equivalently, V is bounded if, in every V, there exists a finite set of rational generators $v_i, i = 1, \dots, l$, with coordinates in a finite set of rational numbers. Vectors v_i generate V if $\langle v_1, \dots, v_l \rangle = V$.

We apply the boundedness to affine $\mathbb{Q}$-spans $\langle D \rangle$ of certain $\mathbb{R}$-divisors D on X. Every such span is in the space $\operatorname{WDiv}_{\mathbb{R}} X$ of $\mathbb{R}$-divisors, actually, in the space of divisors supported on $\operatorname{Supp} D$. The standard basis of $\operatorname{WDiv}_{\mathbb{R}} X$ consists the prime components of $\operatorname{Supp} D$ or prime divisors of X.

Example 15. (1) If D is rational divisor then $\langle D \rangle$ is a rational divisor itself. Those spans or divisors are bounded if their multiplicities belong to a finite set of rational numbers. In general, $\langle D \rangle$ is bounded if there exists a finite set of $\mathbb{Q}$-divisors $D_i, i = 1, \dots, l$, in $\langle D \rangle$ with multiplicities in a finite set of rational numbers, which generate $\langle D \rangle$.

(2) The spaces $x_1 = x_2, \dots, x_{2l-1} = x_{2l}$ are bounded. Every of those spaces has generators $(1,1,0,\dots,0,0), (0,0,1,1,\dots,0,0), \dots, (0,0,0,0,\dots,1,1)$ with $2l$ coordinates 0 or 1.

Due to the rationality of intersection theory and since the lc property is rational, if $(X/Z \ni o, D)$ is a 0-pair then $(X/Z \ni o, D')$ is a possibly nonlc 0-pair for some $D' \in \langle D \rangle$ but a 0-pair in some neighborhood of D in $\langle D \rangle$. The last neighborhood depends on $(X/Z \ni o, D)$. Notice also that the maximal lc property also holds in some neighborhood of D in $\langle D \rangle$ if it holds for $(X/Z \ni o, D)$. Moreover, $a = a(P; X, D) = a(P; X, D')$ holds for all $D' \in \langle D \rangle$ if a is rational.

Let I be a positive integer. We say that I is a *lc index* of a 0-pair $(X/Z \ni o, D)$ if there are rational generators $D_i, i = 1, \dots, l$, of $\langle D \rangle$ such that every $I(K+D_i) \sim 0/Z \ni o$. Note that if $(X \ni o, D)$ is a log pair with $D \in \mathbb{Q}$ then $(X \ni o, D)$ is a 0-pair over $X \ni o$ and an lc index of $(X \ni o, D)$ is a Cartier index of $K+D$.

The same applies to 0-bd-pairs $(X/Z \ni o, B+\mathcal{P})$ (of index m) with $I(K+D_i+\mathcal{P}_X) \sim 0/Z \ni o$.

Corollary 31 (Boundedness of lc index)**.** *Let d be a nonnegative integer and Γ be a dcc subset in $[0,1]$. Then there exists a finite subset $\Gamma(d) \subseteq \Gamma$ and positive integer $I = I(d,\Gamma)$ such that, for every maximal lc 0-pair $(X/Z \ni o, B)$ with wFt $X/Z \ni o$, $\dim X = d$ and $B \in \Gamma$,*

(1) *$B \in \Gamma(d)$;*

(2) *$\langle B \rangle$ is bounded; and*

(3) *I is an lc index of $(X/Z \ni o, B)$.*

The same holds for maximal lc 0-bd-pairs $(X/Z \ni o, B+\mathcal{P})$ of index m with $\Gamma(d,m)$,$I(d,\Gamma,m)$ depending also on m.

In the proof below, we use n-complements for a finite set of rational boundary multiplicities. On the other hand, in construction of complements we use the corollary in very special case $\Gamma = \Phi(\mathfrak{R})$, a hyperstandard set, and (X,B) is a klt 0-pair with wFt (cf. Corollary 32 and What do we use in the proof? in Introduction).

Proof. We suppose that $d \geq 1$. (The case $d=0$ is trivial.)

(1) $\Gamma(d) = \Gamma \cap \mathbb{R}\text{-clct}(d,\Gamma)$, where $\mathbb{R}\text{-clct}(d,\Gamma)$ denotes the set of $\mathbb{R}\text{-clct}(X/Z \ni o, B)$ for $\dim X = d, B \in \Gamma$. By Theorem 21 $\mathbb{R}\text{-clct}(d,\Gamma)$ is an acc set. Thus $\Gamma(d)$ satisfies acc and dcc, and is finite.

By definition, for every prime divisor P on X over $Z \ni o$,

$$\mathbb{R}\text{-clct}(X/Z \ni o, B-bP; P) = b,$$

where $b = \operatorname{mult}_P B$. By construction $B-bP, b \in \Gamma$ and b belongs $\mathbb{R}\text{-clct}(d,\Gamma)$. Hence $b, B \in \Gamma(d)$.

(2) Put l to be the number of elements in $\Gamma(d)$. Then for every 0-pair $(X/Z \ni o, B)$ under the assumptions of the corollary there exist distinct reduced Weil divisors $D_1,\dots,D_l$ such that $B = \sum_{i=1}^l b_i D_i$, where $b_i \in \Gamma(d)$. (Some of multiplicities b_i are 0.) This gives a $\mathbb{Q}$-linear map $A\colon \mathbb{R}^l \to \operatorname{WDiv}_{\mathbb{R}} X$, which transforms the standard basis $e_1 = (1,0,\dots,0),\dots,e_l =$

$(0, 0, \ldots, 1)$ into the divisors $D_1, \ldots, D_l$ respectively. The hight of A is 1. There exists a unique vector $v \in \mathbb{R}^l$ with $A(v) = B$.

By Corollary 21 under the assumption (3) of Theorem 3, there exist vectors $w_0, w_1, \ldots, w_r \in \langle v \rangle, r = \dim \langle v \rangle \leq l$, which generate $\langle v \rangle$ and such that $B_0 = A(w_0), B_1 = A(w_1), \ldots, B_r = A(w_r)$ are rational boundaries.

Thus $\langle B \rangle = A(\langle v \rangle)$ is generated by $B_0, B_1, \ldots, B_r$ and is bounded.

(3) By construction all multiplicities of boundaries B_i belong to a finite set of rational numbers. On the other hand, every pair $(X/Z \ni o, B_i)$ is a 0-pair and has a monotonic n-complement $(X/Z \ni 0, B^+)$ for some n, depending only on the multiplicities of boundaries B_i and on d. We can take $I = n$. Indeed, $n(K + B_i) \sim 0$ over $Z \ni o$ because $B^+ = B_i$.

Similarly we can treat bd-pairs.

□

Corollary 32 (Invariants of adjunction)**.** *Let d be a nonnegative integer and Γ be a dcc set of rational numbers in $[0, 1]$. Then there exists a finite subset $\Gamma(d) \subseteq \Gamma$ and positive integer $I = I(d, \Gamma)$ such that every 0-contraction $f\colon (X, D) \to Z$ as in Theorem 9 has the adjunction index I. Moreover, $D^{\mathrm{h}} \in \Gamma(d)$ and $Ir_P \in \mathbb{Z}$ for every adjunction constant r_P as in 7.2.*

The same holds for every 0-contraction $(X, D + \mathcal{P}) \to Z$ as in Addendum 36. In this situation $I = I(d, \Gamma, m), \Gamma(d, m)$ depend also on the index m of the bd-pair $(X, D + \mathcal{P})$.

However, first we establish the following.

Proof of Theorem 9. (General case.) We suppose that $1 \in \Gamma$ and take $\Gamma(d)$ and $I = I(d, \Gamma)$ as in Corollary 31. We use the same proof as in the hyperstandard case with one improvement. According to Corollary 31 the maximal lc 0-pair $(X/Z, D)$ locally over $\operatorname{Supp} D_{\mathrm{div}}$, in Step 2 of the proof of Theorem 9, has index $I = I(d, \Gamma)$ and $D^{\mathrm{h}} = D \in \Gamma(d)$. Recall, that the vertical multiplicities of D are 0 or 1 locally over $\operatorname{Supp} D_{\mathrm{div}}$.

□

Proof of Corollary 32. Immediate by Theorem 9. □

Accumulations of $\mathbb{R}$-clct **thresholds.** Denote by $\mathbb{R}\text{-clct}(d, \Gamma_b, \Gamma_f)$ the thresholds of Theorem 21. Denote by $\operatorname{act}(d, \Gamma_b, \Gamma_f), \operatorname{lct}(d, \Gamma_b, \Gamma_f)$ corresponding ac and lc thresholds (see Lemma 14 and Corollaries 23, 24; cf. Remark 10 and Example 14). The thresholds satisfies the acc but not the dcc. However, the accumulations have rational constrains it terms of the closures $\overline{\Gamma_b}, \overline{\Gamma_f}$. Notice that both closures $\overline{\Gamma_b}, \overline{\Gamma_f}$ are also dcc (nonnegative) sets if so do Γ_b, Γ_f. Thus for simplicity we can suppose that Γ_b, Γ_f are already closed.

The same applies to bd-pairs of index m. In particular, we can consider $\mathbb{R}\text{-clct}(d, \Gamma_b, \Gamma_f, m), \operatorname{act}(d, \Gamma_b, \Gamma_f, m), \operatorname{lct}(d, \Gamma_b, \Gamma_f, m)$ and their accumulation points.

Corollary 33. *Let d be a nonnegative integer and Γ_b, Γ_f be two closed dcc sets of nonnegative real numbers. Then every accumulation threshold t in every set*

$$\mathbb{R}\text{-clct}(d, \Gamma_b, \Gamma_f), \operatorname{act}(d, \Gamma_b, \Gamma_f), \operatorname{lct}(d, \Gamma_b, \Gamma_f)$$

has a rational constrain between $B = (b_1, \dots, b_l), 0 < tF = t(f_1, \dots, f_l)$, where all $b_i \in \Gamma_b, f_i \in \Gamma_f$, that is, $B + \mathbb{R}F \not\subseteq \langle B + tF \rangle$.

Addendum 85 (cf. [Sh94, Corollary$^+$])**.** *If additionally $\Gamma_b, \Gamma_f \subset \mathbb{Q}$, then*

$$\overline{\mathbb{R}\text{-clct}(d, \Gamma_b, \Gamma_f)}, \overline{\operatorname{lct}(d, \Gamma_b, \Gamma_f)}, \overline{\operatorname{act}(d, \Gamma_b, \Gamma_f)} \subset \mathbb{Q}.$$

Addendum 86. *The same holds for thresholds*

$$\mathbb{R}\text{-clct}(d, \Gamma_b, \Gamma_f, m), \operatorname{act}(d, \Gamma_b, \Gamma_f, m), \operatorname{lct}(d, \Gamma_b, \Gamma_f, m)$$

of bd-pairs of index m.

Proof. By Lemma 14 and dimensional induction it is enough to prove the result for the accumulation points of ac and lc thresholds in the dimension d. We consider only ac thresholds. Lc thresholds can be treated similarly.

Consider now an accumulation point t for $\operatorname{act}(X_i, B_i; F_i) = \mathbb{R}\text{-clct}(X_i, B_i; F_i), i = 1, 2, \dots, i, \dots,$ where

(1) X is $\mathbb{Q}$-factorial, Ft of dimension d with $\rho(X) = 1$;

(2) $B_i \in \Gamma_b, F_i \in \Gamma_f$ and $F_i > 0$; and

(3) $(X_i, B_i + t_i F_i)$ is a 0-pair, in particular maximal lc, where $t_i = \operatorname{act}(X_i, B_i; F_i)$.

Put $t = \lim_{i \to \infty} t_i$. However, this time

$$0 < t < \dots < t_i < \dots < t_1$$

because $\operatorname{act}(d, \Gamma_b, \Gamma_f)$ satisfies the acc and t is an accumulation point. ($t = 0$ is rational and always an accumulation point in dimensions ≥ 1. Cf. with our assumption $0 \in \Phi$ in Hyperstandard sets is Section 3.)

By (1-2) and our assumptions, every $\operatorname{Supp} B_i \cup \operatorname{Supp} F_i$ have at most l prime divisors for some positive integer l. We will use the same notation as in Step 1 of the proof of Theorem 21. Since Γ_b, Γ_f are closed, $A(x) \in \Gamma_b, A(y) \in \Gamma_f$ hold. As in Step 1 ibid $(X_i, A(x+ty))$ has an $\mathbb{R}$-complement and $A(x+ty) = A(x) + tA(y)$ is a boundary for all $i \gg 0$.

By Corollary 21 there exists a neighborhood U of $x + ty$ in $\langle x + ty \rangle$ such that, for every $v \in U$, $(X_i, A(v))$ has an $\mathbb{R}$-complement. Thus if $x + \mathbb{R}y \subseteq \langle x + ty \rangle$ then there exists a real number $\varepsilon > 0$ such that $(X_i, A(x + (t + \varepsilon y))$ has an $\mathbb{R}$-complement, or equivalently, $\mathbb{R}\text{-clct}(X_i, A(x); A(y)) \geq t + \varepsilon$ for all $i \gg 0$. On the other hand, by construction every $x_i \leq x, B_i = A(x_i) \leq A(x)$ and $y_i \leq y, F_i = A(y_i) \leq A(y) > 0$; by (1) $\rho(X) = 1$. Hence $\mathbb{R}\text{-clct}(X_i, A(x); B(x)) \leq t_i$: for every curve C on X and $t' > t_i$,

$$(C.A(x) + t'A(y)) > (C.A(x) + t_iA(y)) \geq (C.B_i + t_iF_i) = 0$$

by (3). So, $t_i \geq t + \varepsilon$, a contradiction. This proves that $B + \mathbb{R}F \not\subseteq \langle B + tF \rangle$, where $B = x \in \Gamma_b, F = y \in \Gamma_f$.

For Addendum 85 notice that if $B, 0 \neq F \in \mathbb{Q}$ and $B + \mathbb{R}F \not\subseteq \langle B + tF \rangle$ then t is rational: $(B + \mathbb{R}F) \cap \langle B + tF \rangle = \{B + tF\}$.

Similarly we can treat bd-pairs.

□

13 Open problems

It is expected that in most of our results and, in particular, in most of our applications we can omit the assumption to have wFt but bd-pairs should be of Alexeev type.

Example 16. Let E be a complete nonsingular curve of genus 1 and $P, Q \in E$ be two closed points on E. Then $\mathcal{P} = p - q$ is a numerically trivial Cartier (b-)divisor, in particular, nef. Thus $(E, \mathcal{P})$ is a bd-pair of index 1 in the Birkar-Zhang sense but in the Alexeev sense only if $p = q$ and $\mathcal{P} = 0$. The pair $(E, \mathcal{P})$ has an $\mathbb{R}$-complement if and only if $\mathcal{P} \sim_{\mathbb{Q}} 0$, that is, a torsion. Since the torsions are not bounded, n-complements of those pairs are not bounded too.

Notice also that $(E, \mathcal{P})$ is a maximal lc pair. But the lc indices of those pairs are also not bounded.

Thus general bd-pairs are a very good instrument for wFt morphisms but for general morphisms it is expected that a generalization of our results works only for Alexeev bd-pairs $(X/Z, D+\mathcal{P})$ of index m. Such a pair or, for short, *Alexeev pair of index m* is an Alexeev pair $(X/Z, D+\mathcal{P})$ with $\mathcal{P} = \sum r_i L_i$ and $0 \le mr_i \in \mathbb{Z}$ (cf. Conjecture 3 and Corollary 34). Notice also that we collect all mobile linear systems in $\mathcal{P}$ and with one fixed prime divisor in D.

Conjecture 1 (Existence and boundedness of n-complements)**.** It is expected that Theorems 1, 2, 3, 17 and 18 hold without the assumption that $X/Z \ni o$ has wFt. Additionally we can relax the connectedness assumption on X_o and suppose instead that the number of connected components of X_o is bounded (cf. Addendum 1).

The same expected for Alexeev pairs of index m (cf. Conjecture 3 below).

The conjecture holds in dimension $d \le 2$. Methods of the paper allows to prove Theorem 3 in dimension $d \le 2$ with the existence of an $\mathbb{R}$-complement instead of a klt $\mathbb{R}$-complement in (1) of the theorem (cf. Examples 11, (1-3) and [Sh95, Inductive Theorem 2.3]). In this situation assumptions (2-3) of Theorem 3 and Theorems 17 and 18 are redundant. See n-complements for $d=1$, essentially, for $\mathbb{P}^1$ in Examples 3, 13.

Most of our applications follows directly from results about n-complements. The following result also follows from the boundedness of n-complements.

Corollary-Conjecture 1 (Boundedness of lc index)**.** *Let d be a nonnegative integer and $\Gamma \subset [0,1]$ be a dcc subset. Then there exists a finite subset $\Gamma(d) \subseteq \Gamma$ and positive integer $I = I(d,\Gamma)$, depending only on d and Γ, such that, for every maximal lc 0-pair $(X/Z \ni o, B)$ with $\dim X = d$ and $B \in \Gamma$,*

(1) *$B \in \Gamma(d)$;*

(2) *$\langle B \rangle$ is bounded; and*

(3) *I is an lc index of $(X/Z \ni o, B)$.*

The same is expected for maximal lc Alexeev 0-pairs $(X/Z \ni o, B+\mathcal{P})$ of index m with $\Gamma(d,m)$,$I(d,\Gamma,m)$ depending also on m.

Proof-derivation. Indeed, we can use the proof of Corollary 31. However, before we need to derive nonwFt versions of Theorem 21 and of Corollary 21. □

In particular, the last conjecture includes the global case with a rational dcc subset $\Gamma \subset [0,1]$: every 0-pair (X,B) with $B \in \Gamma$ of a *bounded* dimension has a *bounded* (global) lc index I, that is,

$$I(K+B) \sim 0.$$

Moreover, for a proof of Conjecture 1, two extreme cases are important: Ft pairs and global 0-pairs with $B=0$ and canonical singularities, known as Calabi-Yau varieties. So, the following very famous conjecture is indispensable for the theory of complements and possibly related to topology (cf. [FM, Theorema 6.1]).

Conjecture 2 (Index conjecture)**.** Let d be a nonnegative integer. Then there exists a positive integer $I = I(d)$ such that, every complete variety or space X with $\dim X = d$, only canonical singularities and $K \equiv 0$ has index I:

$$IK \sim 0.$$

E.g, if we assume additionally that X has only canonical singularities, then $I(1) = 1, I(2) = 12$ (classical) and

$$I(3) = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19,$$

the Bauville number. The next indices $I(d), d \geq 4$, and their existence is unknown. We know that, for every variety or space in Index conjecture, $K \sim_{\mathbb{Q}} 0$, that is, there exists an index [A05, Theorem 0.1,(1)].

A reduction to Ft varieties and varieties of Index conjecture uses the LMMP and the following semiampleness.

Conjecture 3 (Effective b-semiampleness; cf. [PSh08, (7.13.3)])**.** Let d be a nonnegative integer and $\Gamma \subset [0,1]$ be a rational dcc subset. Then there exists a positive integer $I = I(d,\Gamma)$ such that, for every 0-contraction $(X,D) \to Z$ under assumptions 7.1 and with $\dim X = d, D^{\mathrm{h}} \in \Gamma$, $I\mathcal{D}_{\mathrm{mod}}$ is b-free, that is, $I\mathcal{D}_{\mathrm{mod}} = \overline{M}$, where M is a base point free divisor on some model Z' of Z. In other words, $(Z, D_{\mathrm{div}} + \mathcal{D}_{\mathrm{mod}})$ is an Alexeev log pair (not necessarily complete) of index I.

The same is expected for 0-contractions $(X, B+\mathcal{P}) \to Z$ of Alexeev pairs $(X, B+\mathcal{P})$ of index m with $I(d,\Gamma,m)$ depending also on m.

Corollary-Conjecture 1 and construction of n-complements for fibrations can be reduced to the global case as for Ft fibrations. In its turn, if a global 0-pair (X, B) has a nonzero boundary then the lc index conjecture can be reduced to a Ft variety X or to a nontrivial Ft fibration $(X, B) \to Z$. In the latter case we can use dimensional induction for the base Z. If $\dim Z \geq 1$, we get a bd-pair $(Z, B_{\mathrm{div}} + \mathcal{B}_{\mathrm{mod}})$ with possibly nonFt Z. As we already noticed the induction will not work for bd-pairs of Birkar-Zhang pairs but is expected to work for Alexeev pairs. As for Ft varieties in the most critical for us situation B has hyperstandard multiplicities. Then according to Conjecture 3 $(X, B_{\mathrm{div}} + \mathcal{B}_{\mathrm{mod}})$ is actually an Alexeev pair of bounded index.

It is well-known [PSh08, Corollary 7.18] that the Alexeev log pair $(X, B_{\mathrm{div}} + \mathcal{B}_{\mathrm{mod}})$ can be easily converted into a log pair $(X, B_{\mathrm{div}} + B_{\mathrm{mod}})$ for a suitable boundary B_{mod} with a finite set of rational multiplicities (depending only on the index of the Alexeev pair).

Corollary 34. *Let $(X, D + \mathcal{P})$ be an Alexeev (log, lc, klt) pair of index m. Then there exists an effective divisor P such that $(X, D + P)$ is also a (respectively, log, lc klt) pair, $mP \in m\mathcal{P}_X$ and* $\operatorname{Supp} D, \operatorname{Supp} P$ *are disjoint. In particular, $P \in [0, 1] \cap (\mathbb{Z}/m)$.*

Addendum 87. $\mathbb{D}(X, D + P) \geq \mathbb{D}_{\mathrm{div}} = \mathbb{D}(X, D + \mathcal{P}_X) - \mathcal{P}$.

Addendum 88. *Let Γ be a set of real numbers. If additionally $D \in \Gamma$ then*

$$B + P \in \Gamma \cup ([0, 1] \cap \frac{\mathbb{Z}}{m}).$$

Addendum 89. *If additionally Γ satisfies the dcc (in $[0, 1]$, in a hyperstandard set) then*

$$\Gamma \cup ([0, 1] \cap \frac{\mathbb{Z}}{m}).$$

satisfies the dcc (respectively, in $[0, 1]$, in a hyperstandard set).

Proof. (Cf. [PSh08, Corollay 7.18].) Take a crepant model $f\colon (Y, D_Y + \mathcal{P}) \to (X, D + \mathcal{P})$ such that $\mathcal{P}$ is *free over* Y, that is, $\mathcal{P} = \sum r_i L_i$ and every L_i is a base point free linear system on Y. Since $(X, D + \mathcal{P})$ is an Alexeev pair of index$m$, $mr_i = m_i$ is a nonnegative integer. By the Bertini theorem we can pick up m_i rather general effective divisors $E_{i,j}, j = 1, \ldots, m_i$ in every linear system L_i such that

$$P = \sum f(E_{i,j})/m$$

satisfies the required properties. The inclusion $mP \in m\mathcal{P}_X$ means that every summand $f(E_{i,j})$ of mP belongs to the corresponding linear system L_i on X. The klt property of $(X, D+P)$ follows from that of $(X, D+\mathcal{P})$ if $m \geq 2$. Otherwise we replace $m = 1$ in our construction by any positive integer ≥ 2.

Notice that log discrepancies of $(X, D+P)$ are strictly larger than that of $(X, D+\mathcal{P})$ only over $\mathrm{Supp} \cup E_{i,j}$. The other log discrepancies are the same. This proves Addendum 87.

Two other addenda are immediate by definition.

□

The most important case of Index conjecture is the case with a variety X having only canonical and even terminal singularities. Otherwise, we can blow up a prime b-divisor P of X with a log discrepancy $a = a(P; X, 0) < 1$. A result is a crepant 0-pair $(Y, (1-a)P)$ with $0 < 1-a \leq 1$. Every such case can be reduced as above to Ft varieties or varieties of Index conjecture of smaller dimension. However, this works for all lc or klt 0-pairs X of given dimension d if a belongs to an acc set. This follows from the acc of mld's [Sh88, Problem 5] for mld's $a < 1$ because then $1-a$ form a dcc set in the dimension d. Actually, we can use much weaker and already known results about n-complements when a is sufficiently close to 0. In this situation $a = 0, 1-a = 1$ and Index conjecture again can be done by dimensional induction. The same works for a which are not close to 1. But $a < 1$ are actually not close to 1 by the following very special case of the acc for mld's conjecture. If singularities are canonical we can blow up all P with $a(P; X, 0) = 1$; in this case $1-a = 0$.

Conjecture 4 (Gap conjecture)**.** Let d be a nonnegative integer. Then there exists a positive integer $a < 1$ such that if $X \ni P$ with $\dim X = d$ is a-klt, that is, the mld of X at P is $> a$, then $X \ni p$ is canonical, that is the mld of X at P is ≥ 1.

Index and Gap conjectures are established up to dimension $d = 3$ [J, Theorem 1.3 and Corollary 1.7] [LX, Theorem 1.4].

Finally, we state the following ε-lc strengthening of n-complements. However, we use a instead of ε.

Conjecture 5 (a-n-complements; cf. [Sh04b, Conjecture])**.** Let a be a nonnegative real number. Assume additionally in Theorems 1, 2, 3, 17 and 18 that $(X/Z \ni o, B)$ has an a-lc over o $\mathbb{R}$-complement then in all of these theorems it expected the existence an a-n-complement $(X/Z, B^+)$ with $n \in \mathcal{N}$, that is, additionally (X, B^+) is a-lc over o. The set of complementary indices $\mathcal{N}$ is this case depends also on a. However, Restrictions on complementary indices are expected to hold only for I but not for approximations if a is not rational.

The same expected for Alexeev pairs of index m.

To remove the wFt assumption in the conjecture we need the following concept.

Definition 10. Let $(X/Z, D)$ be a log pair and δ be a nonnegative real number. The pair is *strictly* δ-lc if for every b-0-contraction

$$\begin{array}{ccc} (X, D) & \stackrel{\varphi}{\dashrightarrow} & (Y, D_Y) \\ & & f \downarrow \\ & & Z \end{array}$$

such that

(1) φ is a crepant birational 1-contraction;

(2) f is a 0-contraction as in 7.1,

$(Z, \mathbb{D}_{Y,\mathrm{div}})$ is δ-lc, that is, every multiplicity of the b-$\mathbb{R}$-divisor $\mathbb{D}_{Y,\mathrm{div}}$ is $\leq 1-\delta$, or, equivalently, the bd-pair $(Z, D_{Y,\mathrm{div}} + \mathcal{D}_{Y,\mathrm{mod}})$ is δ-lc.

Notice that the strict δ-lc property implies the usual δ-lc property of (X, D). The converse holds for $\delta = 0$ by (6) in 7.5. The converse also is expected for wFt X/Z but with a different $\delta' \leq \delta$ but not in general, e.g., for fibrations of genus 1 curves. Of course, the strict δ-lc property means exactly the boundedness of adjunction constants (multiplicities) $l = l_P$ in 7.2 for every vertical prime b-divisors P of Y with $r = r_P = 1$ (cf. [GH, Theorem 1.6, (3)]).

Addendum 90. *In Conjecture 5 we can replace the wFt assumption by the* strict *a-lc property over o: the exists $\delta \geq 0$ such that*

$\delta > 0$ if $a > 0$; and

there exists a strict δ-lc over o $\mathbb{R}$-complement of $(X/Z \ni o, B)$.

But still we keep the assumption that $(X/Z \ni o, B)$ has an a-lc over o $\mathbb{R}$-complement.

The boundedness of lc indices expected for indices of maximal a-lc 0-pairs under additionally the strict δ-lc property over o.

We already established the conjecture for $a = 0$. So, corresponding n-complements are better to call lc n-complements.

Notice that the a-lc property over a point o means that $a(P; X, B) \geq a$ for every b-divisor of X over o. In particular, if $(X/o, B)$ is global and $a > 0$ then by BBAB projective Ft varieties X are bounded in any fixed dimension d and the conjecture can be easily established by approximations as for lc n-complements in the exceptional case of Section 5 if positive multiplicities of B are bounded from below. However, the case with small positive multiplicities of B is already difficult. This case is interesting because then a can be very close to 1 but not above 1 in the global case. The nonglobal case is much more difficult and related to other nontrivial conjectures [BC, Conjectures 1.1 and 1.2].

Notice also that if a is not rational then Restrictions on complementary indices related to approximations can collide with a because for a-n-complements, the a-lc property implies m/n-lc property with upper approximation m/n of a.

Conjecture 5 does not hold for nonFt morphisms X/Z. For instance, minimal nonsingular fibrations of genus 1 curves over a curve have fibers with unbounded multiplicities. Thus lc indices of a canonical divisor near those fibers are unbounded too. This contradicts to related Index conjecture for 1-lc maximal 0-pairs (for $a = 1$). However this does not give a contradiction if additionally the pair is strictly δ-lc (see Addendum 90).

Possibly, it is better to start from Addendum 90 and [BC, Conjecture 1.2] because they imply Conjecture 5.

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