Introduction to Holomorphic Functions of Several Variables

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Homological Theory

Robert C. Gunning
Princeton University

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Elementary Properties of Sheaves

Sheaves were introduced into complex analysis in the work of H. Cartan and J.-P. Serre in the early 1950s, and they have played an important role since then. In part, through the notion of coherence, they provide a convenient and relatively simple machine for handling some of the rather complicated local properties of holomorphic functions and holomorphic varieties; these properties were discovered and their significance noted in the fundamental work of H. Oka. In part, through the cohomology theory of sheaves, they provide a useful systematic procedure for passing from local results to global results. The basic properties of sheaves and sheaf cohomology will be presented here fairly much from the beginning, not presupposing any previous knowledge of the subject. Those readers already familiar with some or all of these general properties can skip this section.

1. **DEFINITION.** *A* **sheaf** *of abelian groups over a topological space M is a topological space $\mathscr{S}$ with a mapping $\pi\colon \mathscr{S} \to M$ such that:*

(i) *π is a surjective local homeomorphism.*
(ii) *$\pi^{-1}(Z)$ has the structure of an abelian group for each point $Z \in M$.*
(iii) *The group operations are continuous in the topology of $\mathscr{S}$.*

Sheaves of other algebraic structures, such as complex vector spaces, rings, modules over a fixed ring, and so on, can be introduced correspondingly by making the obvious modifications of the preceding definition. To simplify the exposition the discussion here will for the most part be limited to sheaves of abelian groups; the extensions to other cases generally present no problems unless otherwise noted. The mapping π in the preceding definition is called the **projection**. The subset $\pi^{-1}(Z) \subseteq \mathscr{S}$ is called the **stalk** of the sheaf $\mathscr{S}$ over the point $Z \in M$ and is also written $\mathscr{S}_Z$. It is clear that for any subset $X \subseteq M$ the inverse image $\pi^{-1}(X) \subseteq \mathscr{S}$ can be viewed as a sheaf of abelian groups over X with the projection π; this is called the **restriction** of the sheaf $\mathscr{S}$ to the subset X and is denoted by $\mathscr{S}|X$. Another useful auxiliary notion is the following.

2. **DEFINITION.** *If $\mathscr{S}$ is a sheaf of abelian groups over M with projection π, then a* **section** *of the sheaf $\mathscr{S}$ over a subset $X \subseteq M$ is a continuous mapping $f\colon X \to \mathscr{S}$ such that the*

composition $\pi \circ f : X \to X$ is the identity mapping. The set of all sections over X will be denoted by $\Gamma(X, \mathscr{S})$.

Let $\mathscr{S}$ be a sheaf of abelian groups over M with projection π, and consider a point $s \in \mathscr{S}$ lying over a point $Z = \pi(s) \in M$. From property (i) of Definition 1 it follows that there is an open neighborhood U_s of the point s in $\mathscr{S}$ such that the restriction of π to the set U_s is a homeomorphism between U_s and the open neighborhood $U = \pi(U_s)$ of the point Z in M. The inverse of this restriction is then clearly a section of the sheaf $\mathscr{S}$ over U. It is thus evident that for any point of $\mathscr{S}$ there is a section of $\mathscr{S}$ over some open subset of M passing through that point, and that indeed the images $f(U)$ of the sections $f \in \Gamma(U, \mathscr{S})$ over the open subsets $U \subseteq M$ are a basis for the topology of $\mathscr{S}$. Moreover if $f \in \Gamma(U, \mathscr{S})$ is any section over U passing through $s \in \mathscr{S}$, then $f^{-1}(U_s)$ is an open neighborhood of Z, so that f coincides with the inverse of the projection mapping locally. Consequently any two sections $f, g \in \Gamma(U, \mathscr{S})$ that agree at a point $Z \in U$ must also agree in a full open neighborhood of Z.

The meaning of condition (iii) of Definition 1 may not be altogether clear, so a few words of explanation should be inserted here. Note that the group operations in the sheaf $\mathscr{S}$ are defined only on the individual stalks of $\mathscr{S}$. The Cartesian product $\mathscr{S} \times \mathscr{S}$ can be given the natural product topology, and the subset

$$\mathscr{S} \times_\pi \mathscr{S} = \{(s, t) \in \mathscr{S} \times \mathscr{S} : \pi(s) = \pi(t)\} \subseteq \mathscr{S} \times \mathscr{S} \tag{1}$$

which is called the fibred product over π, inherits a topology as a subset of the topological space $\mathscr{S} \times \mathscr{S}$. Now the group structures on the stalks of $\mathscr{S}$ can be used to introduce the mapping $\sigma: \mathscr{S} \times_\pi \mathscr{S} \to \mathscr{S}$ that sends a point $(s, t) \in \mathscr{S} \times_\pi \mathscr{S}$ to the point $\sigma(s, t) = s - t \in \mathscr{S}_Z$ where $Z = \pi(s) = \pi(t)$, and condition (iii) means precisely that this is a continuous mapping. Note that if $f, g \in \Gamma(X, \mathscr{S})$ for some subset $X \subseteq M$, then it is possible to define a mapping $f - g : X \to \mathscr{S}$ by setting $(f - g)(Z) = f(Z) - g(Z) \in \mathscr{S}_Z$. But this is just the composition of the continuous mapping $f \times g : X \to \mathscr{S} \times_\pi \mathscr{S}$ that sends a point $Z \in X$ to the point $(f(Z), g(Z)) \in \mathscr{S} \times_\pi \mathscr{S} \subseteq \mathscr{S} \times \mathscr{S}$ and the continuous mapping $\sigma: \mathscr{S} \times_\pi \mathscr{S} \to \mathscr{S}$ and so is also continuous, and hence $f - g \in \Gamma(X, \mathscr{S})$. That means that the set $\Gamma(X, \mathscr{S})$ has the natural structure of an abelian group as defined by pointwise group operations on the sections. In particular, if $f \in \Gamma(U_A, \mathscr{S})$ is any section over an open neighborhood U_A of a point $A \in M$, then $f - f \in \Gamma(U_A, \mathscr{S})$. Thus the mapping that sends any point $Z \in M$ to the zero element in the group $\mathscr{S}_Z$ is locally and hence globally a section of $\mathscr{S}$ over M. The restriction of this zero section to any subset $X \subseteq M$ is the zero element of the group $\Gamma(X, \mathscr{S})$.

To see at least one example, albeit a trivial one, consider a topological space M and an abelian group G with the discrete topology. Let $\mathscr{S} = M \times G$ have the natural product topology, and let $\pi: \mathscr{S} \to M$ be the natural projection. It is evident that the space $\mathscr{S}$ with the mapping π is a sheaf of abelian groups over M; this is called the **trivial sheaf** over M with stalk G and is often denoted merely by G. The sections of this sheaf over any connected subset of M can naturally be identified with the elements of the group G. In particular, when G is the zero group the resulting trivial sheaf, which might be considered as the trivial trivial sheaf, is usually called the **zero sheaf** over M and is denoted merely by 0.

3. DEFINITION. *A* **subsheaf** *of a sheaf $\mathscr{S}$ of abelian groups over M is an open subset $\mathscr{R} \subseteq \mathscr{S}$ such that $\mathscr{R}_Z$ is a subgroup of $\mathscr{S}_Z$ for each point $Z \in M$.*

It is evident that a subsheaf $\mathscr{R}$ of a sheaf $\mathscr{S}$ of abelian groups is itself a sheaf of abelian groups over M; the condition that $\mathscr{R}$ be an open subset is of course essential for this to be true. For an example, if $\mathscr{S}$ is any sheaf of abelian groups over M, then the image of the zero section $0 \in \Gamma(M, \mathscr{S})$ is a subsheaf of $\mathscr{S}$, the zero subsheaf of $\mathscr{S}$. For another example, if $\mathscr{S}$ is any sheaf of abelian groups over M and U is an open subset of M, then $\mathscr{R} = \bigcup_{Z \in M} \mathscr{R}_Z$ is a subsheaf of $\mathscr{S}$, where

$$\mathscr{R}_Z = \begin{cases} \mathscr{S}_Z & \text{if } Z \in U \\ 0 & \text{if } Z \notin U \end{cases} \tag{2}$$

Here too it is essential that U be an open subset of M. This sheaf $\mathscr{R}$ coincides with $\mathscr{S}$ on the open subset $U \subseteq M$ and is the zero sheaf on the closed complement of U. If $\mathscr{R}$ is a subsheaf of a sheaf $\mathscr{S}$ of abelian groups and $\mathscr{T}_Z = \mathscr{S}_Z/\mathscr{R}_Z$ is the quotient group of the stalks over $Z \in M$, then introduce the set $\mathscr{T} = \bigcup_{Z \in M} \mathscr{T}_Z$. It is a simple matter to verify that, with the mapping $\pi: \mathscr{T} \to M$ defined by $\pi(\mathscr{T}_Z) = Z$ and with the natural quotient topology derived from $\mathscr{S}$, the set $\mathscr{T}$ is also a sheaf of abelian groups over M. There is a natural mapping $\mathscr{S} \to \mathscr{T}$, and a subset of $\mathscr{T}$ is open in the natural quotient topology precisely when its inverse image is an open subset of $\mathscr{S}$. The sheaf $\mathscr{T}$ is called the **quotient sheaf** of $\mathscr{S}$ by $\mathscr{R}$ and is written $\mathscr{T} = \mathscr{S}/\mathscr{R}$. As an example, if $\mathscr{R}$ is the zero subsheaf of $\mathscr{S}$, then $\mathscr{S}/\mathscr{R} = \mathscr{S}$. As another example, if $\mathscr{R} \subseteq \mathscr{S}$ is the subsheaf (2), then the quotient sheaf $\mathscr{T} = \mathscr{S}/\mathscr{R}$ has stalks

$$\mathscr{T}_Z = \begin{cases} 0 & \text{if } Z \in U \\ \mathscr{S}_Z & \text{if } Z \notin U \end{cases} \tag{3}$$

This sheaf coincides with $\mathscr{S}$ on the closed subset $M - U \subseteq M$ and is the zero sheaf on the open subset U.

4. DEFINITION. *If $\mathscr{R}$ and $\mathscr{S}$ are sheaves of abelian groups over M, then a* **sheaf homomorphism** *$\phi: \mathscr{R} \to \mathscr{S}$ is a continuous mapping such that $\phi(\mathscr{R}_Z) \subseteq \mathscr{S}_Z$ for each point $Z \in M$ and the restriction $\phi|\mathscr{R}_Z: \mathscr{R}_Z \to \mathscr{S}_Z$ is a group homomorphism. A* **sheaf isomorphism** *is a sheaf homomorphism with an inverse that is also a sheaf homomorphism.*

For example, if $\mathscr{R}$ is a subsheaf of $\mathscr{S}$, then the inclusion $\mathscr{R} \to \mathscr{S}$ and the natural mapping $\mathscr{S} \to \mathscr{S}/\mathscr{R}$ are both sheaf homomorphisms. Note that for any sheaf homomorphism $\phi: \mathscr{R} \to \mathscr{S}$ the composition $\phi \circ f$ of ϕ with a section $f \in \Gamma(X, \mathscr{R})$ is a section $\phi \circ f \in \Gamma(X, \mathscr{S})$; thus, ϕ induces a homomorphism $\phi^*: \Gamma(X, \mathscr{R}) \to \Gamma(X, \mathscr{S})$ between the groups of sections over any subset $X \subseteq M$. It follows quite easily from this that *a sheaf homomorphism $\phi: \mathscr{R} \to \mathscr{S}$ is always an open mapping.* The **kernel** of a sheaf homomorphism $\phi: \mathscr{R} \to \mathscr{S}$ is the subset $\phi^{-1}(0) \subseteq \mathscr{R}$, where 0 is the zero subsheaf of $\mathcal{S}$; it is clear that the kernel of ϕ is a subsheaf of $\mathcal{R}$. The **image** of a sheaf homomorphism $\phi\colon \mathcal{R} \to \mathcal{S}$ is the subset $\phi(\mathcal{R}) \subseteq \mathcal{S}$; since ϕ is an open mapping, it is also clear that the image of ϕ is a subsheaf of $\mathcal{S}$. A sheaf homomorphism $\phi\colon \mathcal{R} \to \mathcal{S}$ is an isomorphism if it is **injective**—that is, has as kernel the zero subsheaf of $\mathcal{R}$—and is also **surjective**—that is, has as image the full sheaf $\mathcal{S}$. As a very useful terminology, a sequence of sheaves and sheaf homomorphisms

$$\cdots \longrightarrow \mathcal{S}_{n-1} \xrightarrow{\phi_{n-1}} \mathcal{S}_n \xrightarrow{\phi_n} \mathcal{S}_{n+1} \longrightarrow \cdots \tag{4}$$

is called an **exact sequence of sheaves** if for each n the image of ϕ_{n-1} is exactly the kernel of ϕ_n. In particular,

$$0 \xrightarrow{i} \mathcal{R} \xrightarrow{\phi} \mathcal{S}$$

is an exact sequence of sheaves, where 0 denotes the zero sheaf and i is the natural inclusion of 0 as the zero subsheaf of $\mathcal{R}$, precisely when ϕ is injective; and

$$\mathcal{S} \xrightarrow{\psi} \mathcal{T} \longrightarrow 0$$

is an exact sequence of sheaves, where 0 again denotes the zero sheaf, precisely when ψ is surjective. An exact sequence of sheaves of the form

$$0 \longrightarrow \mathcal{R} \xrightarrow{\phi} \mathcal{S} \xrightarrow{\psi} \mathcal{T} \longrightarrow 0 \tag{5}$$

is called a **short exact sequence of sheaves**. The exactness of the sequence (5) means that ϕ is injective, ψ is surjective, and $\phi(\mathcal{R})$ is precisely the kernel of ψ. Thus, ϕ can be viewed as an embedding of $\mathcal{R}$ as a subsheaf of $\mathcal{S}$, and ψ exhibits $\mathcal{T}$ as being isomorphic to the quotient sheaf $\mathcal{S}/\mathcal{R}$. The exact sequence of sheaves (4) can be rewritten as a collection of short exact sequences of sheaves of the form

$$0 \longrightarrow \mathcal{R}_n \xrightarrow{i} \mathcal{S}_n \xrightarrow{\phi_n} \mathcal{R}_{n+1} \longrightarrow 0$$

where $\mathcal{R}_n$ is the kernel of the homomorphism $\phi_n\colon \mathcal{S}_n \to \mathcal{S}_{n+1}$; here $\mathcal{R}_n$ is a subsheaf of $\mathcal{S}_n$, and $i\colon \mathcal{R}_n \to \mathcal{S}_n$ is merely the inclusion mapping.

The actual construction of sheaves in practice usually follows a rather standard pattern, but before describing that construction it is convenient to consider separately and more abstractly the purely algebraic aspect of that construction, since it will be used again later in another context. Suppose that I is a directed set: that means that I is a partially ordered set, so there is a binary relation $\alpha \leqq \beta$ among the elements of I such that $\alpha \leqq \alpha$ and that $\alpha \leqq \beta$ and $\beta \leqq \gamma$ imply $\alpha \leqq \gamma$, and moreover for any elements α, β of I there must be another element γ of I such that $\alpha \leqq \gamma$ and $\beta \leqq \gamma$. A direct system of abelian groups indexed by I is a collection of abelian groups G_α, one for each element $\alpha \in I$, and of group homomorphisms $\phi_{\beta\alpha}\colon G_\alpha \to G_\beta$ whenever $\alpha \leqq \beta$, such that $\phi_{\alpha\alpha}$ is the identity homomorphism and $\phi_{\gamma\beta}\phi_{\beta\alpha} = \phi_{\gamma\alpha}$ whenever $\alpha \leqq \beta \leqq \gamma$. If $\{G_\alpha\}$ is a direct system of abelian groups indexed by I, then on the disjoint union $\bigcup_\alpha G_\alpha$ introduce an equivalence relation by writing $g_\alpha \sim g_\beta$ for elements $g_\alpha \in G_\alpha$, $g_\beta \in G_\beta$ precisely when there is an index γ such that $\alpha \leqq \gamma$, $\beta \leqq \gamma$, and $\phi_{\gamma\alpha}(g_\alpha) = \phi_{\gamma\beta}(g_\beta)$. To verify that this is an equivalence relation in the ordinary sense it is clearly only necessary to demonstrate transitivity, since both symmetry and reflexivity hold quite trivially. Suppose then that $g_\alpha \sim g_\beta$ and $g_\beta \sim g_\gamma$, so that there is an index δ such that $\alpha \leqq \delta$, $\beta \leqq \delta$, and $\phi_{\delta\alpha}(g_\alpha) = \phi_{\delta\beta}(g_\beta)$ and there is an index ε such that $\beta \leqq \varepsilon$, $\gamma \leqq \varepsilon$, and $\phi_{\varepsilon\beta}(g_\beta) = \phi_{\varepsilon\gamma}(g_\gamma)$. There must also be an index ζ for which $\delta \leqq \zeta$ and $\varepsilon \leqq \zeta$, and upon using all these results, it follows that $\phi_{\zeta\alpha}(g_\alpha) = \phi_{\zeta\delta}(\phi_{\delta\alpha}(g_\alpha)) = \phi_{\zeta\delta}(\phi_{\delta\beta}(g_\beta)) = \phi_{\zeta\beta}(g_\beta) = \phi_{\zeta\varepsilon}(\phi_{\varepsilon\beta}(g_\beta)) = \phi_{\zeta\varepsilon}(\phi_{\varepsilon\gamma}(g_\gamma)) = \phi_{\zeta\gamma}(g_\gamma)$, so that $g_\alpha \sim g_\gamma$ and this is an equivalence relation as desired. The set of equivalence classes will be called the **direct limit** of the direct system of abelian groups $\{G_\alpha\}$ and will be denoted by $\operatorname{dir\,lim}_I G_\alpha$. Any two equivalence classes $g', g'' \in \operatorname{dir\,lim}_I G_\alpha$ will have representatives g'_β, g''_β in some common group G_β, and since the mappings $\phi_{\gamma\beta}$ are group homomorphisms, it follows quite readily that the equivalence class of the element $g'_\beta - g''_\beta$ depends only on the equivalence classes g', g'' and not on the particular choice of representatives. That equivalence class will be denoted by $g' - g''$, and it is a straightforward matter to verify that this operation determines the structure of an abelian group on $\operatorname{dir\,lim}_I G_\alpha$. The mapping $\phi_\beta\colon G_\beta \to \operatorname{dir\,lim}_I G_\alpha$ that associates to each element $g_\beta \in G_\beta$ its equivalence class is evidently a group homomorphism, and clearly $\phi_\beta = \phi_\gamma \phi_{\gamma\beta}$ whenever $\beta \leqq \gamma$.

5. DEFINITION. *A* **presheaf** *of abelian groups over a topological space M is a collection of abelian groups $\mathcal{S}_U$ indexed by the open subsets $U \subseteq M$ and group homomorphisms $\rho_{VU}\colon \mathcal{S}_U \to \mathcal{S}_V$ associated to the inclusion relations $U \supseteq V$, such that ρ_{UU} is the identity homomorphism and $\rho_{WV}\rho_{VU} = \rho_{WU}$ whenever $U \supseteq V \supseteq W$.*

To each sheaf $\mathcal{S}$ of abelian groups over M there is the naturally **associated presheaf** for which $\mathcal{S}_U = \Gamma(U, \mathcal{S})$ and $\rho_{VU}\colon \Gamma(U, \mathcal{S}) \to \Gamma(V, \mathcal{S})$ is the restriction to the subset $V \subseteq U$ of any section $f \in \Gamma(U, \mathcal{S})$. More interestingly and less obviously, to each presheaf $\{\mathcal{S}_U, \rho_{VU}\}$ of abelian groups over M there is a naturally **associated sheaf**, constructed as follows. The collection of all open subsets is a partially ordered set, when the ordering $U \leqq V$ is taken to mean that $U \supseteq V$. For any fixed point $Z \in M$ the subcollection I_Z consisting of those open subsets of M containing Z is a directed set under this partial ordering, since whenever $U, V \in I_Z$ then for instance $W = U \cap V \in I_Z$ and $U \leqq W$ and $V \leqq W$. From the definition of a presheaf it follows immediately that the collection of groups $\mathcal{S}_U$ for $U \in I_Z$ and of homomorphisms ρ_{VU} for $U \leqq V$ forms a direct system of abelian groups indexed by I_Z, so it is possible to introduce the direct limit group $\tilde{\mathcal{S}}_Z = \operatorname{dir\,lim}_{I_Z} \mathcal{S}_U$. Note that for any $U \in I_Z$ there is a natural homomorphism $\rho_{ZU}\colon \mathcal{S}_U \to \tilde{\mathcal{S}}_Z$. Then let $\tilde{\mathcal{S}} = \bigcup_{Z \in M} \tilde{\mathcal{S}}_Z$, introduce a mapping $\pi\colon \tilde{\mathcal{S}} \to M$ by setting $\pi(\tilde{\mathcal{S}}_Z) = Z$, and note that $\pi^{-1}(Z) = \tilde{\mathcal{S}}_Z$ has the structure of an abelian group. Finally for any open subset $U \subseteq M$ and any element $f_U \in \mathcal{S}_U$ introduce the subset $\rho(f_U) = \{\rho_{ZU}(f_U)\colon Z \in U\} \subseteq \tilde{\mathcal{S}}$, and topologize $\tilde{\mathcal{S}}$ by taking these subsets $\{\rho(f_U)\}$ as a basis for the open sets. That these subsets can indeed be taken as a basis for a topology follows quite easily: whenever $s \in \rho(f_U) \cap \rho(f_V)$, then $Z = \pi(s) \in U \cap V$ and $s = \rho_{ZU}(f_U) = \rho_{ZV}(f_V)$, so from the definition of a direct limit there must be some $W \in I_Z$ such that $U \leqq W$, $V \leqq W$, and $\rho_{WU}(f_U) = \rho_{WV}(f_V) = f_W \in \mathcal{S}_W$; then $W \subseteq U \cap V$ and $s \in \rho(f_W) \subseteq \rho(f_U) \cap \rho(f_V)$. It is quite clear that with this topology on $\tilde{\mathscr{S}}$ the mapping π is a local homeomorphism, and since $\rho(f_U) - \rho(g_U) = \rho(f_U - g_U)$ for any elements $f_U, g_U \in \mathscr{S}_U$, it is a simple matter to verify that the group operations on $\tilde{\mathscr{S}}$ are continuous. Thus $\tilde{\mathscr{S}}$ is a sheaf, naturally and canonically constructed from the presheaf $\{\mathscr{S}_U, \rho_{VU}\}$.

To each sheaf $\mathscr{S}$ of abelian groups over M there is the naturally associated presheaf $\{\Gamma(U, \mathscr{S}), \rho_{VU}\}$, and to this presheaf there is in turn a naturally associated sheaf $\tilde{\mathscr{S}}$. Since for every point $s \in \mathscr{S}$ there is a section $f_U \in \Gamma(U, \mathscr{S})$ over some open subset $U \subseteq M$ passing through s, and since any two sections of $\mathscr{S}$ that agree at a point necessarily agree in a full open neighborhood of that point, an examination of the construction of the sheaf $\tilde{\mathscr{S}}$ readily shows that there is a natural identification of $\tilde{\mathscr{S}}$ with the original sheaf $\mathscr{S}$. On the other hand, to each presheaf $\{\mathscr{S}_U, \rho_{VU}\}$ of abelian groups over M there is the naturally associated sheaf $\tilde{\mathscr{S}}$, and to this sheaf there is also in turn the naturally associated presheaf $\{\Gamma(U, \tilde{\mathscr{S}}), \tilde{\rho}_{VU}\}$; but the latter presheaf may be quite different from the former presheaf. For example, consider the presheaf of abelian groups $\{\mathscr{S}_U, \rho_{UV}\}$ over some topological space M where $\mathscr{S}_U = \mathbb{Z}$ for each open subset $U \subseteq M$ and ρ_{VU} is the zero homomorphism for each inclusion relation $U \supseteq V$. The associated sheaf is just the zero sheaf $\tilde{\mathscr{S}} = 0$, and its associated presheaf has $\Gamma(U, \tilde{\mathscr{S}}) = 0$ for each open subset $U \subseteq M$. It is evident that this unpleasant behavior can be ruled out by considering only those presheaves that are associated to sheaves, and indeed only by such a limitation. That raises the problem of characterizing such presheaves, for the solution of which problem the following notion is basic.

6. DEFINITION. *A presheaf $\{\mathscr{S}_U, \rho_{VU}\}$ of abelian groups over a topological space M is* **complete** *if:*

(i) *Whenever U is an open subset of M, $\{U_j\}$ is an open covering of U, and $f \in \mathscr{S}_U$ is an element such that $\rho_{U_jU}(f) = 0$ for each j, then $f = 0$.*

(ii) *Whenever U is an open subset of M, $\{U_j\}$ is an open covering of U, and $f_j \in \mathscr{S}_{U_j}$ are elements such that $\rho_{U_j \cap U_k, U_j}(f_j) = \rho_{U_j \cap U_k, U_k}(f_k)$ for all j, k, then there is an element $f \in \mathscr{S}_U$ such that $\rho_{U_jU}(f) = f_j$ for each j.*

7. THEOREM. *A presheaf of abelian groups is the associated presheaf to a sheaf precisely when that presheaf is complete.*

Proof. It is obvious that the associated presheaf to any sheaf is a complete presheaf. On the other hand, let $\{\mathscr{S}_U, \rho_{VU}\}$ be a complete presheaf over some topological space M, and let $\tilde{\mathscr{S}}$ be the associated sheaf. It is clear from the construction of the associated sheaf that for each open subset $U \subseteq M$ there is a natural homomorphism $\rho: \mathscr{S}_U \to \Gamma(U, \tilde{\mathscr{S}})$; indeed, it is merely the mapping that assigns to any element $f_U \in \mathscr{S}_U$ the section associated to the basic open set $\rho(f_U) \subseteq \tilde{\mathscr{S}}$ in the topology of $\tilde{\mathscr{S}}$. It is clear that to complete the proof of the theorem it is sufficient just to show that this homomorphism ρ is an isomorphism. First consider an element $f_U \in \mathscr{S}_U$ such that $\rho(f_U) = 0$. That means that for each point $Z \in U$ the image $\rho_{ZU}(f_U) = 0 \in \tilde{\mathscr{S}}_Z$, and hence from the definition of the direct limit there must be some subneighborhood $U_Z \subseteq U$ of Z such that $\rho_{U_ZU}(f_U) = 0$. A countable subcollection of such neighborhoods $U_j = U_{Z_j}$ will cover U, and it then follows from condition (i) of Definition 6 that $f_U = 0$, so that ρ is an injective homomorphism. Next consider an arbitrary section $\tilde{f} \in \Gamma(U, \tilde{\mathscr{S}})$. In some open subneighborhood $U_Z \subseteq U$ of any point $Z \in U$ this section must coincide with some basic open set $\rho(f_{U_Z})$ in the topology of $\tilde{\mathscr{S}}$. A countable subcollection of such neighborhoods $U_j = U_{Z_j}$ will cover U, and the elements $f_j = f_{U_j}$ have the property that $\rho_{U_j \cap U_k, U_j}(f_j) - \rho_{U_j \cap U_k, U_k}(f_k)$ lies in the kernel of the natural homomorphism $\rho: \mathscr{S}_{U_j \cap U_k} \to \Gamma(U_j \cap U_k, \tilde{\mathscr{S}})$. Hence by what has already been proved, $\rho_{U_j \cap U_k, U_j}(f_j) = \rho_{U_j \cap U_k, U_k}(f_k)$, so it follows from property (ii) of Definition 6 that there is an element $f_U \in \mathscr{S}_U$ for which $\rho(f_U) = \tilde{f}$. That shows that the homomorphism is also surjective and thereby concludes the proof.

The preceding shows how to construct sheaves from presheaves. It is natural then to ask how to construct homomorphisms of sheaves in terms of presheaves.

8. DEFINITION. *A* **homomorphism** *$\phi: \{\mathscr{R}_U, \rho_{VU}\} \to \{\mathscr{S}_U, \sigma_{VU}\}$ between two presheaves of abelian groups over M is a collection of group homomorphisms $\phi_U: \mathscr{R}_U \to \mathscr{S}_U$ such that $\phi_V \rho_{VU} = \sigma_{VU} \phi_U$ whenever $U \supseteq V$. A homomorphism ϕ is an* **isomorphism** *if all the homomorphisms ϕ_U are isomorphisms.*

If $\{\mathscr{S}_U, \sigma_{VU}\}$ is a presheaf of abelian groups and $\mathscr{R}_U$ are subgroups of $\mathscr{S}_U$ such that $\sigma_{VU}(\mathscr{R}_V) \subseteq \mathscr{R}_U$ whenever $U \supseteq V$, then $\{\mathscr{R}_U, \sigma_{VU}\}$ is also a presheaf, called naturally enough a **subpresheaf** of $\{\mathscr{S}_U, \sigma_{VU}\}$. The inclusion mappings $i_U: \mathscr{R}_U \to \mathscr{S}_U$ then determine a presheaf homomorphism. If $\phi: \{\mathscr{R}_U, \rho_{VU}\} \to \{\mathscr{S}_U, \sigma_{VU}\}$ is a homomorphism between two presheaves and $\mathscr{K}_U$ is the kernel of the group homomorphism $\phi_U: \mathscr{R}_U \to \mathscr{S}_U$ for each U, then clearly $\rho_{VU}(\mathscr{K}_U) \subseteq \mathscr{K}_V$ whenever $U \supseteq V$, so $\{\mathscr{K}_U, \rho_{VU}\}$ is a subpresheaf of $\{\mathscr{R}_U, \rho_{VU}\}$; this presheaf is called the **kernel** of the homomorphism ϕ. Furthermore if $\mathscr{T}_U$ is the image of the group homomorphism $\phi_U: \mathscr{R}_U \to \mathscr{S}_U$ for each U, then clearly $\sigma_{VU}(\mathscr{T}_U) \subseteq \mathscr{T}_V$ whenever $U \supseteq V$, so $\{\mathscr{T}_U, \sigma_{VU}\}$ is a subpresheaf of $\{\mathscr{S}_U, \sigma_{VU}\}$; this presheaf is called the **image** of the homomorphism ϕ. A homomorphism of presheaves $\phi: \{\mathscr{R}_U, \rho_{VU}\} \to \{\mathscr{S}_U, \sigma_{VU}\}$ is evidently an isomorphism precisely when its kernel is the zero subpresheaf of $\{\mathscr{R}_U, \rho_{VU}\}$ and its image is all of $\{\mathscr{S}_U, \sigma_{VU}\}$. It is then possible to introduce exact sequences of presheaves in the obvious manner.

9. THEOREM. *A homomorphism $\phi: \{\mathscr{R}_U, \rho_{VU}\} \to \{\mathscr{S}_U, \sigma_{VU}\}$ between presheaves induces a natural homomorphism $\phi^*: \tilde{\mathscr{R}} \to \tilde{\mathscr{S}}$ between the associated sheaves, in such a manner that the identity presheaf homomorphism induces the identity sheaf homomorphism and $(\phi \circ \psi)^* = \phi^* \circ \psi^*$. Moreover if*

$$0 \longrightarrow \{\mathscr{R}_U, \rho_{VU}\} \xrightarrow{\phi} \{\mathscr{S}_U, \sigma_{VU}\} \xrightarrow{\psi} \{\mathscr{T}_U, \tau_{VU}\} \longrightarrow 0 \tag{6}$$

is an exact sequence of presheaf homomorphisms, then the induced sequence of homomorphisms of the associated sheaves

$$0 \longrightarrow \tilde{\mathscr{R}} \xrightarrow{\phi^*} \tilde{\mathscr{S}} \xrightarrow{\psi^*} \tilde{\mathscr{T}} \longrightarrow 0 \tag{7}$$

is also an exact sequence.

Proof. Note that if $Z \in U \cap V$ and if $f_U \in \mathscr{R}_U$, $f_V \in \mathscr{R}_V$ represent the same element in the direct limit $\tilde{\mathscr{R}}_Z = \text{dir lim}_{I_Z} \mathscr{R}_U$, then there must be an open set $W \in I_Z$ such that $U \leq W$, $V \leq W$, and $\rho_{WU}(f_U) = \rho_{WV}(f_V)$. But then $W \subseteq U \cap V$ and $\sigma_{WU}(\phi_U(f_U)) = \phi_W(\rho_{WU}(f_U)) = \varphi_W(\rho_{WV}(f_V)) = \sigma_{WV}(\phi_V(f_V))$, so that $\phi_U(f_U)$ and $\phi_V(f_V)$ represent the same element in the direct limit $\tilde{\mathscr{S}}_Z = \text{dir lim}_{I_Z} \mathscr{S}_U$. Thus the homomorphisms ϕ_U induce a well-defined mapping $\phi_Z^*: \tilde{\mathscr{R}}_Z \to \tilde{\mathscr{S}}_Z$, which is clearly a group homomorphism. These mappings moreover take a basic open set $\rho(f_U)$ in $\tilde{\mathscr{R}}$ to the basic open set $\sigma(\phi_U(f_U))$ in $\tilde{\mathscr{S}}$ and so determine a continuous mapping $\phi^*: \tilde{\mathscr{R}} \to \tilde{\mathscr{S}}$ which must be a sheaf homomorphism. To demonstrate that the exact sequence of presheaf homomorphisms (6) induces an exact sequence (7) of sheaf homomorphisms it is sufficient to show that for each point $Z \in M$ the sequence

$$0 \longrightarrow \tilde{\mathscr{R}}_Z \xrightarrow{\phi_Z^*} \tilde{\mathscr{S}}_Z \xrightarrow{\psi_Z^*} \tilde{\mathscr{T}}_Z \longrightarrow 0$$

is an exact sequence of abelian groups. If $f_U \in \mathscr{R}_U$ represents an element in the kernel of ϕ_Z^*, then $\phi_U(f_U)$ represents zero in $\tilde{\mathscr{S}}_Z$; hence, there is an open set V such that $Z \in V \subseteq U$ and $0 = \sigma_{VU}(\phi_U(f_U)) = \phi_V(\rho_{VU}(f_U))$. However since ϕ_V is injective by the exactness of (6), it follows that $\rho_{VU}(f_U) = 0$ and hence that f_U necessarily represents the zero element of $\tilde{\mathscr{R}}_Z$. It is of course obvious that $\psi_Z^* \phi_Z^* = (\psi\phi)_Z^* = 0$. If $g_U \in \mathscr{S}_U$ represents an element in the kernel of ψ_Z^*, then $\psi_U(g_U)$ represents zero in $\tilde{\mathscr{T}}_Z$; hence, there is an open set V such that $0 = \tau_{VU}(\psi_U(g_U)) = \psi_V(\sigma_{VU}(g_U))$. However from the exactness of (6) it follows that there is an element $f_V \in \mathscr{R}_V$ such that $\sigma_{VU}(g_U) = \phi_V(f_V)$ and hence that g_U represents an element in the image of ϕ_Z^*. Finally since the homomorphisms $\psi_U: \mathscr{S}_U \to \mathscr{T}_U$ are surjective by the exactness of (6), it is obvious that ψ_Z^* is surjective. What remains to be shown is rather clear, and that suffices for the proof.

The relation between sheaves and presheaves is actually rather subtler than might be imagined from the simplicity of the preceding discussion. In fact the exactness of a sequence of presheaf homomorphisms as defined above is much more restrictive than the exactness of a sequence of sheaf homomorphisms, and much less useful though convenient in some constructions. To see this, consider the problem of obtaining a converse to the preceding theorem; part at least is also trivial, as follows.

10. THEOREM. *For any short exact sequence of sheaves of abelian groups over M of the form*

$$0 \longrightarrow \mathscr{R} \xrightarrow{\phi} \mathscr{S} \xrightarrow{\psi} \mathscr{T} \longrightarrow 0 \tag{8}$$

and any subset $X \subseteq M$, there is an induced exact sequence of sections of the form

$$0 \longrightarrow \Gamma(X, \mathscr{R}) \xrightarrow{\phi^*} \Gamma(X, \mathscr{S}) \xrightarrow{\psi^*} \Gamma(X, \mathscr{T}) \tag{9}$$

Proof. In the exact sequence of sheaves (8) the homomorphism ϕ can be viewed as an embedding of $\mathscr{R}$ as a subsheaf of $\mathscr{S}$, and the homomorphism ψ can be viewed as the natural mapping from $\mathscr{S}$ to the quotient sheaf $\mathscr{S}/\phi(\mathscr{R})$. For a subsheaf $\mathscr{R} \subseteq \mathscr{S}$ it is obvious that the inclusion $i: \mathscr{R} \to \mathscr{S}$ induces an injective homomorphism $i^*: \Gamma(X, \mathscr{R}) \to \Gamma(X, \mathscr{S})$ over any subset $X \subseteq M$, and that the image of i^* is exactly the kernel of the homomorphism $\Gamma(X, \mathscr{S}) \to \Gamma(X, \mathscr{S}/\mathscr{R})$ induced by the natural mapping $\mathscr{S} \to \mathscr{S}/\mathscr{R}$. That is all that is needed to conclude the proof.

Almost trivial examples show that (9) cannot always be extended to a short exact sequence—that is, that ψ^* is not necessarily surjective. Consider for instance the trivial sheaf $\mathbb{C}$ over a connected topological space M. Let A and B be two distinct points of M and $\mathscr{R} \subseteq \mathbb{C}$ be the subsheaf having as stalks

$$\mathscr{R}_Z = \begin{cases} \mathbb{C} & \text{if } Z \neq A \text{ and } Z \neq B \\ 0 & \text{if } Z = A \text{ or } Z = B \end{cases}$$

and let $\mathscr{T}$ be the quotient sheaf $\mathscr{T} = \mathbb{C}/\mathscr{R}$, so that

$$\mathscr{T}_Z = \begin{cases} 0 & \text{if } Z \neq A \text{ and } Z \neq B \\ \mathbb{C} & \text{if } Z = A \text{ or } Z = B \end{cases}$$

Sections in $\Gamma(M, \mathscr{T})$ can have distinct values at the points A and B, but any section in $\Gamma(M, \mathbb{C})$ must have the same value at all points of M; thus, the homomorphism $\Gamma(M, \mathbb{C}) \to \Gamma(M, \mathscr{T})$ is not surjective. On the other hand, though, if $U = M - B$, then the homomorphism $\Gamma(U, \mathbb{C}) \to \Gamma(U, \mathscr{T})$ is clearly surjective. One of the principal parts of the theory of sheaves is devoted to examining obstructions measuring the extent to which surjective sheaf homomorphisms do not induce surjective homomorphisms of sections; that leads to the cohomology theory of sheaves, to be developed in subsequent sections. Before turning to that theory, though, it is instructive to introduce some additional and more interesting examples of sheaves, after one preliminary general remark.

It is useful to note that if $\phi: \{\mathscr{R}_U, \rho_{UV}\} \to \{\mathscr{S}_U, \sigma_{UV}\}$ is a homomorphism of presheaves and if both $\{\mathscr{R}_U, \rho_{UV}\}$ and $\{\mathscr{S}_U, \sigma_{UV}\}$ are complete presheaves, then the kernel of ϕ is also a complete presheaf; the verification is trivial and will be left as an exercise. On the other hand, the image of ϕ is not necessarily a complete presheaf. Indeed if $\phi: \mathscr{R} \to \mathscr{S}$ is a surjective sheaf homomorphism such that $\phi^*: \Gamma(M, \mathscr{R}) \to \Gamma(M, \mathscr{S})$ is not surjective, then the induced homomorphism ϕ^* of the associated presheaves has an image that is clearly not a complete presheaf. The question of the completeness of images is thus closely related to the question of extending Theorem 10, and the example of the preceding paragraph illustrates that.

Examples. Let M be an arbitrary topological space. To each open subset $U \subseteq M$ associate the ring $\mathscr{C}_U$ of continuous complex-valued functions in U, and to each inclusion relation $U \subseteq V$ associate the natural restriction homomorphism $\rho_{UV}: \mathscr{C}_V \to \mathscr{C}_U$. It is obvious that this is a presheaf of rings over M, even a complete presheaf of rings. The associated sheaf $\mathscr{C}$ is called the **sheaf of germs of continuous complex-valued functions** over M. Upon comparing the construction of the associated sheaf and the discussion of germs of functions in section II-A, it is evident that the stalk $\mathscr{C}_Z$ consists of the ring of germs of continuous complex-valued functions at the point Z on M; that explains the terminology. Since the presheaf is complete, there is a natural identification $\mathscr{C}_U = \Gamma(U, \mathscr{C})$. Note that each ring $\mathscr{C}_U$ can also be viewed as an algebra over the complex numbers, and $\mathscr{C}$ can thus also be considered as a sheaf of complex algebras.

Next let M be a differentiable manifold of dimension n and of class C^∞. To each open subset $U \subseteq M$ associate the complex vector space $\mathscr{E}_U^r$ of complex-valued C^∞ differential forms of degree r in U, and to each inclusion relation $U \subseteq V$ associate the natural restriction homomorphism $\rho_{UV}: \mathscr{E}_V^r \to \mathscr{E}_U^r$. It is obvious that this too is a complete presheaf of complex vector spaces over M. The associated sheaf $\mathscr{E}^r$ is the **sheaf of germs of C^∞ complex-valued differential forms of degree r** over M. For the special case $r = 0$ the sheaf $\mathscr{E}^0 = \mathscr{E}$ of germs of C^∞ complex-valued functions can also be viewed as a sheaf of complex algebras over M. Note that exterior differentiation $d: \mathscr{E}_U^r \to \mathscr{E}_U^{r+1}$ commutes with restriction and is thus a homomorphism of presheaves. The induced homomorphism of sheaves $d: \mathscr{E}^r \to \mathscr{E}^{r+1}$ is also called exterior differentiation. In the special case $d: \mathscr{E}^0 \to \mathscr{E}^1$, the kernel of this homomorphism is just the subsheaf of $\mathscr{E}^0$ consisting of functions that are constant in an open neighborhood of each point and hence is the constant sheaf $\mathbb{C}$. It is familiar from advanced calculus that every differential form f of degree $r > 0$ satisfying $df = 0$ can be written locally as $f = dg$ for some differential form g of degree $r - 1$. There results the **de Rham exact sequence of sheaves**

$$0 \to \mathbb{C} \to \mathscr{E}^0 \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \to \cdots \to \mathscr{E}^{n-1} \xrightarrow{d} \mathscr{E}^n \to 0 \tag{10}$$

If M is a complex manifold of complex dimension n, it is possible to introduce in addition to the sheaves of vector spaces $\mathscr{E}^r$ the subsheaves $\mathscr{E}^{p,q} \subseteq \mathscr{E}^r$ defined by the presheaves $\{\mathscr{E}_U^{p,q}\}$ of differential forms of bidegree (p, q); $\mathscr{E}^{p,q}$ is the **sheaf of germs of C^∞ complex-valued differential forms of bidegree (p, q)** over M. The decomposition of differential forms of degree r into sums of differential forms of bidegrees (p, q) with $p + q = r$ was discussed in section I-E. It is clear that this decomposition is possible on an arbitrary complex manifold as well. Correspondingly the exterior derivative can be decomposed into the sum of differential operators $d = \partial + \bar\partial$, where $\partial: \mathscr{E}^{p,q} \to \mathscr{E}^{p+1,q}$ and $\bar\partial: \mathscr{E}^{p,q} \to \mathscr{E}^{p,q+1}$. The kernel of the sheaf homomorphism $\bar\partial: \mathscr{E}^{p,0} \to \mathscr{E}^{p,1}$ is called the **sheaf of germs of holomorphic p-forms** and is denoted by $\mathcal{O}^{p,0}$. In particular the kernel of the sheaf homomorphism $\bar\partial: \mathscr{E} \to \mathscr{E}^{0,1}$ is the **sheaf of germs of holomorphic functions** and is denoted merely by $\mathcal{O}$. This is the associated sheaf to the complete presheaf $\{\mathcal{O}_U, \rho_{UV}\}$, where $\mathcal{O}_U$ is the ring of holomorphic functions in U and ρ_{UV} is the natural restriction mapping; there is thus a natural identification $\Gamma(U, \mathcal{O}) = \mathcal{O}_U$. Note that Dolbeault's lemma, Theorem I-E3, implies that the following is an exact sequence of sheaves for any index p, $0 \leqq p \leqq n$:

$$0 \to \mathcal{O}^{p,0} \xrightarrow{\bar\partial} \mathscr{E}^{p,1} \xrightarrow{\bar\partial} \mathscr{E}^{p,2} \to \cdots \to \mathscr{E}^{p,n-1} \xrightarrow{\bar\partial} \mathscr{E}^{p,n} \to 0 \tag{11}$$

This is called the **Dolbeault exact sequence of sheaves.**

Next consider a holomorphic variety V. To each open subset $U \subseteq V$ associate the ring ${}_V\mathcal{O}_U$ of holomorphic functions in U, and to each inclusion relation $U_1 \subseteq U_2$ associate the natural restriction homomorphism $\rho_{U_1U_2}: {}_V\mathcal{O}_{U_2} \to {}_V\mathcal{O}_{U_1}$. This is

obviously a complete presheaf over V. The associated sheaf ${}_V\mathcal{O}$ is the **sheaf of germs of holomorphic functions** over V. This can be considered either as a sheaf of rings or as a sheaf of algebras over the complex numbers. Here too there is a natural identification $\Gamma(U, {}_V\mathcal{O}) = {}_V\mathcal{O}_U$ for any open subset $U \subseteq V$.

It should be pointed out particularly that in these examples the construction of the associated sheaf is really a very familiar construction. The stalks ${}_V\mathcal{O}_Z$ of the sheaves ${}_V\mathcal{O}$ are precisely the rings of germs of holomorphic functions on V at the point Z, as discussed in section II-B. On the one hand, that should serve to clarify the abstract construction of the sheaves associated to presheaves, and on the other hand, it should also serve to indicate the significance of sheaves, at least as sets. It should also help explain the terminology. The topology is introduced on sheaves in order to permit the reconstruction of the original presheaves as continuous sections of the associated sheaves. The sheaves themselves may provide a good deal of local information, as for example all the local properties of the rings of germs of holomorphic functions discussed in Volume II; but the sections of these sheaves, the rings of global holomorphic functions for example, are really the primary objects of interest. The question whether an exact sequence of sheaves leads to an exact sequence of sections was already considered in section I-E for the special case of the Dolbeault exact sequence of sheaves. Indeed the Dolbeault cohomology groups of Definition I-E4 are measures of the extent to which the associated sequence of sections fails to be exact. Theorem I-E5 demonstrates that the associated sequence of sections is exact in some cases, and Theorems I-E6 and I-E7 show the usefulness of this exactness.