Separate Analyticity and Hartogs Theorems

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Introduction. Generalizations of the 1906 theorem of Hartogs [Ha] on separately holomorphic functions have been given by various authors. Terada [Te] showed that if $f(z,w)$ is a complex-valued function defined for $z \in U \subset \mathbf{C}^M$, $w \in V \subset \mathbf{C}^N$ and is holomorphic in w for all fixed $z \in U$ and is holomorphic in z for all fixed w in some non-pluripolar set $A \subset V$, then f is jointly holomorphic. If one assumes only that A is not contained in an analytic hypersurface of V, then the result is false (unless f is also assumed to be bounded, in which case the conclusion is an easy consequence of the Cauchy integral formula). The author [Sh] showed that if f is holomorphic (or meromorphic) in z for almost all fixed w and is holomorphic (or meromorphic) in w for almost all fixed z, then f is jointly holomorphic (or meromorphic). Bernstein [Br], Siciak [Si1], [Si2], Zaharjuta [Za], and Nguyen and Zeriahi [NZ] gave results on the holomorphic extendibility of separately holomorphic functions defined on sets of the form $(U \times F) \cup (E \times V)$. In this paper we give a general form of these results (Lemma 4), where we assume only that E, F are non-pluripolar. (We do not need the assumption used in [Si2] and [Za] that E, F are L-regular.) As a consequence, we give an improvement of Terada's theorem (Corollary 3) which we generalize to separately meromorphic functions (Theorem 2). Our methods are based on the "pluripotential theory" developed by Siciak [Si2], [Si3].

If $h(x,t)$ is a separately real-analytic function, then of course h does not even have to be continuous. However, Lelong [L, Theorem 11] gave a condition guaranteeing that h is jointly real-analytic; in particular Lelong showed that separately harmonic functions are harmonic. We show that if $h(x,w)$ is holomorphic in $w \in V \subset \mathbf{C}^N$ for all fixed x and is real-analytic in x for all fixed w in a non-pluripolar subset of V, then x is jointly real-analytic (Theorem 1). A special case of this result is used by Katok, Knieper, Pollicott, and Weiss [KKPW] to obtain a regularity result for real-analytic deformations of Anosov flows. The author would like to thank Howard Weiss for suggesting the conjecture that led to Theorem 1.

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1. Pluripotential Theory. We first summarize some basic facts on pluripotential theory which we shall use later. For more complete details, see [Be], [Sa], [Si2], [Si3].

For $z = (z_1,\dots,z_N) \in \mathbf{C}^N$, we let $|z| = (|z_1|^2 + \cdots + |z_N|^2)^{1/2}$. We write

$$B(a,r) = \{z \in \mathbf{C}^N : |z-a| < r\}, \qquad B_r = B(0,r),$$

for $a \in \mathbf{C}^N$, $r > 0$, and we let $d(z,A) = \inf\{|z-a| : a \in A\}$ for $A \subset \mathbf{C}^N$. For U an open set in $\mathbf{C}^N$, we let $\mathrm{PSH}(U)$ and $\mathcal{O}(U)$ denote the classes of plurisubharmonic functions and holomorphic functions, respectively, on U. We also consider the class

$$\mathcal{L} = \{u \in \mathrm{PSH}(\mathbf{C}^N) : u(z) \le \log(1+|z|) + O(1)\}. \tag{1.1}$$

A set $E \subset \mathbf{C}^N$ is said to be pluripolar if for all $a \in E$ there exist a neighborhood U of a and a function $u \in \mathrm{PSH}(U)$ such that $u = -\infty$ on $E \cap U$. Using a result of Josefson [J], Siciak [Si, 3.10] proved that E is pluripolar if and only if there exists $u \in \mathcal{L}$ with $u = -\infty$ on E. It follows that a countable union of pluripolar sets is pluripolar. Suppose U is an open set in $\mathbf{C}^N$ and $\mathcal{F} \subset \mathrm{PSH}(U)$ such that $\mathcal{F}$ is locally bounded above on U, and let

$$\begin{aligned} u(z) &= \sup\{\varphi(z) : \varphi \in \mathcal{F}\}, \\ u^*(z) &= \limsup_{\zeta \to z} u(\zeta), \end{aligned} \tag{1.2}$$

for $z \in U$. Then $u^* \in \mathrm{PSH}(U)$, and we have the following important result of Bedford and Taylor [BT, Theorem 7.1]:

Proposition 1. *Let u, u^* be given by* (1.2). *Then the set $\{z \in U;\ u^*(z) > u(z)\}$ is pluripolar.*

For $E \subset \mathbf{C}^N$, we define the extremal function V_E by

$$V_E(z) = \sup\{u(z) : u \in \mathcal{L},\ u(a) \le 0 \text{ for } a \in E\}. \tag{1.3}$$

The set E is pluripolar if and only if $V_E^* \equiv +\infty$; if E is not pluripolar, then $V_E^* \in \mathcal{L}$ [Si1, 3.9, 3.10]. Furthermore, if P is a pluripolar set and E is bounded, then by [Si1, 3.11]

$$V_E^* = V_{E \cup P}^*. \tag{1.4}$$

The following fact is stated by Siciak [Si2, p. 8]:

Proposition 2. *Let $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset \mathbf{C}^N$, let $E = \bigcup_{n=1}^{\infty} E_n$, and suppose that E is bounded. Then*

$$V_{E_n}^*(z) \to V_E^*(z)$$

for all $z \in \mathbf{C}^n$.

Proof. Let $u = \lim V_{E_n}^* \in \mathcal{L}$. We must show that $u \leq V_E^*$. Let

$$P = \bigcup_{n=1}^{\infty} \{z \in \mathbf{C}^N : V_{E_n}(z) < V_{E_n}^*(z)\}.$$

The set P is pluripolar by Proposition 1. Since $u(z) = 0$ for $z \in E - P$, we have by (1.4)

$$u \leq V_{E-P} \leq V_{E-P}^* = V_E^*. \qquad \square$$

For a subset E of a bounded open set U in $\mathbf{C}^N$, we define

$$h_{EU} = \sup\{u(z) : u \in \mathrm{PSH}(U),\ u(a) \leq 0 \text{ for } a \in E,\ u \leq 1 \text{ on } U\}. \tag{1.5}$$

Then $h_{EU}^* \in \mathrm{PSH}(U)$. One easily checks that $h_{EU}^* \equiv 1$ if and only if E is pluripolar. Furthermore if P is a pluripolar set in U, then

$$h_{EU}^* = h_{E\cup P,U}^*. \tag{1.6}$$

The following well-known result follows by the method of proof of Proposition 2:

Proposition 3. *Let U be a bounded open set in $\mathbf{C}^N$, let $E_1 \subset E_2 \subset \cdots \subset E_n \subset U$, and let $E = \cup_{n=1}^{\infty} E_n$. Then*

$$h_{E_n U}^*(z) \to h_{EU}^*(z)$$

for $z \in U$.

Let $E \subset \mathbf{C}^N$, $a \in \bar{E}$; we say that E is L-regular at a if $V_E^*(a) = 0$. Note that if E is L-regular at a, then E is not pluripolar since $V_P^* \equiv +\infty$ for pluripolar sets P. We say that E is locally L-regular at a if $E \cap B(a,r)$ is L-regular at a for all $r > 0$. Note that if $\mathcal{N}$ is an arbitrary neighborhood basis for a, then E is locally L-regular at a if and only if $E \cap U$ is L-regular at a for all $U \in \mathcal{N}$. It follows by considering a countable basis for the open sets in $\mathbf{C}^N$ that the set

$$\{a \in E : E \text{ is not locally } L\text{-regular at } a\}$$

is pluripolar. (Sadullaev [Sa] uses the phrase "globally pluriregular" in place of "L-regular," and "pluriregular" in place of "locally L-regular.")

Another description of L-regularity is given by the following fact:

Proposition 4 ([Sa, 12.3]). *Let $E \subset U$, where U is a bounded, open set in $\mathbf{C}^N$, and let $a \in U$. If $h^*_{EU}(a) = 0$, then $V^*_E(a) = 0$. Furthermore, if U contains the polynomial hull of $\bar{E}$ and $V^*_E(a) = 0$, then $h^*_{EU}(a) = 0$.*

Corollary 1. *Let $E \subset \mathbf{C}^N$ and let $a \in \bar{E} \subset D$, where D is a bounded, polynomially convex, open set in $\mathbf{C}^N$. Then E is L-regular at a if and only if $h^*_{ED}(a) = 0$.*

Corollary 2. *Let $E \subset \mathbf{C}^N$, let $a \in \bar{E}$, and let $\mathcal{N}$ be a neighborhood basis for a. Then E is locally L-regular at a if and only if $h^*_{E\cap U,U}(a) = 0$ for all $U \in \mathcal{N}$.*

Proof. (See also [Sa, 12.5].) If $h^*_{E\cap U,U}(a) = 0$ for all $U \in \mathcal{N}$, then by Proposition 4, $E \cap U$ is L-regular at a for all $U \in \mathcal{N}$ and thus E is locally L-regular at a. Conversely, suppose E is locally L-regular at a, and let U be an arbitrary open set containing a. Choose $r > 0$ such that $\overline{B(a,r)} \subset U$. Then $V^*_{E\cap B(a,r)}(a) = 0$ and therefore by Proposition 4,

$$h^*_{E\cap U,U}(a) \leq h^*_{E\cap B(a,r),U}(a) = 0. \qquad \square$$

We now describe Siciak's interpolating polynomials [Si, 4.1]. For a complex-valued function φ on a set S, we write

$$\|\varphi\|_E = \sup_{x\in E} |\varphi(x)| \leq +\infty$$

for $E \subset S$. We consider the polynomial ring $\mathcal{P} = \mathbf{C}[X_1,\ldots,X_N]$ and we let $\mathcal{P}_n$ denote the space of polynomials in $\mathcal{P}$ of total degree at most n. Let $m_n = \dim \mathcal{P}_n = \binom{N+n}{n}$, for $n = 0,1,2\ldots$. Let $\{e_1,e_2,e_3,\ldots\}$ denote the monomials in $\mathcal{P}$ ordered in such a way that $\deg e_j \leq \deg e_{j+1}$ $(j \geq 0)$ and therefore $\{e_1,\ldots,e_{m_n}\}$ is a basis for $\mathcal{P}_n$, for each $n \geq 0$. Define the polynomial functions $V_n : (\mathbf{C}^N)^{m_n} \to \mathbf{C}$ by

$$V_n(\xi^1,\ldots,\xi^{m_n}) = \det[e_j(\xi_k)] \qquad (1 \leq j,\ k \leq m_n), \tag{1.7}$$

for $\xi^1,\ldots,\xi^{m_n} \in \mathbf{C}^N$, $n \geq 0$. (Note that $V_0 = 1$.) If $\xi = (\xi^1,\ldots,\xi^{m_n}) \in (\mathbf{C}^N)^{m_n}$ such that $V_n(\xi) \neq 0$, we define the polynomials $L^j_n(\xi) \in \mathcal{P}_n$ by

$$L^j_n(\xi)(z) = \frac{V_n(\xi^1,\ldots,\xi^{j-1},z,\xi^{j+1},\ldots,\xi^{m_n})}{V_n(\xi)} \quad \text{for } 1 \leq j \leq m_n. \tag{1.8}$$

We have Siciak's "Lagrange interpolation formula"

$$f = \sum_{j=1}^{m_n} f(\xi^j) L_n^j(\xi) \qquad \text{for } f \in \mathcal{P}_n. \tag{1.9}$$

Now let E be a compact subset of $\mathbf{C}^N$, and suppose that E is not contained in any algebraic hypersurface. It easily follows that

$$\|V_n\|_{E^{m_n}} \neq 0$$

for $n \geq 0$ [Sil, 4.3]. For $n \geq 0$ choose $\xi_{(n)} \in E^{m_n}$ such that

$$|V_n(\xi_{(n)})| = \|V_n\|_{E^{m_n}}. \tag{1.10}$$

Let $\mathcal{F}(E)$ denote the space of complex-valued functions on E, and define the linear mappings $S_n : \mathcal{F}(E) \to \mathcal{P}_n$ by

$$S_n f = \sum_{j=1}^{m_n} f(\xi_{(n)}^j) L_n^j(\xi_{(n)}) \tag{1.11}$$

for $f \in \mathcal{F}(E)$, $n \geq 0$. We call (S_n) a *Siciak interpolation sequence* for E. Note that by (1.9), $S_n|\mathcal{P}_n$ is the identity. We define

$$\Phi_E(z) = \sup\{|f(z)|^{1/\deg f} : f \in \mathcal{P},\ \|f\|_E \leq 1\} \tag{1.12}$$

for $z \in \mathbf{C}^n$. Clearly, $\log \Phi_E \leq V_E$. (In fact, it is a theorem of Siciak [Sil, 4.12] that $\log \Phi_E = V_E$.) Note that

$$|f| \leq \|f\|_E \Phi_E \qquad \text{for } f \in \mathcal{P}_n. \tag{1.13}$$

It follows from (1.10) that

$$\|L_n^j(\xi_{(n)})\|_E = 1. \tag{1.14}$$

and therefore by (1.11)

$$|S_n f| \leq m_n \|f\|_E \Phi_E^n \qquad \text{for } f \in \mathcal{F}(E). \tag{1.15}$$

2. Separately holomorphic functions. In order to prove our result on "separately-analytic-holomorphic" functions (Theorem 1), we give a modification (Lemma 4) of a result of Siciak [Sil, Theorem 9.5] on separately holomorphic functions on special subsets of $\mathbf{C}^M \times \mathbf{C}^N$. As a consequence, we also give an optimal Hartogs theorem on separately holomorphic functions (Corollary 3).

We shall use the following notation: Suppose U and V are sets, $X \subset U \times V$ and $f : X \to \mathbf{C}$. We write

$$X_z = \{w \in V : (z,w) \in X\} \qquad \text{for } z \in U,$$

$$X^w = \{z \in U : (z,w) \in X\} \qquad \text{for } w \in V,$$

and we define the functions $f_z : X_z \to \mathbf{C}$, $f^w : X^w \to \mathbf{C}$ by

$$f_z(w) = f^w(z) = f(z,w) \qquad \text{for } (z,w) \in X.$$

We begin with the following elementary estimate for the rate of convergence of Siciak interpolating polynomials:

Lemma 1. *Let $f \in \mathcal{O}(B_1)$ and let $E \subset \bar{B}_r$, $r < 1$, where E is compact and is not contained in any algebraic hypersurface. Let $f_n = S_n f$, where $\{S_n\}$ is a Siciak interpolation sequence for E. Then*

$$\limsup_{n\to\infty} |f_n - f_{n-1}|^{1/n} \le r\Phi_E.$$

Proof. Let $s > r$ be arbitrary. Let

$$f(z) = \sum a_{j_1 \dots j_N} z_1^{j_1} \dots z_N^{j_N} = \sum a_J z^J \qquad (J = (j_1, \dots, j_N))$$

be the power series expansion of f, and let

$$p_n(z) = \sum_{|J| \le n} a_J z^J \tag{2.1}$$

for $n = 1,2,\dots$, where $|J| = j_1 + \cdots + j_n$. Then

$$\|p_n - p_{n-1}\|_E \le cs^n, \ \|f - p_n\|_E \le cs^n \tag{2.2}$$

for some positive constant c independent of n. Therefore by (1.15),

$$|f_n - p_n| = |S_n(f - p_n)| \le m_n \|f - p_n\|_E \Phi_E^n \le cm_n s^n \Phi_E^n. \tag{2.3}$$

Furthermore by (2.2) and (1.13)

$$|p_n - p_{n-1}| \le cs^n \Phi_E^n. \tag{2.4}$$

The conclusion follows from (2.3) and (2.4). □

Lemma 2. *Let $a \in E \subset U \subset \mathbf{C}^M$ where E is locally L-regular at a and U is open. Let $f \in \mathcal{O}(U \times B_r)$ where $0 < r < 1$, and suppose that f_z extends holomorphically to B_1, for all $z \in E$. Then there exists a neighborhood Ω of $\{a\} \times B_1$ and $\tilde{f} \in \mathcal{O}(\Omega)$ such that $\tilde{f} = f$ on $\Omega \cap (U \times B_r)$.*

Proof. Let $s < r < \rho < R < 1$. It suffices to find $\tilde{f} \in \mathcal{O}(U_0 \times B_\rho)$ when U_0 is a neighborhood of a. By shrinking U we can assume that $\|f\|_{U \times B_s} = K < +\infty$. We can write

$$f = \sum_{n=0}^{\infty} g^n \qquad \text{on } U \times B_r, \tag{2.5}$$

where $g^n \in \mathcal{O}(U \times \mathbf{C}^N)$ such that g_z^n is a homogeneous polynomial of degree n for $z \in U$. We then have

$$\|g^n\|_{B_1} \leq Ks^{-n} \tag{2.6}$$

for $n = 0,1,2,\ldots$. Since $f_z \in \mathcal{O}(B_1)$ for $z \in E$, we have

$$\|g_z^n\|_{B_1} R^n \to 0 \qquad \text{for } z \in E. \tag{2.7}$$

Let

$$u_n(z) = \frac{1}{n} \log \|g_z^n\|_{B_1} + \log R \tag{2.8}$$

for $z \in U$. Note that u_n is a (continuous) plurisubharmonic function on U, and by (2.7)

$$\limsup_{n\to\infty} u_n(z) \leq 0 \qquad \text{for } z \in E. \tag{2.9}$$

Furthermore, by (2.6)

$$u_n \leq c \qquad \text{for } n \geq 1, \tag{2.10}$$

where $c = \log K + \log \frac{R}{s}$. Let $\varepsilon = \frac{1}{3} \log \frac{R}{\rho}$, and define

$$E_n = \{z \in E : u_k(z) \leq \varepsilon \text{ for } k \geq n\}. \tag{2.11}$$

Then $E_n \nearrow E$ and hence by Proposition 3

$$h^*_{E_n U} \searrow h^*_{EU}. \tag{2.12}$$

Since E is locally L-regular at a, $h^*_{EU}(a) = 0$ and thus there exists a neighborhood $U_1 \subset U$ of a such that $h^*_{EU} < \frac{\varepsilon}{c}$ on U_1. Choose a neighborhood U_0 of a with $U_0 \subset\subset U_1$. Then by Harnack's principle there exists an integer n_0 such that

$$h^*_{E_n U}(z) < 2\frac{\varepsilon}{c} \qquad \text{for } z \in U_0,\ n \geq n_0. \tag{2.13}$$

By (2.10) and (2.11)

$$u_n \leq c h_{E_n U} + \varepsilon \tag{2.14}$$

and hence

$$u_n(z) < 3\varepsilon = \log \frac{R}{\rho} \qquad \text{for } z \in U_0,\ n \geq n_0. \tag{2.15}$$

Therefore

$$\|g^n\|_{U_0 \times B_1} \leq \frac{1}{\rho^n} \qquad \text{for } n \geq n_0, \tag{2.16}$$

which gives the convergence of (2.5) on $U_0 \times B_1$. □

Lemma 3. *Let a, E, U be as in Lemma 2 and let V_0, V be open sets in $\mathbf{C}^N$ with $V_0 \subset V$. Suppose $f \in \mathcal{O}(U \times V_0)$ such that f_z extends holomorphically to V for all $z \in E$. Then there exist a neighborhood Ω of $\{a\} \times V$ and $\tilde{f} \in \mathcal{O}(\Omega)$ such that $\tilde{f} = f$ on $\Omega \cap (U \times V_0)$.*

Proof. Suppose W is a connected open set with $W \subset\subset V$. It suffices to find a neighborhood U' of a and a function $f' \in \mathcal{O}(U' \times W)$ such that $f' = f$ on $U' \times (V_0 \cap W)$. One can easily find a sequence of open sets $V_1, \ldots, V_m$ such that $V_m \supset \bar{W}$ and

$$V_j = V_{j-1} \cup B(b_j, r_j) \tag{2.17}$$

where $b_j \in V_{j-1}$, $r_j > 0$, and $\overline{B(b_j,r)} \subset V$, for $1 \le j \le m$. We shall construct, by induction on j, connected neighborhoods U_j of a and functions $f_j \in \mathcal{O}(U_j \times V_j)$ such that $f_j = f$ on $U_j \times V_0$. We let $U_0 = U$ and $f_0 = f$. Suppose $1 \le j \le m$ and U_{j-1}, f_{j-1} are given. Choose $\varepsilon > 0$ such that $B(b_j, \varepsilon) \subset V_{j-1}$. Since $\overline{B(b_j,r_j)} \subset V$, it follows from Lemma 2 that there exists a connected neighborhood $U_j \subset U_{j-1}$ of a and a function $f' \in \mathcal{O}\big(U_j \times B(b_j,r)\big)$ agreeing with f_{j-1} on $U_j \times B(b_j,\varepsilon)$. We must show that $f' = f_{j-1}$ on $U_j \times [B(b_j,r_j) \cap V_{j-1}]$. Suppose $w \in B(b_j,r_j) \cap V_{j-1}$ is fixed and let $g \in \mathcal{O}(U_j)$ be given by

$$g(z) f'(z,w) - f_{j-1}(z,w). \tag{2.18}$$

Suppose $z \in E \cap U_j$ and let $\tilde{f}_z \in \mathcal{O}(V)$ denote the holomorphic extension of f_z. Then

$$f_{j-1}(z,w) = \tilde{f}_z(w) = f'(z,w) \tag{2.19}$$

and therefore $g(z) = 0$. Since $E \cap U_j$ is not pluripolar, it follows that $g = 0$. Therefore f' and f_{j-1} define $f_j \in \mathcal{O}(U_j \times V_j)$. Then $f' = f_m | U_m \times W$ is our desired function. □

We use Siciak's method of polynomial interpolation to extend Theorem 9.5 of [Si2] on separately holomorphic functions on $(U \times F) \cup (E \times V)$ to the case where E and F are not assumed to be L-regular:

Lemma 4. *Let $E \subset U \subset \mathbf{C}^M$, $F \subset V \subset \mathbf{C}^N$ where E and F are non-pluripolar, U and V are open, and V is connected. Let*

$$E' = \{z \in \bar{E} \cap U : E \textit{ is locally } L\textit{-regular at } z\}$$

and let $X = (U \times F) \cup (E \times V)$. Suppose $f : X \to \mathbf{C}$ such that $f_z \in \mathcal{O}(V)$ for all $z \in E$ and $f^w \in \mathcal{O}(U)$ for all $w \in F$. Then there exists a neighborhood Ω of $E' \times V$ and a function $\tilde{f} \in \mathcal{O}(\Omega)$ such that $\tilde{f} = f$ on $\Omega \cap X$.

Proof. Let $a \in E'$ be arbitrary. It suffices to extend f holomorphically to a neighborhood of $\{a\} \times V$. Let U' be a connected neighborhood of a with $U' \subset\subset U$. Let

$$F_n = \{w \in F : \|f^w\|_{U'} \leq n\} \tag{2.20}$$

for $n = 1,2,\ldots.$ Choose k such that F_k is not pluripolar and let $A = \bar{F}_k \cap V$. We define $\hat{f} : U' \times A \to \mathbf{C}$ as follows: Let $w_0 \in A$ be arbitrary. Choose a sequence $\{w_n\} \subset F_k$ such that $w_n \to w_0$ and let

$$\hat{f}(z,w_0) = \lim_{n \to \infty} f(z,w_n) \tag{2.21}$$

for $z \in U'$. Then $\hat{f}^{w_0} \in \mathcal{O}(U')$ and $\hat{f}^{w_0} = f^{w_0}$ on $E \cap U'$. Since $E \cap U'$ is not pluripolar, it follows that $\hat{f}^{w_0}$ does not depend on the choice of the sequence $\{w_n\}$. Thus $\hat{f}$ agrees with f on $X \cap (U' \times A)$, $\hat{f}^w \in \mathcal{O}(U')$ for $w \in A$, and $|\hat{f}| \leq k$ on $U' \times A$.

We can assume without loss of generality that $V \supset B_1$ and $A \cap \bar{B}_r$ is not pluripolar, for some $r < 1$. By replacing E with $E \cap U'$, U with U', and F with $A \cap \bar{B}_r$, we can assume without loss of generality that F is a closed subset of $\bar{B}_r$ and f is bounded on $U \times F$.. We first construct a neighborhood U_0 of a, an open set $W \subset B_1 \subset V$ such that $F \cap W$ is not pluripolar, and a holomorphic function $\hat{f}$ on $U_0 \times W$ such that $\hat{f} = f$ on $X \cap (U_0 \times W)$:

Let $\{S_n\}$ be a Siciak interpolation sequence for F given by (1.10) and (1.11) (with E replaced by F). Define $f_n \in \mathcal{O}(U \times \mathbf{C}^N)$ by

$$f_n(z,w) = (S_n f_z)(w) \tag{2.22}$$

for $n \geq 0$. Let $Q_n = f_n - f_{n-1}$, for $n \geq 1$, and set $Q_0 = f_0$. Let $K = \|f\|_{U \times F} < +\infty$. By (1.14) we have

$$|(f_n)_z| \leq m_n K \Phi_F^n. \tag{2.23}$$

Choose a point $w_0 \in F$ such that F is locally L-regular at w_0 and therefore

$$\Phi_F^*(w_0) = 1. \tag{2.24}$$

Choose a neighborhood $W \subset\subset V$ of w_0 such that $\|\Phi_F\|_W < \frac{1}{r}$, and let

$$\gamma = r\|\Phi_F\|_W < 1. \tag{2.25}$$

The set $F \cap W$ is not pluripolar since it is locally L-regular at w_0. By (2.23)

$$\|(Q_n)_z\|_W \leq 2m_n K r^{-n} \tag{2.26}$$

for all $z \in U$. By (2.25) and Lemma 1,

$$\limsup_{n\to\infty} \|(Q_n)_z\|_W^{1/n} \le \gamma \qquad \text{for all } z \in E. \tag{2.27}$$

Furthermore, by the proof of Lemma 1 (see (2.3)),

$$\|(f_n)_z - f_z\|_W \to 0 \qquad \text{for all } z \in E. \tag{2.28}$$

Let

$$u_n(z) = \sup_{k\ge n,\, w\in W} \frac{1}{k}\log|Q_k(z,w)| = \sup_{k\ge n}\frac{1}{k}\log\|(Q_k)_z\|_W. \tag{2.29}$$

Let $a = -\log r + 1$, $b = -\frac{1}{2}\log\gamma$. By (2.26) there exists $n_0 \in \mathbf{Z}$ such that

$$u_n(z) \le a \qquad \text{for } z \in U,\ n \ge n_0. \tag{2.30}$$

Let

$$E_n = \{z \in E : u_n(z) \le -b\}, \tag{2.31}$$

for $n = 1,2,\ldots.$ Then by (2.30)

$$u_n + b \le (a+b)h_{E_nU} \qquad \text{for } n \ge n_0. \tag{2.32}$$

By (2.27)

$$\begin{aligned}\lim_{n\to\infty} u_n(z) &= \limsup_{n\to\infty}\frac{1}{n}\log\|(Q_n)_z\|_W \\ &\le \log\gamma = -2b \qquad \text{for } z \in E,\end{aligned} \tag{2.33}$$

and thus

$$E_n \nearrow E. \tag{2.34}$$

By Proposition 3,

$$h^*_{E_nU} \searrow h^*_{EU}. \tag{2.35}$$

By Corollary 2, $h^*_{EU}(a) = 0$. Let

$$U' = \{z \in U : h^*_{EU} < \delta\}, \tag{2.36}$$

where

$$\delta = \frac{b}{4(a+b)}. \tag{2.37}$$

Let U_0 be a connected neighborhood of a such that $U_0 \subset\subset U'$. We shall show that $\sum Q_n$ converges uniformly on $U_0 \times W$. By Harnack's principle, there exists $n_1 \geq n_0$ such that

$$h^*_{E_n U}(z) < 2\delta \qquad \text{for } z \in U_0,\ n \geq n_1. \tag{2.38}$$

Thus by (2.32)

$$u_n(z) \leq 2\delta(a+b) - b = -\frac{b}{2} \qquad \text{for } z \in U_0,\ n \geq n_1.$$

Therefore, recalling (2.29)

$$\|Q_n\|_{U_0 \times W} \leq \gamma^{n/4} \qquad \text{for } n \geq n_1. \tag{2.39}$$

Thus $\sum Q_n$ converges uniformly on $U_0 \times W$ and we can define

$$\hat{f} = \sum_{n=0}^{\infty} Q_n = \lim_{n\to\infty} f_n \in \mathcal{O}(U_0 \times W). \tag{2.40}$$

By (2.28) $\hat{f} = f$ on $(E \cap U_0) \times W$. To show that $\hat{f} = f$ on $U_0 \times (F \cap W)$, we let $w \in F \cap W$ be arbitrary. The functions f^w and $\hat{f}^w$ are holomorphic on U_0 and agree on $E \cap U_0$. Since $E \cap U_0$ is not pluripolar, it follows that $\hat{f}^w = f^w | U_0$. Therefore $\hat{f} = f$ on $X \cap (U_0 \times W)$.

To complete the proof of the lemma, we apply Lemma 3 (with E, U, V_0 replaced by $E \cap U_0$, U_0, W) to extend $\hat{f}$ to a holomorphic function $\tilde{f}$ on a neighborhood Ω of $\{a\} \times V$. One easily verifies that $\hat{f} = f$ on $X \cap \Omega$. □

The following improvement of Terada's version of Hartogs' Theorem [Te, Proposition 2] is a consequence of Lemma 4.

Corollary 3. *Let* $f : U \times V \to \mathbf{C}$ *where* U, V *are domains in* $\mathbf{C}^M$, $\mathbf{C}^N$ *respectively. If* $f_z \in \mathcal{O}(V)$ *for almost all* $z \in U$ *and* $f^w \in \mathcal{O}(U)$ *for all* w *in a non-pluripolar subset of* V, *then* f *is equal a.e. to a holomorphic function on* $U \times V$.

Proof. Let E denote the set of $z \in U$ such that $f_z \in \mathcal{O}(V)$, and F the set of $w \in V$ with $f^w \in \mathcal{O}(U)$. Since $U - E$ has Lebesque measure zero, $h_{EU} \equiv 0$ and thus the set E' of Lemma 4 coincides with U. Thus by Lemma 4, there exists $\tilde{f} \in \mathcal{O}(U \times V)$ such that $\tilde{f} = f$ on $E \times V$. □

Theorem 1. *Let $f : \Omega \times V \to \mathbf{C}$ where Ω and V are domains in $\mathbf{R}^M$ and $\mathbf{C}^N$ respectively, and let A be a non-pluripolar subset of V. If $f_x \in \mathcal{O}(V)$ for all $x \in \Omega$ and f^w is real-analytic on Ω for all $w \in A$, then f is real-analytic on $\Omega \times V$.*

Proof. Let Ω_0 be an arbitrary relatively-compact open subset of Ω. Let

$$U_n = \left\{ z \in \mathbf{C}^M : d(z,\Omega_0) < \frac{1}{n} \right\}, \tag{2.41}$$

$$A_n = \{ w \in A : f^w \text{ extends holomorphically to } U_n \}, \tag{2.42}$$

for $n = 1,2,\ldots$. Then $A_n \nearrow A$. Choose k such that A_k is not pluripolar and let $(f^w)\tilde{} \in \mathcal{O}(U_k)$ denote the holomorphic extension of f^w for $w \in A_k$. Write $X = (U_k \times A_k) \cup (\Omega_0 \times V)$. Let $f' : X \to \mathbf{C}$ be given by $f' = f$ on $\Omega_0 \times V$, and $f'(z,w) = (f^w)\tilde{}(z)$ for $z \in U_k$, $w \in A_k$. Since Ω_0 is locally L-regular (in $\mathbf{C}^M$) at each of its points (see for example [Si, 6.8]), it follows from Lemma 4 applied to f' that f extends to a holomorphic function on a neighborhood of $\Omega_0 \times V$ and hence is real-analytic on $\Omega_0 \times V$. □

Remark. Theorem 1 is false if "real-analytic" is replaced by "$\mathcal{C}^\infty$." For example, define $f : \mathbf{R} \times \mathbf{C} \to \mathbf{C}$ as follows: Choose $\varphi \in \mathcal{C}^\infty(\mathbf{R})$ such that $\operatorname{Supp}\varphi \subset [1,3]$, $\varphi \geq 0$, and $\varphi(2) = 1$. Let

$$f(x,w) = \sum_{n=1}^{\infty} \varphi(nx) w^n \tag{2.43}$$

for $x \in \mathbf{R}$, $w \in \mathbf{C}$. For each $x \in \mathbf{R}$, f_x is a polynomial and thus is an entire function. Also, one easily checks that $f^w \in \mathcal{C}^\infty(\mathbf{R})$ for $|w| < 1$. However, f^1 is not continuous at 0 since $f(0,1) = 0$ but $f\left(\frac{1}{n},1\right) \geq 1$ for all positive integers n.

3. Separately meromorphic functions. Rothstein [R] proved in 1950 that Hartog's Theorem is valid as well for separately meromorphic functions. Extensions of Siciak's results to meromorphic functions were given by Kazarian [Ka1], [Ka2]. The author [Sh, Theorem 1] generalized Rothstein's Theorem by showing that if a function is meromorphic in each variable for almost all values of the other variables, then it is meromorphic. We conclude by giving a generalization of Corollary 3 to meromorphic functions:

Theorem 2. *Let U and V be domains in $\mathbf{C}^M$ and $\mathbf{C}^N$ respectively, and let $f : U \times V - I \to \mathbf{C}$ where $I \subset U \times V$. Suppose that*

- (i) *I_z is pluripolar (in $\mathbf{C}^N$) and f_z extends meromorphically to V for almost all $z \in U$;*
- (ii) *there exists a non-pluripolar set A in V such that I^w has Lebesgue measure 0 (in $\mathbf{C}^M$) and f^w extends meromorphically to U for all $w \in A$.*

Then there exists a meromorphic function $\tilde{f}$ on $U \times V$ such that $\tilde{f} = f$ a.e. on $U \times V - I$.

Proof. Let G be the set of full measure in V such that (i) is valid for all $z \in G$. By replacing I with $I \cup [(U - G) \times V]$ we can assume without loss of generality that $I_z = V$ for all $z \in U - G$. We shall show in four steps that f extends to a meromorphic function on $U \times V$:

▷ *Step 1.*

We construct $E \subset U_1 \subset U$, $V_1 \subset V$, $f \subset A \cap V_1$, where U_1, V_1 are open and E, F are non-pluripolar, and $f' : X \to \mathbf{C}$, where $X = (E \times V_1) \cup (U_1 \times F)$, such that f' is separately holomorphic:

By the method of proof of Lemma 2 in [Sh], there exist a constant K, an open set $U_1 \subset U$ and a non-pluripolar set $A_1 \subset A$ such that f^w has an extension $(f^w)\tilde{} \in \mathcal{O}(U_1)$ with $|(f^w)\tilde{}| < K$ on U_1 for all $w \in A_1$. Furthermore, there exist $K' \geq K$, $Q \subset G \cap U_1$ and $V_1 \subset V$ such that Q has positive Lebesgue outer measure, $A_1 \cap V_1$ is not pluripolar, and f_z has an extension $(f_z)\tilde{} \in \mathcal{O}(V_1)$ with $|(f_z)\tilde{}| < K'$ on V_1 for all $z \in Q$. Let E be the set of points $z \in Q$ such that $Q \cap D$ has positive Lebesgue outer measure for all neighborhoods D of z. Then E also has positive Lebesgue outer measure and therefore is not pluripolar. Let $F = A_1 \cap V_1$. It remains to show that

$$(f_z)\tilde{}(w) = (f^w)\tilde{}(z) \qquad \text{for } (z,w) \in (E \times F) \cap I. \tag{3.1}$$

Let $(z,w) \in (E \times F) \cap I$. Let $\{w_n\}$ be a countable dense subset of $F - I_z$ and choose a sequence

$$\{z_n\} \subset Q - I^w - \bigcup_{m=1}^{\infty} I^{w_m}$$

such that $z_n \to z$. Then $(z_n,w_m) \in Q \times V_1 - I$ for m, $n \geq 1$ and thus $(f_{z_n})\tilde{} \to (f_z)\tilde{}$ on V_1. Therefore

$$(f_z)\tilde{}(w) = \lim_{n\to\infty} (f_{z_n})\tilde{}(w) = \lim_{n\to\infty} f(z_n,w) = (f^w)\tilde{}(z), \tag{3.2}$$

which completes Step 1.

▹ *Step 2.*

We construct $U_2 \subset U_1$, $V_2 \subset V_1$ and $\hat{f} \in \mathcal{O}(U_2 \times V_2)$ such that $E \cap U_1$ and $F \cap V_2$ are not pluripolar and $\hat{f} = f$ on $U_2 \times V_2 - I$:

Choose $V_2 \subset\subset V_1$ such that $F \cap V_2$ is not pluripolar. Let $a \in E$ such that E is not locally L-regular at a. Then by Lemma 4, there exist a neighborhood U_2 of a and a function $\hat{f} \in \mathcal{O}(U_2 \times V_2)$ such that $\hat{f} = f'$ on $(U_2 \times V_2) \cap X$. It easily follows that $\hat{f} = f$ on $U_2 \times V_2 - I$.

▹ *Step 3.*

Let $U' \subset\subset U$ be an arbitrary open set. We construct $V_3 \subset\subset V_2$ such that $F \cap V_3$ is not pluripolar and a meromorphic function $\hat{f}$ on $U' \times V_3$ such that $\hat{f} = f$ on $U' \times V_3 - I$.

We obtain Step 3 as a consequence of Step 2 and the following analogue of Lemma 3 for meromorphic functions:

Lemma 5. *Let U_0, U be open sets in $\mathbf{C}^M$ with $U_0 \subset U$, and let $F \subset V \subset \mathbf{C}^N$ where F is non-pluripolar and V is open. Let $f \in \mathcal{O}(U_0 \times V)$ and suppose that f^w extends meromorphically to U for all $w \in F$. Then for all open $U' \subset\subset U$ there exist an open set $W \subset V$ such that $F \cap W$ is non-pluripolar and a meromorphic function g on $U' \times W$ such that $g = f$ on $(U_0 \cap U') \times W$.*

Proof. By the argument in the proof of Lemma 3, it suffices to consider the case where U_0 and U are polydisks. By induction it suffices to consider the case $M = 1$. The conclusion for this case follows from the argument in [HS, Theorem 2.9] or [Ka2, Lemma 9] using Proposition 1 as in the proof of Lemma 4. □

▹ *Step 4.*

We construct a meromorphic function $\tilde{f}$ on $U \times V$ such that $\tilde{f}|(U \times V - I) = f$:

The function $\hat{f}$ from Step 3 has the property that $\hat{f}_z$ extends meromorphically to V for all $z \in G$. It follows from Rothstein's theorem [R] (see [HS, Theorem 2.9]) and induction on N as in Step 3 that $\hat{f}$ extends meromorphically to $U' \times V$. Since $U' \subset\subset U$ is arbitrary it follows that $\hat{f}$ has a meromorphic extension $\tilde{f}$ to $U \times V$. □

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