Synthetic Projective Geometry and Poincaré's Theorem on Automorphisms of the Ball

by

BERNARD SHIFFMAN[1]

1. Introduction.

Let B_n denote the unit ball in $\mathbb{C}^n$. In 1907, Poincaré [Po] showed that any nonconstant holomorphic map f from a neighborhood $U \subset \mathbb{C}^2$ of a point $z_0 \in \partial B_2$ into $\mathbb{C}^2$ which maps $U \cap \partial B_2$ into ∂B_2 must be the restriction of an element of the Möbius group of automorphisms of B_2. This result was generalized to n variables by Tanaka [Ta] and was given new proofs by Pelles [Pe], Alexander [Al], Rudin [Ru], and others, and recently by Chern and Ji [CJ]. Chern and Ji considered the "Segre family" of ∂B_n,

$$\mathcal{M}_{B_n} = \left\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n : \textstyle\sum_{j=1}^n z_j w_j = 1\right\} ,$$

and showed that if $(z_0, w_0) \in \mathcal{M}_{B_n}$ and if f, g are nondegenerate holomorphic maps from neighborhoods U, V of z_0, w_0, respectively, into $\mathbb{C}^n$ such that $f \times g$ maps $\mathcal{M}_{B_n} \cap (U \times V)$ into $\mathcal{M}_{B_n}$, then both f and g are restrictions of elements of the Möbius group [CJ, Theorem 2]. The Poincaré-Tanaka theorem follows easily from this result by considering the point $(z_0, \bar{z}_0) \in \mathcal{M}_{B_n}$ and taking $g(w) = \overline{f(\bar{w})}$ (see §3). The method of Segre families was also used in this context by S. Webster [We], who showed that local holomorphic maps of nondegenerate real-algebraic hypersurfaces in $\mathbb{C}^n$ are algebraic.

In this paper, we show how the methods of Desarguesian projective geometry provide an elementary proof of the Chern-Ji theorem. Since our methods are "synthetic," we do not use any differential geometry, and apart from some complex analysis used in the proof of the Poincaré-Tanaka theorem, our proofs use only linear algebra and point-set topology and are self-contained (except for the omission of the proofs of the fundamental theorems of Desargues and Pappus, which can be found in most texts on plane projective geometry, e.g., [Co]). In fact we show (Theorem 6) that the Chern-Ji theorem extends to the case of continuous f, g (where the conclusion holds either for f, g or for their conjugates). Our method is based on the principle that a continuous local self-map of real or complex projective space is projective-linear or anti-projective-linear (in the complex case) if it maps each line in a sufficiently large family $\mathcal{L}_0$ of lines into a line. For the case of the real projective plane $\mathbb{P}^2_{\mathbb{R}}$, this principle was stated by Blaschke and his co-workers in the 1920s (see [BB, p. 91]) when $\mathcal{L}_0$ is a "4-web;" i.e., $\mathcal{L}_0$ consists of four pairwise transversal families of lines, each covering

[1]Research partially supported by National Science Foundation Grant No. DMS-9204037.

the domain of the map. A complete proof of this fact was given in 1935 by W. Prenowitz [Pr] (see also [Re]). We give a simple proof of this principle for the case where $\mathcal{L}_0$ is an open set in the Grassmannian of projective lines in real or complex projective n-space (Theorem 3).

Various other results on extending local collineations have appeared in the literature. For example, E. Cartan [Ca] showed that a self-map of the boundary of the 2-ball B_2 that takes any linear section in ∂B_2 into a complex line must be either projective-linear or anti-projective-linear. Rado (see [Ra]) observed that a collineation on any subset of a projective plane $\mathbb{P}^2_K$ (over any field K) that contains three generic lines and a generic point extends to a collineation of the entire projective plane. Mok and Yeung [MY, pp. 257-258] showed that local holomorphic collineations are projective-linear; a generalization of this result to biholomorphisms of complex manifolds preserving the geodesics of a projective connection was recently given by Molzon and Mortensen [MM, Theorem 9.1]. Some applications of Blaschke's theory of webs to algebraic geometry can be found in Chern-Griffiths [CG]. (For an overview of the theory of webs, see [Go].) Also, the Poincaré-Tanaka theorem was generalized by Alexander and Rudin to the case where f is a holomorphic map from a domain $\Omega \subset B_n$ whose boundary contains an open subset of ∂B_n onto a similar domain. Alexander [Al] showed that if f has a C^∞ extension to $\bar{\Omega}$ that maps $\bar{\Omega} \cap \partial B_n$ into ∂B_n, then f extends to an automorphism of B_n; Rudin [Ru, Theorem 15.3.4] replaced Alexander's hypothesis by a much weaker condition that is satisfied, for example, when f has a continuous extension to $\bar{\Omega}$ mapping $\bar{\Omega} \cap \partial B_n$ into ∂B_n. (For discussions of related results, see [Fo, pp. 325-326] and [Ru, §15.3].)

Acknowledgements: I would like to thank Valery Alexeev, Shiing-Shen Chern, Shanyu Ji and Sid Webster for their helpful suggestions and references to the literature.

2. The local collineation theorem.

In this section, we show that continuous local collineations of real or complex projective space are projective-linear or anti-projective-linear (Theorem 3). Our methods involve using Desargues' Theorem to extend to a global collineation and then applying the fundamental description of collineations over an arbitrary field (Proposition 1).

We let $\mathcal{L}^n_K$ denote the set of projective lines in projective n-space $\mathbb{P}^n_K$ over a field K. (We are interested here in the cases $K = \mathbb{R}$ or $\mathbb{C}$.) Note that $\mathcal{L}^n_K$ can be identified with the Grassmannian of 2-dimensional subspaces of K^{n+1}. A *collineation* on $\mathbb{P}^n_K$ is a bijective self-map $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ such that $f(L) \in \mathcal{L}^n_K$ for all $L \in \mathcal{L}^n_K$. Examples of collineations on $\mathbb{P}(K^{n+1})$ are provided by elements of the projective linear group $\mathrm{PGL}(n+1, K) = \mathrm{GL}(n+1, K)/(K \setminus \{0\})$ However, these are not the only collineations. We let the group $\mathrm{Gal}(K)$ of automorphisms of K (the Galois group of K over its prime field, $\mathbb{Z}_p$ or $\mathbb{Q}$) act on $\mathbb{P}^n_K$ by

$$g(z) = (gz_0 : \ldots : gz_n) \quad \text{for} \quad g \in \mathrm{Gal}(K)\,,\ z = (z_0 : \cdots : z_n) \in \mathbb{P}^n_K\ ;$$

then elements of $\mathrm{Gal}(K)$ also give collineations on $\mathbb{P}^n_K$. The following well-known result (see [Ar, Theorem 2.26]) states that these examples provide all the collineations on $\mathbb{P}^n_K$:

Proposition 1. *Let $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be a collineation, where $n \geq 2$ and K is an arbitrary field. Then there exist a unique $A \in \mathrm{PGL}(n+1, K)$ and a unique $g \in \mathrm{Gal}(K)$ such that $f = g \circ A$.*

We shall use of the following immediate consequence of Proposition 1:

Corollary 2. *Let $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be a collineation, where $K = \mathbb{R}$ or $\mathbb{C}$, $n \geq 2$. Suppose f is continuous on a nonempty open subset of $\mathbb{P}^n_K$. If $K = \mathbb{R}$, then $f \in \mathrm{PGL}(n+1, \mathbb{R})$. If $K = \mathbb{C}$, then either f or $\bar{f}$ is in $\mathrm{PGL}(n+1, \mathbb{C})$.*

We let $\langle a_1, \ldots, a_m \rangle$ denote the projective linear subspace of $\mathbb{P}^n_K$ determined by the points $a_1, \ldots, a_m \in \mathbb{P}^n_K$. In particular, $\langle a, b \rangle$ is the projective line through a and b (for $a \neq b \in \mathbb{P}^n_K$). We also let a denote the one-point set $\langle a \rangle = \{a\}$. We now give a short proof of Proposition 1. First we need two well-known, elementary lemmas:

Lemma (a). *Let $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be a collineation. If $a_1, \ldots, a_m$ are points in general position in $\mathbb{P}^n_K$, then $f(a_1), \ldots, f(a_m)$ are in general position and $f(\langle a_1, \ldots, a_m \rangle) = \langle f(a_1), \ldots, f(a_m) \rangle$.*

Proof: It suffices to consider $m \leq n+1$. If $m = 1$ the conclusion is just the definition of a collineation. So let $2 \leq m \leq n+1$ and assume by induction that the lemma has been verified for $m-1$ points. We write $f(a) = \hat{a}$. Since $f(\langle a_1, \ldots, a_{m-1} \rangle) = \langle \hat{a}_1, \ldots, \hat{a}_{m-1} \rangle$ and f is injective, it follows that $\hat{a}_m \notin \langle \hat{a}_1, \ldots, \hat{a}_{m-1} \rangle$ and thus $\hat{a}_1, \ldots, \hat{a}_m$ are in general position. The second conclusion follows from the fact that $\langle \hat{a}_1, \ldots, \hat{a}_m \rangle$ is the union of lines $\langle \hat{a}_m, b \rangle$, where b runs through the points of $\langle \hat{a}_1, \ldots, \hat{a}_{m-1} \rangle$. $\square$

Lemma (b). *Let $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be a collineation. If there exists a line $L \in \mathcal{L}^n_K$ such that $f|_L : L \to f(L)$ is projective-linear, then $f \in \mathrm{PGL}(n+1, K)$.*

Proof: Let $\widetilde{e_j} = (0, \ldots, \overset{j\text{-th}}{1}, \ldots, 0) \in K^{n+1}$, $0 \leq j \leq n$, $\widetilde{\delta} = \widetilde{e_0} + \cdots + \widetilde{e_n}$, and let $e_0, \ldots, e_n, \delta$ be the corresponding points in $\mathbb{P}^n_K$. Let $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be as in the hypothesis; we can assume without loss of generality that $f|_{\langle e_0, e_1 \rangle}$ is projective-linear. By Lemma (a), the points $f(e_0), \ldots, f(e_n), f(\delta)$ are in general position. Choose representatives $\widetilde{f(e_0)}, \ldots, \widetilde{f(e_n)}, \widetilde{f(\delta)}$ in $K^{n+1} \setminus \{0\}$ of $f(e_0), \ldots, f(e_n), f(\delta)$ respectively. Let $\lambda_j \in K \setminus \{0\}$ $(0 \leq j \leq n)$ be given by $\sum \lambda_j \widetilde{f(e_j)} = \widetilde{f(\delta)}$, and let $T \in GL(n+1, K)$ be given by $T(\widetilde{e_j}) = \lambda_j \widetilde{f(e_j)}$. Then $T(\widetilde{\delta}) = \sum \lambda_j \widetilde{f(e_j)} = \widetilde{f(\delta)}$.

Let $\varphi = T^{-1} \circ f$. Thus the lemma is reduced to the following statement:

(A_n) *Let $\varphi : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be a collineation such that $\varphi|_{\langle e_0, e_1 \rangle}$ is projective-linear, $\varphi(e_j) = e_j$ $(0 \leq j \leq n)$, and $\varphi(\delta) = \delta$. Then φ is the identity.*

We verify (A_n) by induction on n. For $n = 1$ the conclusion is immediate. So let $n \geq 2$ and assume (A_{n-1}). We write $\mathbb{P}^{n-1}_K = \langle e_0, \ldots, e_{n-1} \rangle$ and let $\delta' = (1 : \ldots : 1 : 0) \in \mathbb{P}^{n-1}_K$; thus $\langle e_n, \delta \rangle \cap \mathbb{P}^{n-1}_K = \{\delta'\}$. By Lemma (a), $\varphi(\mathbb{P}^{n-1}_K) = \mathbb{P}^{n-1}_K$ and thus $\varphi(\delta') = \delta'$. Hence by ($\mathrm{A}_{n-1}$), φ is the identity on $\mathbb{P}^{n-1}_K$. If a line $L \in \mathcal{L}^n_K$ contains a point $b \notin \mathbb{P}^{n-1}_K$ such that $\varphi(b) = b$, then $\varphi(L) = L$, since L must contain another fixed point of φ in $\mathbb{P}_K^{n-1}$. Let $a \in \langle e_0, e_n \rangle$, $a \neq e_0$, be arbitrary. Since $\{a\} = \langle a, \delta \rangle \cap \langle e_0, e_n \rangle$ and the points δ, e_n are fixed by φ, it follows that $\varphi(\langle a, \delta \rangle) = \langle a, \delta \rangle$ and $\varphi(\langle e_0, e_n \rangle) = \langle e_0, e_n \rangle$ and thus $\varphi(a) = a$. Finally, let $x \in \mathbb{P}_K^n \setminus \langle e_0, e_n \rangle$ be arbitrary. Since $\{x\} = \langle a, x \rangle \cap \langle e_n, x \rangle$, where a is as above and φ fixes a, e_n, it follows as before that $\varphi(x) = x$. $\square$

Proof of Proposition 1: Consider the usual embeddings $\mathbb{P}_K^1 \subset \mathbb{P}_K^2 \subset \mathbb{P}_K^n$. By Lemma (a), $f(\mathbb{P}_K^2)$ is a projective 2-plane. Hence there exists a projective linear map $T : f(\mathbb{P}_K^2) \to \mathbb{P}_K^2$ such that the map $f' = T \circ f|_{\mathbb{P}_K^2} : \mathbb{P}_K^2 \to \mathbb{P}_K^2$ leaves the points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and $(1:1:1)$ fixed. Then, for each $a \in K$, we can write $f'(1:a:0) = (1:\hat{a}:0)$, where $\hat{a} \in K$. We observe that the map $a \mapsto \hat{a}$ is an element of $\mathrm{Gal}(K)$. This follows from the fact that if $a, b \in K$, then $a-b$ and a/b can be constructed from the following "projective straight edge" constructions:

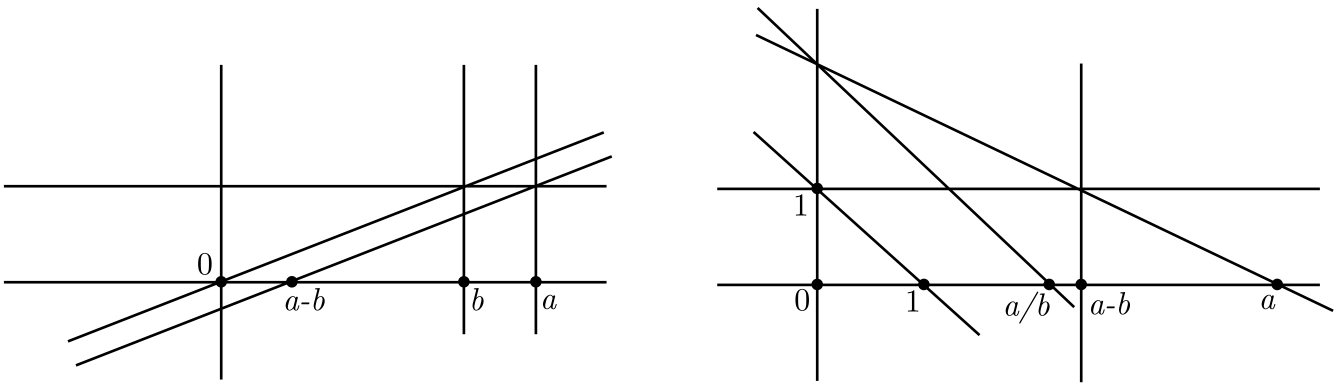

Figure 0

(Figure 0 shows the affine plane $K^2 \subset \mathbb{P}_K^2$.) Let $g \in \mathrm{Gal}(K)$ with $g(a) = \hat{a}$. Then $f' \circ g^{-1}|_{\mathbb{P}_K^1}$ is the identity map, and it follows that the map $f \circ g^{-1}|_{\mathbb{P}_K^1} : \mathbb{P}_K^1 \to f(\mathbb{P}_K^1)$ is projective-linear. Therefore by Lemma (b), $f \circ g^{-1} = A' \in \mathrm{PGL}(n+1, K)$, and thus $f = A' \circ g = g \circ A$, where $A = g^{-1} A' g \in \mathrm{PGL}(n+1, K)$. $\square$

For a subset $U \subset \mathbb{P}_K^n$, we write

$$\mathcal{L}(U) = \{L \in \mathcal{L}_K^n : L \cap U \neq \emptyset\} .$$

We give the projective spaces $\mathbb{P}_{\mathbb{R}}^n, \mathbb{P}_{\mathbb{C}}^n$ and the Grassmannians $\mathcal{L}_{\mathbb{R}}^n, \mathcal{L}_{\mathbb{C}}^n$ the usual metric topologies. The main result of this section gives a condition for a local collineation to be projective-linear:

Theorem 3. *Let U be a connected open set in $\mathbb{P}_K^n$ $(n \geq 2)$, where K denotes either $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{L}_0$ be an open subset of $\mathcal{L}(U)$ such that $\bigcup \mathcal{L}_0 \supset U$. Suppose that $f : U \to \mathbb{P}_K^n$ is a continuous injective map such that $f(L \cap U)$ is contained in a projective line for all $L \in \mathcal{L}_0$. Then there exists $A \in \mathrm{PGL}(n+1, K)$ such that*

(i) $f = A|_U$, *if* $K = \mathbb{R}$,

(ii) $f = A|_U$ *or* $\bar{f} = A|_U$, *if* $K = \mathbb{C}$.

The case $K = \mathbb{R}$ of Theorem 3 follows easily from Prenowitz's theorem [Pr, Theorem V], which provides a much stronger result for $n = 2$. (We include an elementary proof of the case $K = \mathbb{R}$ below.)

We begin by proving the following weaker form of Theorem 3:

Lemma 4. *Let U be an open set in $\mathbb{P}^n_K$ $(n \geq 2)$, where K denotes either $\mathbb{R}$ or $\mathbb{C}$, and let $f : U \to \mathbb{P}^n_K$ be a continuous injective map. If $f(L \cap U)$ is contained in a projective line for all $L \in \mathcal{L}(U)$, then the conclusion of Theorem 3 holds.*

Proof: Let $f : U \to \mathbb{P}^n_K$ be as in the statement of the lemma, and let $f(U) = \widehat{U}$. We write $\hat{a} = f(a)$ for $a \in U$. Note that if three points a_1, a_2, a_3 of U are not collinear, then $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are not collinear, since otherwise the sets $f(\langle a_1, a_2\rangle \cap U)$ and $f(\langle a_1, a_3\rangle \cap U)$ would both be neighborhoods of a_1 in the line $\langle \hat{a}_1, \hat{a}_2\rangle$ and hence f would not be injective. We also observe that if $L = \langle a, b\rangle$, where a, b are distinct points of U, then by hypothesis, $f(L \cap U) \subset \langle \hat{a}, \hat{b}\rangle$, and in fact we have $f(L \cap U) = \langle \hat{a}, \hat{b}\rangle \cap \widehat{U}$. To verify this equality, let $\chi \in \langle \hat{a}, \hat{b}\rangle \cap \widehat{U}$ be arbitrary and write $\chi = \hat{x}$, where $x \in U$. Since $\hat{a}, \hat{b}, \hat{x}$ are collinear, it follows from the above that x, a, b are collinear and thus $x \in L$.

We first consider the case $n = 2$. Choose a connected open set $U_0 \subset U$. Let $x \in \mathbb{P}^2_K$. We want to define $\hat{x} = \tilde{f}(x)$. Choose $a, b \in U_0$ such that a, b, x are not collinear. Let $\widehat{L}_a, \widehat{L}_b \in \mathcal{L}(\widehat{U})$ be given by $f(\langle a, x\rangle \cap U) = \widehat{L}_a \cap \widehat{U}$, $f(\langle b, x\rangle \cap U) = \widehat{L}_b \cap \widehat{U}$. We define $\hat{x}(a, b) \in \mathbb{P}^2_K$ by

$$\widehat{L}_a \cap \widehat{L}_b = \hat{x}(a, b)\ .$$

(Note that $\widehat{L}_a \neq \widehat{L}_b$ since $\langle a, x\rangle \neq \langle b, x\rangle$ and f is injective.)

We observe that if $a' \in \langle a, x\rangle \cap U_0$, $b' \in \langle b, x\rangle \cap U_0$ with $a' \neq a$, $b' \neq b$, then

$$\hat{x}(a, b) = \langle \hat{a}, \hat{a}'\rangle \cap \langle \hat{b}, \hat{b}'\rangle\ .$$

In particular if $x \in U$, then

$$\hat{x}(a, b) = \langle \hat{a}, \hat{x}\rangle \cap \langle \hat{b}, \hat{x}\rangle = \hat{x}\ .$$

Step 1. *$\hat{x}(a, b)$ is independent of the choice of $a, b \in U_0$.*

We can assume by the above that $x \notin U$. Let $a \in U_0$ and let $b_0, b_1 \in U_0 \setminus \langle a, x\rangle$ be arbitrary. It suffices to show that $\hat{x}(a, b_0) = \hat{x}(a, b_1)$.

We first consider the case $K = \mathbb{C}$. Let C be a real curve from b_0 to b_1 in $U_0 \setminus \langle a, x\rangle$. Let $\varepsilon > 0$, and suppose that b_2, b_3 are points in C such that $\mathrm{dist}(b_2, b_3) < \varepsilon$ (with respect to some metric on $\mathbb{P}^2_{\mathbb{C}}$ defining the usual topology). Choose $a', a'' \in \langle a, x\rangle \cap U_0$ with a, a', a'' distinct. Then let

$$b_3',\ b_3'',\ b_2',\ c,\ b_2''$$

be constructed (in the above order) as in Figure 1 below.

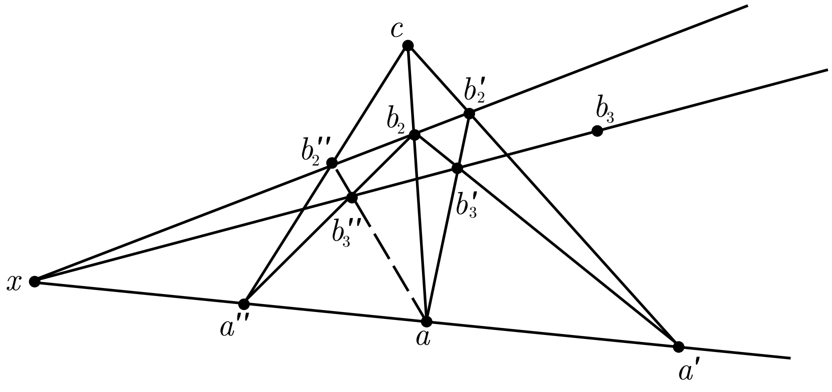

Figure 1

We claim that a, b_2'', b_3'' are collinear: Let $b_3^\star = \langle a, b_2'' \rangle \cap \langle a'', b_2 \rangle$; to verify the claim, we must show that $b_3^\star = b_3''$. By Desargues' Theorem [Co, 2.32; see Fig. 4.4a on p. 39] $b_3', b_3^\star, x$ are collinear and thus

$$b_3^\star \in \langle b_3', x \rangle \cap \langle a'', b_2 \rangle = b_3'' ,$$

as desired.

We note that if $b_3 = b_2$, then

$$b_2 = b_3' = b_3'' = b_2' = c = b_2'' .$$

Since C is compact, it follows that we can choose ε small enough so that all the labeled points in Figure 1 except x lie in U_0 whenever b_2, b_3 are points of C with $\mathrm{dist}(b_2, b_3) < \varepsilon$. Again by Desargues' Theorem, $\langle \hat{a}, \hat{a}' \rangle$, $\langle \hat{b}_2, \hat{b}_2' \rangle$ and $\langle \hat{b}_3', \hat{b}_3'' \rangle$ are coincident. Thus

$$\hat{x}(a, b_2) = \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_2, \hat{b}_2' \rangle = \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_3', \hat{b}_3'' \rangle = \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_3, \hat{b}_3' \rangle = \hat{x}(a, b_3) .$$

It follows that $\hat{x}(a, b_0) = \hat{x}(a, b_1)$, which completes Step 1 for the case $K = \mathbb{C}$.

We now suppose that $K = \mathbb{R}$. (The proof must be modified for the case $K = \mathbb{R}$, since $U_0 \setminus \langle a, x \rangle$ may not be connected.) We may assume without loss of generality that the line segment

$$C \stackrel{\text{def}}{=} \{ t b_0 + (1-t) b_1 : 0 \le t \le 1 \}$$

is contained in U_0. If $C \cap \langle a, x \rangle = \emptyset$, then we conclude that $\hat{x}(a, b_0) = \hat{x}(a, b_1)$, by the proof for the case $K = \mathbb{C}$ above. On the other hand, if $C \cap \langle a, x \rangle = b'$, then

$$\hat{x}(b_0, a) = \hat{x}(b_0, b') = \hat{x}(b_0, b_1) = \hat{x}(b', b_1) = \hat{x}(a, b_1) ,$$

which completes Step 1 for the case $K = \mathbb{R}$.

We now write $\hat{x} = \hat{x}(a, b) = \tilde{f}(x)$ for all $x \in \mathbb{P}^2_K$.

Step 2. *$\tilde{f}$ is a collineation.*

Let x, y, z be collinear. We must show that $\hat{x}, \hat{y}, \hat{z}$ are collinear. Choose collinear points $a, b, c \in U_0 \setminus \langle x, y \rangle$. Let a', b', c' be as in Figure 2 below. We note that if $a = b = c$, then $a' = b' = c' = a$. Thus we can choose distinct collinear $a, b, c \in U_0 \setminus \langle x, y \rangle$ such that a', b', c' are in U_0. By moving the line $\langle a, b \rangle$ slightly if necessary, we can assume further that $x, y, z \notin \langle a, b \rangle$, and hence a', b', c' are distinct. By Pappas' Theorem (see for example [Co, 4.41 and Fig. 4.4a]), a', b', c' are collinear. It further follows from the above that no four of the nine labeled points in Figure 2 are collinear. By the collinearity of f on U, the points $\hat{a}, \hat{b}, \hat{c}$ are collinear and distinct, and the same is true for $\hat{a}', \hat{b}', \hat{c}'$; furthermore, no four of the points $\hat{a}, \hat{b}, \hat{c}, \hat{a}', \hat{b}', \hat{c}'$ are collinear. Hence $\hat{x}, \hat{y}, \hat{z}$ are distinct, and thus $\tilde{f}$ is injective. Applying Pappas' Theorem again (with $a, b, c, x, y, z, a', b', c'$ replaced by $\hat{a}, \hat{b}, \hat{c}, \hat{a}', \hat{b}', \hat{c}', \hat{x}, \hat{y}, \hat{z}$, respectively), we conclude that $\hat{x}, \hat{y}, \hat{z}$ are collinear.

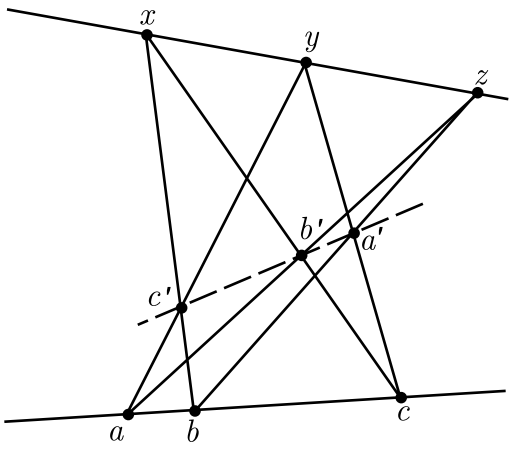

Figure 2

Finally, to show that $\tilde{f}$ is surjective, let $\chi \in \mathbb{P}^2_K$ be arbitrary. Choose points $\alpha, \alpha', \beta, \beta' \in \widehat{U}_0 = f(U_0)$ such that $\chi = \langle \alpha, \alpha' \rangle \cap \langle \beta, \beta' \rangle$. The points $\alpha, \alpha', \beta, \beta'$ are the respective images of points $a, a', b, b' \in U_0$. If we set $x = \langle a, a' \rangle \cap \langle b, b' \rangle$, then $\chi = \hat{x}$.

Hence $\tilde{f}$ is a collineation. The case $n = 2$ then follows from Corollary 2.

Step 3. *The proof for $n > 2$.*

Let $n > 2$. We easily see that f takes 2-planes in U to 2-planes in $\widehat{U}$. Let $L \in \mathcal{L}(U)$ be arbitrary. By applying the case $n = 2$ to a projective 2-plane containing L, we see that $f|_{L \cap U} : L \cap U \to \widehat{L} \cap \widehat{U}$ is either projective-linear or anti-projective-linear. If $f|_{L \cap U}$ is anti-projective-linear for one L, it must be anti-projective-linear for all L (by the case $n = 2$), so by replacing f with $\bar{f}$ if necessary, we can assume that $f|_{L \cap U}$ is projective-linear for all $L \in \mathcal{L}(U)$. Now fix $a \in U$. For $x \in \mathbb{P}^n_K$, define $\hat{x} = T(x)$ where $T : \langle a, x \rangle \to \langle \hat{a}, \hat{x} \rangle$ is the projective-linear transformation extending $f|_{\langle a, x \rangle \cap U}$. By applying the case $n = 2$ to the plane determined by a, a', x (for an arbitrary point $a' \notin \langle a, x \rangle$), we see that $\hat{x}$ is independent of a. Thus we can define $\tilde{f}(x) = \hat{x}$. If x, y, z are collinear and $a \notin \langle x, y \rangle$, then the case $n = 2$ applied to the plane determined by a, x, y implies that $\hat{x}, \hat{y}, \hat{z}$ are collinear. The injectivity of $\tilde{f}$ similarly follows from the case $n = 2$. To show surjectivity, let $\chi \in \mathbb{P}^n_K$ be arbitrary, and choose a point $\alpha \in \langle \hat{a}, \chi \rangle \cap \widehat{U} \setminus \{\hat{a}\}$. Then α is the image of a point $a' \in U$ and $\tilde{f}(\langle a, a' \rangle) = \langle \hat{a}, \alpha \rangle$. Hence $\chi \in \langle \hat{a}, \alpha \rangle \subset \text{image } \tilde{f}$.

Thus $\tilde{f}$ is a collineation. The conclusion of the lemma follows as before from Corollary 2. $\square$

Definition: A subset U of $\mathbb{P}^n_{\mathbb{R}}$ or $\mathbb{P}^n_{\mathbb{C}}$ is said to be *projectively convex* if $L \cap U$ is connected for all projective lines $L \in \mathcal{L}(U)$. (Note that if $U \subset \mathbb{R}^n \subset \mathbb{P}^n_{\mathbb{R}}$, then U is projectively convex if and only if U is convex.)

We use the following lemma to complete the proof of Theorem 3:

Lemma 5. *Let U be a projectively convex, open set in $\mathbb{P}^n_K$, where K denotes either $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{L}_0$ be an open subset of $\mathcal{L}(U)$ such that $\bigcup \mathcal{L}_0 \supset U$. Suppose that $f : U \to \mathbb{P}^n_K$ is a continuous injective map such that $f(L \cap U)$ is contained in a projective line for each $L \in \mathcal{L}_0$. Then $f(L \cap U)$ is contained in a projective line for every $L \in \mathcal{L}(U)$.*

Proof: We again write $\hat{p} = f(p)$, for $p \in U$. Let $L \in \mathcal{L}(U)$ be arbitrary, and let $x \in L \cap U$. Since $L \cap U$ is connected, it suffices to show that there is a neighborhood $V \subset U$ of x such that $\hat{x}, \hat{y}, \hat{z}$ are collinear whenever $y, z \in L \cap V$. Choose a line $L_x \in \mathcal{L}_0$ containing x. We can assume that $L_x \neq L$, since otherwise we are done. Choose $w \in L_x \cap U$, $w \neq x$. Next choose a neighborhood $V \subset U$ of x such that $\langle y, w \rangle \in \mathcal{L}_0$ for all $y \in V$.

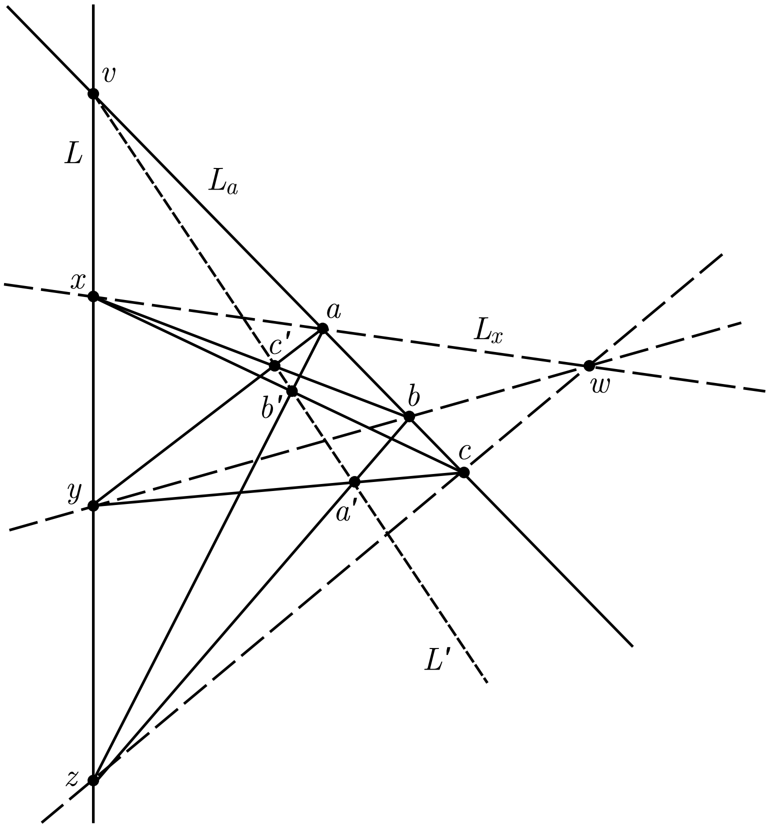

Figure 3

Let $y, z \in L \cap V$. We must show that $\hat{x}, \hat{y}, \hat{z}$ are collinear. We can assume that x, y, z are distinct points. Choose $v \in L \cap V$ distinct from x, y, z (see Figure 3). Since $\langle v, w \rangle \in \mathcal{L}_0$, we can choose $a \in L_x \setminus \{x, w\}$ sufficiently close to w so that the line $L_a = \langle v, a\rangle \in \mathcal{L}_0$. Let $b = \langle y, w\rangle \cap L_a$, $c = \langle z, w\rangle \cap L_a$. By choosing a close enough to w, we can assume further that $a, b, c \in U$ and the six lines

$$\langle x, b\rangle, \quad \langle x, c\rangle, \quad \langle y, a\rangle, \quad \langle y, c\rangle, \quad \langle z, a\rangle, \quad \langle z, b\rangle$$

are in $\mathcal{L}_0$. Let a', b', c' be as in Figure 3. Since all the points and lines of Figure 3 lie in a plane, we can use Desargues' Theorem to conclude that v, a', b', c' are collinear. Write $L' = \langle v, c'\rangle$; thus $a', b' \in L'$. Since a', b', c' (as well as b, c) converge to w as $a \to w$, by choosing a sufficiently close to w we can assume also that $a', b', c' \in U$ and $L' \in \mathcal{L}_0$. Since all the labeled points in Figure 3 lie in U and all the lines in Figure 3 except L are in $\mathcal{L}_0$, we conclude that the f-images of the points in Figure 3 lie in the plane determined by the image lines $\widehat{L_a}$ and $\widehat{L_x}$. We now apply Pappas' Theorem to the image to conclude (as in Step 2 of the proof of Lemma 4) that $\hat{x}, \hat{y}, \hat{z}$ are collinear. $\square$

Proof of Theorem 3: Choose a sequence $\{U_1, U_2, \ldots\}$ of projectively convex, open subsets of U such that $U = \bigcup_{j=1}^{\infty} U_j$ and $U_1 \cup \cdots \cup U_j$ is connected for each $j \geq 1$. If $K = \mathbb{R}$, let $G = \mathrm{PGL}(n+1, \mathbb{R})$; if $K = \mathbb{C}$, let $G = \{e, \tau\} \cdot \mathrm{PGL}(n+1, \mathbb{C})$, where $\tau : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ is given by $\tau(z) = \bar{z}$ and e is the identity map. By Lemmas 5 and 4 applied to the restrictions $f|_{U_j}$, there are transformations $A_j \in G$ such that $f|_{U_j} = A_j|_{U_j}$. Since an element of G is uniquely determined by its values on a nonempty open subset of $\mathbb{P}^n_K$ and $(U_1 \cup \cdots \cup U_j) \cap U_{j+1} \neq \emptyset$, it follows by induction that $A_j = A_1$ for all j. Hence $f = A_1|_U$. $\square$

3. The Poincaré-Tanaka and Chern-Ji theorems.

The Segre family $\mathcal{M}_{B_n}$ mentioned in the introduction has the projective analogue

$$\mathcal{M}^n_K = \{(z, w) \in \mathbb{P}^n_K \times \mathbb{P}^n_K : \sum_{j=0}^{n} z_j w_j = 0\} .$$

(In fact $\mathcal{M}^n_K$ is a compactification of $\mathcal{M}_{B_n}$; see the proof of Corollary 8.) We let $\pi_i : \mathbb{P}^n_K \times \mathbb{P}^n_K \to \mathbb{P}^n_K$ denote the projection to the i-th factor, for $i = 1, 2$. The main result of this section is the following generalization of the Chern-Ji theorem [CJ, Theorem 2]; our generalization says that a pair of local homeomorphisms of $\mathbb{P}^n_K$ ($K = \mathbb{R}$ or $\mathbb{C}$) mapping $\mathcal{M}^n_K$ into itself must be projective-linear, or possibly anti-projective-linear (if $K = \mathbb{C}$):

Theorem 6. *Let $(a^1, a^2) \in \mathcal{M}^n_K$, where $K = \mathbb{R}$ or $\mathbb{C}$, $n \geq 2$. Let U_1, U_2 be open sets in $\mathbb{P}^n_K$ containing a^1, a^2 respectively, and let V_i be the connected component of $\pi_i(\mathcal{M}^n_K \cap U_1 \times U_2)$ containing a_i, for $i = 1, 2$. If $f_i : U_i \to \mathbb{P}^n_K$ $(i = 1, 2)$ are continuous injective maps such that*

$$(f_1 \times f_2)(\mathcal{M}^n_K \cap U_1 \times U_2) \subset \mathcal{M}^n_K ,$$

then there exists $A \in \mathrm{PGL}(n+1, K)$ such that

(i) *$f_1 = A$ on V_1 and $f_2 = {}^tA^{-1}$ on V_2, if $K = \mathbb{R}$,*

(ii) *either* (i) *holds or* $\bar{f}_1 = A$ *on* V_1 *and* $\bar{f}_2 = {}^tA^{-1}$ *on* V_2, *if* $K = \mathbb{C}$.

Remark: If the sets $\pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$ are connected, then $V_i = \pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$ and we have $\mathcal{M}_K^n \cap U_1 \times U_2 = \mathcal{M}_K^n \cap V_1 \times V_2$. In fact, if we assume that only one of the projections $\pi_1(\mathcal{M}_K^n \cap U_1 \times U_2)$ is connected, then by the uniqueness of A it follows that the conclusion of Theorem 6 holds with $V_i = \pi_i(\mathcal{M}_K^n \cap U_1 \times U_2)$, for $i = 1, 2$.

Proof of Theorem 6: For a point $w \in \mathbb{P}_K^n$ we write $w^\perp = \{z \in \mathbb{P}_K^n : z \cdot w = 0\}$, where $z \cdot w = \sum_{j=0}^n z_j w_j$. For a subset $S \subset \mathbb{P}_K^n$ we also write $S^\perp = \{z \in \mathbb{P}_K^n : z \cdot w = 0 \ \ \forall w \in E\}$.

We consider the collection of lines

$$\mathcal{L}_0 = \{L \in \mathcal{L}(V_1) : L^\perp \cap U_2 \neq \emptyset\} ,$$

which is open in $\mathcal{L}(V_1)$. If z is an arbitrary point of V_1, then by hypothesis we can choose $w \in U_2$ such that $(z, w) \in \mathcal{M}_K^n$. If we let L be any projective line in $w^\perp$ containing z, then $w \in L^\perp \cap U_2$ and hence $L \in \mathcal{L}_0$. Therefore $\bigcup \mathcal{L}_0 \supset V_1$.

Now let $L \in \mathcal{L}_0$ be arbitrary. We claim that we can choose points $w^1, \ldots, w^{n-1} \in L^\perp \cap U_2$, such that $f_2(w^1), \ldots, f_2(w^{n-1})$ are in general position: If $n = 2$, the claim is a tautology, so suppose $n \geq 3$. If the claim were false, then $f_2(L^\perp \cap U_2)$ must lie in a projective linear subspace $\mathbb{P}(E)$ of dimension $n-3$ (where E is a linear subspace of K^{n+1} of dimension $n-2$). But then f_2 would be a continuous injection from $(L^\perp \cap U_2)$, which has topological dimension $n-2$ or $2n-4$ (depending on whether K equals $\mathbb{R}$ or $\mathbb{C}$), into $\mathbb{P}(E)$, which has topological dimension $n-3$ or $2n-6$. This contradicts dimension theory.

Let $w^1, \ldots, w^{n-1} \in L^\perp \cap U_2$, such that $f_2(w^1), \ldots, f_2(w^{n-1})$ are in general position, as above. By moving the points slightly if necessary, we can assume also that $w^1, \ldots, w^{n-1}$ are in general position, and hence $L = \langle w^1, \ldots, w^{n-1}\rangle^\perp$. We note that by hypothesis, $f_1(w^\perp \cap U_1) \subset f_2(w)^\perp$ for all $w \in U_2$. Therefore

$$f_1(L \cap U_1) = \bigcap_{j=1}^{n-1} f_1(w^{j\perp} \cap U_1) \subset \bigcap_{j=1}^{n-1} f_2(w^j)^\perp = \langle f_2(w^1), \ldots, f_2(w^{n-1})\rangle^\perp \in \mathcal{L}_K^n .$$

Let G be the group of projective-linear, and if $K = \mathbb{C}$, anti-projective linear, transformations of $\mathbb{P}_K^n$ as in the proof of Theorem 3. By Theorem 3, there exists $A \in G$ such that $f_1 = A$ on V_1; similarly, there exists $B \in G$ such that $f_2 = B$ on V_2. By replacing $f_1 \times f_2$ with $\bar{f}_1 \times \bar{f}_2$ if necessary, we can assume that $A \in \mathrm{PGL}(n+1, K)$. We now show that $B = {}^tA^{-1}$: Let M be the connected component of $\mathcal{M}_K^n \cap U_1 \times U_2$ containing (a^1, a^2). Fix a point $w \in \pi_2(M) \subset V_2$, and choose $z^1, \ldots, z^n \in w^\perp \cap V_1$ in general position. Then $(Az^j, Bw) = (f_1(z^j), f_2(w)) \in \mathcal{M}_K^n$ since $(z^j, w) \in \mathcal{M}_K^n$, and thus

$$0 = Az^j \cdot Bw = z^j \cdot {}^tABw ,$$

for $j = 1, \ldots, n$. Therefore ${}^tABw \in w^{\perp\perp} = \{w\}$. Since w is an arbitrary point of $\pi_2(M)$ and since elements of G are uniquely determined by their values on the open set $\pi_2(M)$, it follows that ${}^tAB$ is the identity $e \in G$, and therefore $B = {}^tA^{-1} \in \mathrm{PGL}(n+1, K)$. $\square$

Corollary 7. (Chern-Ji [CJ, Theorem 2]) *Suppose $U, \widehat{U}, V, \widehat{V}$ are connected open sets in $\mathbb{P}^n_{\mathbb{C}}$ such that $\mathcal{M}^n_{\mathbb{C}} \cap U{\times}V \neq \emptyset$. If $f : U \to \widehat{U}$, $g : V \to \widehat{V}$ are biholomorphic maps such that*

$$(f \times g)(\mathcal{M}^n_{\mathbb{C}} \cap U{\times}V) \subset \mathcal{M}^n_{\mathbb{C}} ,$$

then f and g are restrictions of elements of $\mathrm{PGL}(n+1, \mathbb{C})$.

We conclude this paper by demonstrating how the following theorem of Poincaré and Tanaka is obtained from Corollary 7.

Corollary 8. (Poincaré-Tanaka Theorem) [Po], [Ta] *Let B_n denote the unit ball in $\mathbb{C}^n$, $n \geq 2$. Suppose that U is a connected open set in $\mathbb{C}^n$ such that $U \cap \partial B_n \neq \emptyset$. If $f : U \to \mathbb{C}^n$ is a nonconstant holomorphic map such that $f(U \cap \partial B_n) \subset \partial B_n$, then $f|_{U \cap B_n}$ extends to an automorphism of B_n.*

Proof: By an elementary argument given by H. Alexander ([Al, p. 250]), we can assume that the Jacobian matrix of f is nonsingular at some point $z_0 \in U \cap \partial B_n$. (We shall give Alexander's argument later.) By replacing U with a neighborhood of z_0, we can assume that f is injective. Let $\tau : \mathbb{C}^n \to \mathbb{C}^n$ be the conjugation $z \mapsto \bar{z}$. Let $V = \tau(U)$ and consider the holomorphic map $g = \tau \circ f \circ \tau : V \to \mathbb{C}^n$. We let $\widehat{U} = f(U)$, $\widehat{V} = g(V) = \tau(\widehat{U})$ so that the maps $f : U \to \widehat{U}$, $g : V \to \widehat{V}$ are biholomorphic. We let ψ denote the function on $\mathbb{C}^n \times \mathbb{C}^n$ given by $\psi(z, w) = \sum_{j=1}^n z_j w_j - 1$ and we consider the "Segre family"

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n : \psi(z, w) = 0\} .$$

Let $S : \mathbb{C}^n \to \mathbb{C}^{2n}$ be given by $S(z) = (z, \bar{z})$, so that $S^{-1}(\mathcal{M}_{B_n}) = \partial B_n$ and $S \circ f = (f \times g) \circ S$. Let $\Omega = U \times V$ and $N = S(\partial B_n) = \mathcal{M}_{B_n} \cap S(\mathbb{C}^n)$. Then

$$(f \times g)(\Omega \cap N) = S \circ f(U \cap \partial B_n) \subset S(\partial B_n) = N \subset \mathcal{M}_{B_n} .$$

Choose a point $z_0 \in U \cap \partial B_n$; then $(z_0, \bar{z}_0) \in \Omega \cap N$. Since $\psi \circ (f \times g)$ vanishes on $\Omega \cap N$ and N is a totally real submanifold of (real) dimension $2n - 1$ in $\mathcal{M}_{B_n}$, it follows that $\psi \circ (f \times g)$ vanishes on the connected component of $\Omega \cap \mathcal{M}_{B_n}$ containing $(z_0, \bar{z}_0)$. After shrinking U if necessary, we can assume that $\psi \circ (f \times g)$ vanishes on $\Omega \cap \mathcal{M}_{B_n}$ and thus $(f \times g)(\Omega \cap \mathcal{M}_{B_n}) \subset \mathcal{M}_{B_n}$. We consider the embedding $\iota \times \iota : \mathbb{C}^n \times \mathbb{C}^n \hookrightarrow \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^n_{\mathbb{C}}$ given by $\iota(z_1, \ldots, z_n) = (\sqrt{-1} : z_1 : \ldots : z_n)$, which maps $\mathcal{M}_{B_n}$ onto a (dense open) subset of $\mathcal{M}^n_{\mathbb{C}}$. By Corollary 7 applied to the maps

$$\tilde{f} = \iota \circ f \circ \iota^{-1} : \iota(U) \to \iota(\widehat{U}) , \quad \tilde{g} = \iota \circ g \circ \iota^{-1} : \iota(V) \to \iota(\widehat{V}) ,$$

there exists $A \in \mathrm{PGL}(n+1, \mathbb{C})$ such that $\tilde{f} = A|_{\iota(U)}$. Thus f extends to the fractional linear map $\iota^{-1} \circ A \circ \iota$, which gives an automorphism of B_n.

We now give a simplified form of Alexander's proof [Al, p. 250] that the Jacobian matrix of the map f must be nonsingular at some point of $U \cap \partial B_n$. We begin by observing that $f^{-1}(\partial B_n)$ is nowhere dense. Indeed, suppose on the contrary that $f^{-1}(\partial B_n)$ contains a connected open set U_0 and assume without loss of generality that $f(z_0) = (1, 0, \ldots, 0)$ for some point $z_0 \in U_0$. Then by the maximum principle, $f_1 \equiv 1$ and hence $f \equiv (1, 0, \ldots, 0)$

on U_0 and thus on U, contradicting the assumption that f is nonconstant. Now suppose on the contrary that the Jacobian determinant of f vanishes identically on $U \cap \partial B_n$. Since the zero set of the Jacobian determinant is an analytic subvariety, the Jacobian determinant must vanish identically on U. As a consequence, the fibers of f contain no isolated points. Assume without loss of generality that $(1, 0, \ldots, 0) \in U$ and choose $r < 1$ such that the spherical cap $W := \{z \in B_n : \text{Re } z_1 > r\}$ is contained in U. Choose a point $p \in W$ such that $f(p) \notin \partial B_n$. Let A be the connected component of $f^{-1}(f(p)) \cap W$ that contains p; A is an analytic subvariety of W of positive dimension. Furthermore $\bar{A} \setminus A \subset \{z \in \mathbb{C}^n : \text{Re } z_1 = r\}$. By the maximum principle (see for example [Gu, Theorem H2]) applied to the holomorphic function $\varphi : A \to \mathbb{C}$ given by $\varphi(z) = \exp z_1$, we conclude that φ is constant and thus $\bar{A} \setminus A = \emptyset$ so that A is a compact subvariety of W of positive dimension, which is impossible. □

References

[Al] H. Alexander, Holomorphic mappings from the ball and polydisc, *Math. Ann.* **209** (1974), 249-256.

[Ar] E. Artin, *Geometric Algebra*, Interscience Publishers, New York, 1957.

[BB] W. Blaschke and G. Bol, *Geometrie der Gewebe*, Springer, Berlin, 1938.

[Ca] E. Cartan, Sur le groupe de la géométrie hypersphérique, *Comm. Math. Helv.* **4** (1932), 158-171.

[CG] S.-S. Chern and P. Griffiths, Abel's theorem and webs, *Jahresber. Deutsch. Math.-Verein.* **80** (1978), 13-110; Corrections and addenda, *Jahresber. Deutsch. Math.-Verein.* **83** (1981), 78-83.

[CJ] S.-S. Chern and S. Ji, Projective geometry and Riemann's mapping problem, preprint, 1994.

[Co] H. S. M. Coxeter, *Projective Geometry*, Univ. of Toronto Press, Toronto, 1974.

[Fo] F. Forstnerič, Proper holomorphic mappings: a survey, *Several Complex Variables: Proceedings of the Mittag-Leffler Institute, 1987-1988* (J. E. Fornaess, ed.), Princeton Univ. Press, Princeton, 1993, pp. 297-363.

[Go] V. Goldberg, *Theory of Multicodimensional* $(n+1)$*-Webs*, Kluwer, Dordrecht, 1988.

[Gu] R. C. Gunning, *Introduction to Holomorphic Functions of Several Variables, Volume II: Local Theory*, Brooks/Cole, Pacific Grove, CA, 1990.

[MY] N. Mok and S. K. Yeung, Geometric realizations of uniformization of conjugates of hermitian locally symmetric manifolds, *Complex Analysis and Geometry* (V. Ancona and A. Silva, eds.), Plenum Press, New York, 1993, pp. 253-270.

[MM] R. Molzon and K. P. Mortensen, The Schwarzian derivative for maps between manifolds with complex projective connections, preprint, 1994.

[Pe] D. Pelles, Proper holomorphic self-maps of the unit ball, *Math. Ann.* **190** (1971), 298-305; Correction, *Math. Ann.* **202** (1973), 135-136.

[Po] H. Poincaré, Les fonctions analytiques de deux variables et la représentation conforme, *Rend. Circ. Mat. Palermo* (1907), 185-220.

[Pr] W. Prenowitz, The characterization of plane collineations in terms of homologous families of lines, *Trans. Amer. Math. Soc.* **38** (1935), 564-599.

[Ra] F. Radó, Non-injective collineations on some sets in Desarguesian projective planes and extension of non-commutative valuations, *Aequationes Math.* **4** (1970), 307-321.

[Re] K. Reidemeister, Topologische Fragen der Differentialgeometrie, V, *Math. Z.* **29** (1929), 427-435.

[Ru] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[Ta] N. Tanaka, On pseudo-conformal geometry of hypersurfaces of the space of n complex variables, *J. Math. Soc. Japan* **14** (1962), 397-429.

[We] S. Webster, On the mapping problem for algebraic real hypersurfaces, *Inventiones Math.* **43** (1977), 53-68.

Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218