1. (15 points)

10 pts

a) Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & -1 \end{bmatrix} = A$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3/7 & -1/7 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/7 & 2/7 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3/7 & -1/7 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/7 & 2/7 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3/7 & -1/7 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/7 & 2/7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3/7 & -1/7 \end{array}\right].$$

Answer: $A^{-1} = \begin{bmatrix} 1/7 & 2/7 & 0 & 0 \\ 3/7 & -1/7 & 0 & 0 \\ 0 & 0 & 1/7 & 2/7 \\ 0 & 0 & 3/7 & -1/7 \end{bmatrix}$

5 pts

b) Use part (a) to solve the system of equations $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/7 & 2/7 & 0 & 0 \\ 3/7 & -1/7 & 0 & 0 \\ 0 & 0 & 1/7 & 2/7 \\ 0 & 0 & 3/7 & -1/7 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 5/7 \\ 1/7 \\ 3/7 \end{bmatrix}$$

Answer: $x_1 = 4/7$, $x_2 = 5/7$, $x_3 = 1/7$, $x_4 = 3/7$

2. (15 points) Prove that the functions

$$f_1(x) = e^x, \; f_2(x) = \sin x, \; f_3(x) = \sqrt{1+x}$$

are linearly independent in the space $C(-\frac{1}{10}, \frac{1}{10})$ of continuous functions on the interval $[-\frac{1}{10}, \frac{1}{10}]$.

Suppose $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

i.e, $c_1 e^x + c_2 \sin x + c_3 \sqrt{1+x} = 0$

Set $x=0$: $c_1 + c_3 = 0$

differentiate: $c_1 e^x + c_2 \cos x + \frac{1}{2} c_3 (1+x)^{-1/2} = 0$

set $x=0$: $c_1 + c_2 + \frac{1}{2} c_3 = 0$

differentiate again:

$c_1 e^x - c_2 \sin x - \frac{1}{4} c_3 (1+x)^{-3/2} = 0$

set $x=0$ $c_1 - \frac{1}{4} c_3 = 0$

$\therefore c_1 = \frac{1}{4} c_3 = -\frac{1}{4} c_1$

$\therefore c_1 = c_3 = 0 \quad \therefore c_2 = 0.$

$\therefore f_1, f_2, f_3$ are linearly independent

3. (15 points) Let

$$V = \{c_1 + c_2 t + c_3 t^2 \mid c_1, c_2, c_3 \in \mathbf{R}\}$$

be the linear space of quadratic polynomials (polynomials of degree ≤ 2), with the inner product

$$(f, g) = \int_0^1 f(t)g(t)dt .$$

Apply the Gram-Schmidt process to the elements u_1, u_2, u_3 of V given by

$$u_1(t) = 1, \; u_2(t) = t, \; u_3(t) = t^2$$

to obtain an orthogonal basis for V.

$$y_1 = u_1 = 1$$

$$y_2 = u_2 - \frac{(y_1, u_2)}{\|y_1\|^2} y_1$$

$$\|y_1\|^2 = \int_0^1 1\,dt = 1 \quad , \quad (y_1, u_2) = \int_0^1 t\,dt = \tfrac{1}{2}t^2\Big|_0^1 = \tfrac{1}{2}$$

$$y_2 = t - \tfrac{1}{2}\cdot 1 = t - \tfrac{1}{2}$$

$$y_3 = u_3 - \frac{(y_1, u_3)}{\|y_1\|^2} y_1 - \frac{(y_2, u_3)}{\|y_2\|^2} y_2$$

$$(y_1, u_3) = \int_0^1 t^2 dt = \tfrac{1}{3} .$$

$$(y_2, u_3) = \int_0^1 (t - \tfrac{1}{2}) t^2 dt = \int_0^1 (t^3 - \tfrac{1}{2}t^2) dt = \tfrac{1}{4}t^4 - \tfrac{1}{6}t^3\Big|_0^1 = \tfrac{1}{12}$$

$$\|y_2\|^2 = \int_0^1 (t - \tfrac{1}{2})^2 dt = \tfrac{1}{3}(t - \tfrac{1}{2})^3\Big|_0^1 = \tfrac{1}{24} - (-\tfrac{1}{24}) = \tfrac{1}{12}$$

$$y_3 = t^2 - \tfrac{1}{3}\cdot 1 - 1(t - \tfrac{1}{2}) = t^2 - \tfrac{1}{3} - t + \tfrac{1}{2} = t^2 - t + \tfrac{1}{6}$$

<u>Answer</u>: $y_1 = 1$, $y_2 = t - \tfrac{1}{2}$, $y_3 = t^2 - t + \tfrac{1}{6}$

4. (15 points) Consider the function $f(x,y) = \sqrt{2+xy^2}$.

a) Find the maximum directional derivative of f (derivative with respect to a vector of **length 1**) at the point $(2,-1)$.

$$\frac{\partial f}{\partial x} = \frac{1}{2}(2+xy^2)^{-1/2}\cdot y^2 , \quad \left.\frac{\partial f}{\partial x}\right|_{(2,-1)} = \frac{1}{2}\cdot 4^{-1/2}\cdot 1 = \frac{1}{4}$$

$$\frac{\partial f}{\partial y} = xy(2+xy^2)^{-1/2} \quad \left.\frac{\partial f}{\partial y}\right|_{(2,-1)} = -2\cdot 4^{-1/2} = -1$$

$$\nabla f(2,-1) = \left(\frac{1}{4},-1\right)$$

The max. directional derivative $= \|\nabla f\|_{(2,-1)}$

$$= \sqrt{\frac{1}{16}+1} = \frac{1}{4}\sqrt{17}$$

b) Find a vector $u \in \mathbf{R}^2$ of length 1 such that the directional derivative of f with respect to u at the point $(2,-1)$ equals 0.

$u \perp \nabla f|_{2,-1}$, $\|u\| = 1$.

Let $A = (4,1)$ $\quad A\cdot\left(\frac{1}{4},-1\right) = 0 \quad \therefore A \perp \nabla f|_{2,-1}$

$$u = \frac{1}{\|A\|}A = \frac{1}{\sqrt{17}}(4,1) = \left(\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}}\right)$$

$$\left(\text{or} \quad u = \left(\frac{-4}{\sqrt{17}}, \frac{-1}{\sqrt{17}}\right)\right)$$

5. (10 points) By the rank and nullity of an $m \times n$ matrix A, we mean the rank and nullity of the linear transformation $\mathbf{A} : \mathbf{R}^n \to \mathbf{R}^m$ given by

$$\mathbf{A}(X) = AX\ , \quad X \in \mathbf{R}^n = \text{ the space of } \ n \times 1 \text{ matrices (column vectors).}$$

Suppose that A is a 99×2 matrix whose two columns are linearly independent. Determine the following quantities:

$$\text{rank } A, \quad \text{nullity } A, \quad \text{rank } A^t, \quad \text{nullity } A^t,$$

where A^t denotes the transpose of A. (Note that $\mathbf{A} : \mathbf{R}^2 \to \mathbf{R}^{99}$ and $\mathbf{A}^t : \mathbf{R}^{99} \to \mathbf{R}^2$.)

The range of A is the linear span of the columns of A

$\therefore$ rank $A = \dim(\text{range } A) = 2$

rank A + nullity $A = \dim R^2 = 2$

$\therefore$ nullity $A = 0$

Since the columns of A are independent, we can choose j, k with $1 \le j < k \le 99$ such that

$$\text{rank} \begin{bmatrix} A_{j1} & A_{j2} \\ A_{k1} & A_{k2} \end{bmatrix} = 2 \qquad \therefore \begin{vmatrix} A_{j1} & A_{j2} \\ A_{k1} & A_{k2} \end{vmatrix} \neq 0$$

$$\therefore \text{rank} \begin{bmatrix} A_{j1} & A_{k1} \\ A_{j2} & A_{k2} \end{bmatrix} = 2 \qquad \therefore \text{rank } A^t = 2$$

(General fact: for any $m \times n$ matrix B, rank B = rank B^t)

rank A^t + nullity $A^t = \dim R^{99} = 99$

$\therefore$ nullity $A^t = 97$

Answer:
rank $A =$ 2, nullity $A =$ 0, rank $A^t =$ 2, nullity $A^t =$ 97.

6. (15 points) Let $S = \{X \in \mathbf{R}^n \mid X \neq O\}$ and suppose that $f : S \to \mathbf{R}$ is a differentiable scalar field such that

$$f(tX) = t^p f(X)\,, \quad \text{for } X \in S,\ t \neq 0\,,$$

where p is a real number. Prove *Euler's formula:*

$$X \cdot \nabla f(X) = pf(X)\,.$$

(Hint: Differentiate $f(tX)$ with respect to t.)

Let $X \in S$ be a fixed vector.

Let $G : R \to R^n$ be given by $G(t) = tX$

Let $h(t) = f \circ G(t) = f(tX) = t^p f(X)$

$\therefore\ h'(t) = pt^{p-1} f(X)$.

By the chain rule

$h'(t) = Df|_{G(t)} G'(t) = (Df|_{tX}) X = \nabla f|_{tX} \cdot X$

$\therefore\ X \cdot \nabla f(tX) = h'(t) = pt^{p-1} f(X)$

Let $t=1$: $X \cdot \nabla f(X) = pf(X)$.

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alternately: $h(t) = f(tx_1, \ldots, tx_n)$

$h'(t) = D_1 f(tx_1, \ldots, tx_n) \frac{d}{dt}(tx_1) + \cdots + D_n f(tx_1, \ldots, tx_n) \frac{d}{dt}(tx_n)$

$= D_1 f(tX) x_1 + \cdots + D_n f(tX) x_n$

$= \nabla f(tX) \cdot X$.

7. (15 points) Let r be the scalar field on $S = \{X \in \mathbf{R}^n \mid X \neq O\}$ given by

$$r(X) = \|X\| .$$

Find a scalar function f on $(0, +\infty)$ such that

$$\|\nabla(f \circ r)\| = \log \|X\| .$$

$$\nabla(f \circ r) = (f' \circ r)\, \nabla r$$

$$r(X) = (x_1^2 + \dots + x_n^2)^{1/2} = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$$

$$\frac{\partial r}{\partial x_j} = \frac{1}{2}\left(\sum x_j^2\right)^{-1/2} \cdot 2x_j = \frac{1}{\|X\|} x_j$$

$$\therefore \nabla r = \frac{1}{\|X\|} X$$

$$\therefore \nabla(f \circ r) = f'(\|X\|) \frac{1}{\|X\|} X$$

$$\therefore \|\nabla(f \circ r)\| = |f'(\|X\|)| \cdot \frac{1}{\|X\|} \|X\| = |f'(\|X\|)|$$

$$\|\nabla(f \circ r)\| = \log \|X\| \iff |f'(\|X\|)| = \log \|X\|$$

$$\iff f'(t) = \pm \log t$$

So let $f(t) = \int \log t \, dt = t \log t - t$

__Answer:__ $f(t) = t \log t - t$

(General Solution: $f(t) = \pm(t \log t - t) + C$)