110.211 Honors Multivariable Calculus and Linear Algebra – B. Shiffman

Answers to problems on the 1996 Final Exam

1a. $\begin{pmatrix} -19/2 & 59/2 & -3/2 \\ 5/2 & -15/2 & 1/2 \\ -1 & 3 & 0 \end{pmatrix}$

2a. $\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2$

3. $(0, -3/2, 3/2, 0)$

4a. $\frac{1}{3}\sqrt{5}$

4b. $z = \frac{2}{3}x + \frac{1}{3}y + \log 3 - \frac{4}{3}$

5. $.9 + .3\,x + 1.5\,x^2$

6a. $(-1/3, 0)$

6b. $2/9$

7b. rank $= 2$, nullity $= 1$

Solutions to problems 6 and 7a are on pp. 2–3.

6. Let $H(x,y) = G(x,y,f(x,y))$.

a) Since $H(x,y) = 0$,

$$0 = \frac{\partial H}{\partial x} = D_1 G(x,y,f(x,y)) + D_3 G(x,y,f(x,y)) \frac{\partial f}{\partial x}(x,y).$$

$$(*) \qquad 0 = f(x,y)^3 + \sqrt{1+x^3} + \left(3x\, f(x,y)^2 + y\right) \frac{\partial f}{\partial x}(x,y)$$

$$0 = \frac{\partial H}{\partial y} = D_2 G(x,y,f(x,y)) + D_3 G(x,y,f(x,y)) \frac{\partial f}{\partial y}(x,y).$$

Substitute $x=2$, $y=0$, $f(x,y) = f(2,0) = -1$:

$$0 = -1 + \sqrt{1+8} + (6+0)\frac{\partial f}{\partial x}(2,0) = 2 + 6\frac{\partial f}{\partial x}(2,0)$$

$$\therefore \frac{\partial f}{\partial x}(2,0) = -\frac{1}{3}.$$

$$0 = \frac{\partial H}{\partial y} = f(x,y) + e^{y^2} + (3x\, f(x,y)^2 + y)\frac{\partial f}{\partial y}(x,y)\Big|_{2,0} = 6\frac{\partial f}{\partial y}(2,0)$$

$$\therefore \nabla f(2,0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Big|_{2,0} = \left(-\frac{1}{3}, 0\right)$$

b) Differentiate $(*)$ with respect to x:

$$0 = 3f^2 \frac{\partial f}{\partial x} + \frac{3}{2}x^2(1+x^3)^{-\frac{1}{2}} + (3xf^2+y)\frac{\partial^2 f}{\partial x^2}$$

$$+ \left(3f^2 + 6xf\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial x}.$$

Substitute $x=2$, $y=0$, $f=-1$, $\frac{\partial f}{\partial x} = -\frac{1}{3}$:

$$0 = 3\cdot 1\cdot\left(-\tfrac{1}{3}\right) + \tfrac{3}{2}\cdot 4\cdot \tfrac{1}{3} + (3\cdot 2\cdot 1 + 0)\frac{\partial^2 f}{\partial x^2}(2,0)$$

$$+ \left(3\cdot 1 + 6\cdot 2(-1)\left(-\tfrac{1}{3}\right)\right)\left(-\tfrac{1}{3}\right)$$

$$= -1 + 2 + 6\frac{\partial^2 f}{\partial x^2}(2,0) - 7/3 = -\frac{4}{3} + 6\frac{\partial^2 f}{\partial x^2}(2,0)$$

$$\therefore \frac{\partial^2 f}{\partial x^2}(2,0) = 2/9$$

7 a) Since $A^2 \neq O$, there exists $X \in R^3$ such that $A^2X \neq O$.

Suppose $c_1 X + c_2 AX + c_3 A^2 X = O$.

Since $A^3 = O$, we also have $A^4 = AA^3 = O$,

and $O = A^2(c_1 X + c_2 AX + c_3 A^2 X)$

$= c_1 A^2 X + c_2 A^3 X + c_3 A^4 X = c_1 A^2 X$

$\therefore c_1 = O$.

$\therefore O = A(c_2 AX + c_3 A^2 X) = c_2 A^2 X + O$

$\therefore c_2 = O$

$\therefore c_3 A^2 X = O \quad \therefore c_3 = O$

$\therefore \{X, AX, A^2X\}$ is linearly independent

$\dim R^3 = 3$

Since any set of 3 independent elements in R^3 is a basis of R^3 (Theorem 1.7b), it follows that $\{X, AX, A^2X\}$ is a basis for R^3.