# Midterm Exam November 12, 1996

**Part I.**

1. (20 points) Find bases for the following spaces: (In this problem, you do **not** need to prove that your answers are bases.)

a) the plane

$$M = \{(x, y, z) | 3x + 2y - z = 0\} \subseteq \mathbf{R}^3$$

b) the subspace of the space $\mathcal{C}(0,1)$ of continuous functions on the open interval $(0,1)$ spanned by

$$\{1, x, \frac{1}{x+1}, \frac{2}{x+1}\}$$

c) the space $M_{2,2}$ of $2 \times 2$ matrices

d) The space of functions on the interval $[0, 2\pi]$ of the form

$$g(t) = \sum_{j=1}^{5} c_j \sin jt \qquad (c_j \in \mathbf{R})$$

2. (30 points) Apply the Gram-Schmidt process to the three polynomials

$$f_1 = 1 \ , \ f_2 = t \ , \ f_3 = t^2$$

to obtain three polynomials $g_1, \ g_2, \ g_3$ of degree $\leq 2$ that are orthogonal with respect to the inner product

$$(f, g) = \int_0^{+\infty} f(t)g(t)e^{-t} dt \ .$$

(Useful formula: $\int_0^{+\infty} t^n e^{-t} dt = n!$)

**Part II. Do 2 out of the following 3 problems.** In these problems, give reasons for all statements in your proofs. You may refer to any theorem given in the text, but you may not refer to any of the problems in the text. (Each problem is worth 25 points; credit will be given for only 2 problems.)

3. Let $A$ be a $2 \times 2$ matrix. Suppose that $AX = AY = O$ for (linearly) independent vectors

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ , \ Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \ .$$

Prove that $A = O$.

4. Let $\mathbf{r}(t) = (x(t), y(t))$ be a smooth curve in the **plane** with unit tangent $T(t)$ and principal (unit) normal $N(t)$. Prove that

$$N'(t) = \lambda(t)T(t) \ ,$$

for a scalar function $\lambda(t)$, and give a formula for $\lambda(t)$ in terms of the speed $v(t)$ and curvature $\kappa(t)$ of the curve.

5. Let $\{f, g, h\}$ be a (linearly) dependent set of continuous functions on $\mathbf{R}$. Consider the vectors in $\mathbf{R}^3$:

$$A = (f(0), f(1), f(4)), \ B = (g(0), g(1), g(4)), \ C = (h(0), h(1), h(4)) \ .$$

Prove that the triple scalar product $[ABC] = 0$.