

**Assignment #6 due October 17, 2002:****Reading:**

Sections 14.1–14.8

**Problems:**

Part I:

14.4: 1, 7, 16, 20

14.7: 2, 3, 7, 10

Part II:

Let  $C$  be the *Cantor set* constructed as follows: Writing all numbers in base 3—e.g.,  $.1=1/3$ ,  $.2=2/3$ ,  $.01=1/9$ , etc. (italic fractions are in base 10)—we define:

$$\begin{aligned} C_1 &= [0, .1] \cup [.2, 1] \\ C_2 &= [0, .01] \cup [.02, .1] \cup [.2, .21] \cup [.22, 1] \\ C_3 &= [0, .001] \cup [.002, .01] \cup [.02, .021] \cup [.022, .1] \cup [.2, .201] \cup [.202, .21] \cup [.22, .221] \cup [.222, 1] \end{aligned}$$

Continuing this procedure,  $C_n$  is the union of  $2^n$  closed intervals of the form  $[a, a + 3^{-n}]$  where  $a = .a_1a_2a_3 \cdots a_n$  (in base 3) with  $a_j = 0$  or  $2$ .

We have  $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots$ . We define the Cantor set

$$C = \bigcap_{n=1}^{\infty} C_n .$$

Equivalently,  $C$  is the set of  $x \in [0, 1]$  that can be written in base 3 as  $x = .a_1a_2a_3 \cdots$  with  $a_j = 0$  or  $2$ . This expansion can be infinite. Note for example that  $.1 = .022222 \cdots$ , so  $.1 \in C$ .

1. a)  $[0, 1] - C_n$  is the union of  $2^n - 1$  disjoint open intervals. (Assume this fact without giving a proof.) What is the sum of the lengths of these intervals? (Hint: What is the sum of the lengths of the closed intervals that make up  $C_n$ ?)

b)  $[0, 1] - C$  is an infinite union of disjoint open intervals. (Assume this fact without giving a proof.) What is the sum of the lengths of these intervals?

Let  $f(x)$  be the function on  $0 \leq x \leq 1$  defined as follows: Write  $x = (.a_1a_2a_3 \cdots)_3$  in base 3. If  $x \in C$ , let

$$f(x) = (.b_1b_2b_3 \cdots)_2$$

(expansion in base 2), where  $b_j = a_j/2$ . If  $x \in [0, 1] - C$ , let  $k$  be the smallest integer with  $a_k = 1$  and let

$$f(x) = (.b_1b_2 \cdots b_{k-1}1)_2, \quad b_j = a_j/2 .$$

Note that  $f(0) = 0$ ,  $f(1) = 1$ .

2. a) Draw (as well as you can) the graph of  $f$ .

b) Explain why  $f'(x) = 0$  for  $x \in [0, 1] - C$ .