110.211 Honors Multivariable Calculus and Linear Algebra – B. Shiffman

FINAL EXAM December 18, 1996

Show all of your work. No books, notes, or calculators are allowed.

There are 7 problems making a total of 120 points.

Time: 3 hours

1. (20 points)

a) Find the inverse of the matrix $\begin{bmatrix} 3 & 9 & -7 \\ 1 & 3 & -2 \\ 0 & 2 & 5 \end{bmatrix}$.

b) Let A and B be $n \times n$ nonsingular (i.e., invertible) matrices. Prove that AB is nonsingular.

2. (15 points) Consider the theorem from geometry: *The sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of the lengths of its four sides.*

a) Draw a figure illustrating this theorem and give a vector-algebra identity which expresses the theorem. Label the illustration with the vectors from your identity. (You do not need to prove that your identity gives the theorem.)

b) Give a proof of your identity from part (a).

3. (15 points) Find the orthogonal projection of $u = (7, -4, -1, 2)$ on the solution space of the system:

$$\begin{aligned} 2x_1 + x_2 + x_3 + 3x_4 &= 0 \\ 3x_1 + 2x_2 + 2x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 2x_3 - 4x_4 &= 0 \end{aligned}$$

4. (15 points) Consider the function $f(x, y) = \log(1 + xy)$.

a) Find the maximum directional derivative (derivative with respect to a unit vector) of f at the point $(1, 2)$.

b) Find an equation for the tangent plane to $z = f(x, y)$ at the point $(1, 2, \log 3)$.

5. (20 points) Find the quadratic polynomial that gives the best least-squares approximation to the data points:

$$(-2,6), \quad (-1,3) \quad (0,0), \quad (1,3)$$

6. (20 points) Let $f(x,y)$ be a differentiable function on an open set S in R^2 containing the point $(2,0)$. Suppose that $f(2,0) = -1$ and

$$G(x,y,f(x,y)) = 0$$

where $G(x,y,z)$ is a differentiable function with

$$\frac{\partial G}{\partial x} = z^3 + \sqrt{1+x^3}, \quad \frac{\partial G}{\partial y} = z + e^{y^2}, \quad \frac{\partial G}{\partial z} = 3xz^2 + y\ .$$

a) Determine the value of ∇f at the point $(2,0)$.

b) Determine the value of $\frac{\partial^2 f}{\partial x^2}$ at the point $(2,0)$.

7. (15 points) Let A be a 3×3 matrix such that $A^2 \neq 0$ and $A^3 = 0$.

a) Prove that there is a vector $X \in R^3$ such that $\{X, AX, A^2X\}$ is a basis for R^3.

b) Determine the rank of A and the nullity of A. Give reasons for your answers.