MODULI STACKS OF TWO-DIMENSIONAL GALOIS REPRESENTATIONS

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Abstract. We construct moduli stacks of two-dimensional mod p representations of the absolute Galois group of a p-adic local field, and relate their geometry to the weight part of Serre's conjecture for GL_2.

Contents

1. Introduction

1.1. **Moduli of Galois representations.** Let $K/\mathbf{Q}_p$ be a finite extension, let $\overline{K}$ be an algebraic closure of K, and let $\overline{r} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ be a continuous representation. The theory of deformations of $\overline{r}$ — that is, liftings of $\overline{r}$ to continuous representations $r : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_d(A)$, where A is a complete local ring with residue field $\overline{\mathbf{F}}_p$ — is extremely important in the Langlands program, and in particular is crucial for proving automorphy lifting theorems via the Taylor–Wiles method. Proving such theorems often comes down to studying the moduli spaces of those deformations which satisfy various p-adic Hodge-theoretic conditions.

From the point of view of algebraic geometry, it seems unnatural to study only formal deformations of this kind, and Kisin observed about fifteen years ago that results on the reduction modulo p of two-dimensional crystalline representations

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suggested that there should be moduli spaces of p-adic representations in which the residual representations $\bar{r}$ should be allowed to vary. In particular, the special fibres of these moduli spaces would be would be moduli spaces of mod p representations of $\mathrm{Gal}(\bar{K}/K)$.

In this paper we construct such a space (or rather stack) $\mathcal{Z}$ of mod p representations in the case $d = 2$, and describe its geometry. In particular, we show that their irreducible components are naturally labelled by Serre weights, and that our spaces give a geometrisation of the weight part of Serre's conjecture. More precisely, we prove the following theorem (see Proposition 3.10.19 and Theorem 5.2.2; we explain the definition of a Serre weight, and see Section 1.3 below for the notion of a Serre weight associated to a Galois representation).

Theorem 1.1.1. *The stack $\mathcal{Z}$ is an algebraic stack of finite type over $\overline{\mathbf{F}}_p$, and is equidimensional of dimension $[K : \mathbf{Q}_p]$. The irreducible components of $\mathcal{Z}$ are labelled by the Serre weights $\overline{\sigma}$, in such a way that the $\overline{\mathbf{F}}_p$-points of the component $\mathcal{Z}(\overline{\sigma})$ labelled by $\overline{\sigma}$ are precisely the representations $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ having $\overline{\sigma}$ as a Serre weight.*

We also show that generic points of the irreducible components admit a simple description (they are extensions of characters whose restrictions to inertia are determined by the corresponding Serre weight).

In the course of proving Theorem 1.1.1, we study a partial resolution of the moduli spaces inspired by a construction of Kisin [Kis09] in the setting of formal deformations, and show that its irreducible components are also naturally labelled by Serre weights. We use this resolution to show that our moduli spaces are generically reduced, and as an illustration of the utility of our constructions, we use this to prove the corresponding result for the special fibres of tamely potentially Barsotti–Tate deformation rings (see Proposition 5.1.1). It seems hard to prove this result purely in the setting of formal deformations, and we anticipate that it will have applications to the theory of mod p Hilbert modular forms.

1.2. **The construction.** The reason that we restrict to the case of two-dimensional representations is that in this case one knows that most mod p representations are "tamely potentially finite flat"; that is, after restriction to a finite tamely ramified extension, they come from the generic fibres of finite flat group schemes. Indeed, the only representations not of this form are the so-called très ramifiée representations, which are twists of extensions of the trivial character by the mod p cyclotomic character, and can be described explicitly in terms of Kummer theory. (This is a local Galois-theoretic analogue of the well-known fact that, up to twist, modular forms of any weight and level $\Gamma_1(N)$, with N prime to p, are congruent modulo p to modular forms of weight two and level $\Gamma_1(Np)$; the corresponding modular curves acquire semistable reduction over a tamely ramified extension of $\mathbf{Q}_p$.)

These Galois representations, and the corresponding finite flat group schemes, can be described in terms of semilinear algebra data. Such descriptions also exist for more general p-adic Hodge theoretic conditions (such as being crystalline of prescribed Hodge–Tate weights), although they are more complicated, and can be used to construct analogues, for higher dimensional representations, of the moduli stacks we construct here; this construction is the subject of the forthcoming paper [EG19a].

The semilinear algebra data that we use in the present paper are Breuil–Kisin modules and étale φ-modules. A Breuil–Kisin module is a module with Frobenius over a power series ring, satisfying a condition on the cokernel of the Frobenius which depends on a fixed integer, called the *height* of the Breuil–Kisin module. Inverting the formal variable in the power series ring gives a functor from the category of Breuil–Kisin modules to the category of étale φ-modules. By Fontaine's theory [Fon90], these étale φ-modules correspond to representations of $\mathrm{Gal}(\overline{K}/K_\infty)$, where K_∞ is an infinite non-Galois extension of K obtained by extracting p-power roots of a uniformiser. By work of Breuil and Kisin (in particular [Kis09]), for étale φ-modules that arise from a Breuil–Kisin module of height at most 1 the corresponding representations admit a natural extension to $\mathrm{Gal}(\overline{K}/K)$, and in this way one obtains precisely the finite flat representations. This is the case that we will consider throughout this paper, extended slightly to incorporate descent data from a finite tamely ramified extension K'/K and thereby allowing us to study tamely potentially finite flat representations.

Following Pappas and Rapoport [PR09], we then consider the moduli stack $\mathcal{C}$ of rank two projective Breuil–Kisin modules, and the moduli stack $\mathcal{R}$ of étale φ-modules, together with the natural map $\mathcal{C} \to \mathcal{R}$. We deduce from the results of [PR09] that the stack $\mathcal{C}$ is algebraic (that is, it is an Artin stack); however $\mathcal{R}$ is not algebraic, and indeed is infinite-dimensional. (In fact, we consider versions of these stacks with p-adic coefficients, in which case $\mathcal{C}$ is a p-adic formal algebraic stack, but we suppress this for the purpose of this introduction.) The analogous construction without tame descent data was considered in [EG19b], where it was shown that one can define a notion of the "scheme-theoretic image" of the morphism $\mathcal{C} \to \mathcal{R}$, and that the scheme-theoretic image is algebraic. Using similar arguments, we define our moduli stack $\mathcal{Z}$ of two-dimensional Galois representations to be the scheme-theoretic image of the morphism $\mathcal{C} \to \mathcal{R}$.

By construction, we know that the closed points of $\mathcal{Z}$ are in bijection with the (non-très ramifiée) representations $\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, and by using standard results on the corresponding formal deformation problems, we know that $\mathcal{Z}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$. The closed points of $\mathcal{C}$ correspond to potentially finite flat models of these Galois representations, and we are able to deduce that $\mathcal{C}$ is also equidimensional of dimension $[K : \mathbf{Q}_p]$ (at least morally, this is by Tate's theorem on the uniqueness of prolongations of p-divisible groups).

These constructions are relatively formal. To go further, we combine results from the theory of local models of Shimura varieties and Taylor–Wiles patching with an explicit construction of families of extensions of characters. We begin by describing the last of these.

Intuitively, a natural source of "families" of representations $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is given by the extensions of two fixed characters. Indeed, given two characters $\chi_1, \chi_2 : \mathrm{Gal}(\overline{K}/K) \to \overline{\mathbf{F}}_p^\times$, the $\overline{\mathbf{F}}_p$-vector space $\mathrm{Ext}^1_{\mathrm{Gal}(\overline{K}/K)}(\chi_2, \chi_1)$ is usually $[K : \mathbf{Q}_p]$-dimensional, and a back of the envelope calculation suggests that this should give a $([K : \mathbf{Q}_p]-2)$-dimensional substack of $\mathcal{Z}$ (the difference between an extension and a representation counts for a -1, as does the $\mathbf{G}_m$ of endomorphisms). Twisting χ_1, χ_2 independently by unramified characters gives a candidate for a $[K : \mathbf{Q}_p]$-dimensional family; since $\mathcal{Z}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$, the closure of such a family should be an irreducible component of $\mathcal{Z}$.

Since there are only finitely many possibilities for the restrictions of the χ_i to the inertia subgroup $I(\overline{K}/K)$, this gives a finite list of maximal-dimensional families. On the other hand, there are up to unramified twist only finitely many irreducible two-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$, which suggests that the irreducible representations should correspond to 0-dimensional substacks. Together these considerations suggest that the irreducible components of our moduli stack should be given by the closures of the families of extensions considered in the previous paragraph, and in particular that the irreducible representations should arise as limits of reducible representations. This could not literally be the case for families of Galois representations, rather than families of étale φ-modules, and may seem surprising at first glance, but it is indeed what happens.

1.3. **Serre weights.** In the body of the paper we make this analysis rigorous, and we show that the different families that we have constructed exhaust the irreducible components. We can therefore label the irreducible components of $\mathcal{Z}$ as follows. Let k be the residue field of K; a *Serre weight* is then an irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(k)$ (or rather an isomorphism class thereof). Such a representation is specified by its highest weight, which can be thought of as a pair of characters $k^\times \to \overline{\mathbf{F}}_p^\times$, which via local class field theory corresponds to a pair of characters $I(\overline{K}/K) \to \overline{\mathbf{F}}_p^\times$, and thus to an irreducible component of $\mathcal{Z}$ (in fact, we need to make a shift in this dictionary, corresponding to half the sum of the positive roots of $\mathrm{GL}_2(k)$, but we ignore this for the purposes of this introduction).

This might seem artificial, but in fact it is completely natural, for the following reason. Following the pioneering work of Serre [Ser87] and Buzzard–Diamond–Jarvis [BDJ10] (as extended in [Sch08] and [Gee11]), we now know how to associate a set $W(\overline{r})$ of Serre weights to each continuous representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, with the property that if F is a totally real field and $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is an irreducible representation coming from a Hilbert modular form, then the possible weights of Hilbert modular forms giving rise to $\overline{\rho}$ are precisely determined by the sets $W(\overline{\rho}|_{G_{F_v}})$ for places $v|p$ of F (see for example [BLGG13, GK14, GLS15]).

Going back to our labelling of irreducible components above, we have associated a Serre weight $\overline{\sigma}$ to each irreducible component of $\mathcal{Z}$. One of our main theorems is that the representations $\overline{r}$ on the irreducible component labelled by $\overline{\sigma}$ are precisely the representations with $\overline{\sigma} \in W(\overline{r})$,

We emphasise that the existence of such a geometric interpretation of the sets $W(\overline{r})$ is far from obvious, and indeed we know of no direct proof using any of the explicit descriptions of $W(\overline{r})$ in the literature; it seems hard to understand in any explicit way which Galois representations arise as the limits of a family of extensions of given characters, and the description of the sets $W(\overline{r})$ is very complicated (for example, the description in [BDJ10] relies on certain Ext groups of crystalline characters). Our proof is indirect, and ultimately makes use of a description of $W(\overline{r})$ given in [GK14], which is in terms of potentially Barsotti–Tate deformation rings of $\overline{r}$ and is motivated by the Taylor–Wiles method. We interpret this description in the geometric language of [EG14], which we in turn interpret as the formal completion of a "geometric Breuil–Mézard conjecture" for our stacks.

We also study the irreducible components of the stack $\mathcal{C}$. This stack admits a decomposition as a disjoint union of substacks $\mathcal{C}^\tau$, indexed by the tame inertial types τ (the substack $\mathcal{C}^\tau$ is the moduli of those Breuil–Kisin modules which have descent data given by τ). The inertial local Langlands correspondence assigns a finite set of Serre weights $\mathrm{JH}(\overline{\sigma}(\tau))$ to τ (the Jordan–Hölder factors of the reduction mod p of the representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$ corresponding to τ), and we show that the scheme-theoretic image of the morphism $\mathcal{C}^\tau \to \mathcal{Z}$ is $\mathcal{Z}^\tau = \cup_{\overline{\sigma}\in \mathrm{JH}(\overline{\sigma}(\tau))}\mathcal{Z}(\overline{\sigma})$.

The set $\mathrm{JH}(\overline{\sigma}(\tau))$ can be identified with a subset of the power set $\mathcal{S}$ of the set of embeddings $k \hookrightarrow \overline{\mathbf{F}}_p$. For generic choices of τ, it is equal to $\mathcal{S}$, and in this case we show that the morphism $\mathcal{C}^\tau \to \mathcal{Z}^\tau$ is a generic isomorphism on the source. We are able to show (using the theory of Dieudonné modules) that for any non-scalar type τ, the irreducible components of $\mathcal{C}^\tau$ can be identified with $\mathcal{S}$, and those irreducible components not corresponding to elements of $\mathrm{JH}(\overline{\sigma}(\tau))$ have image in $\mathcal{Z}^\tau$ of positive codimension. (In the case of scalar types, both $\mathcal{C}^\tau$ and $\mathcal{Z}^\tau$ are irreducible.) It follows from the results described that $\mathcal{Z}^\tau$ is generically reduced, which is not at all obvious from its definition.

An important tool in our proofs is that $\mathcal{C}$ has rather mild singularities, and in particular is Cohen–Macaulay and reduced. We show this by relating the singularities of the various $\mathcal{C}^\tau$ to the local models at Iwahori level of Shimura varieties of GL_2-type; such a relationship was first found in [Kis09] (in the context of formal deformation spaces, with no descent data) and [PR09] (in the context of the stacks $\mathcal{C}$, although again without descent data) and developed further by the first author and Levin in [CL18].

1.4. **An outline of the paper.** In Section 2 we recall the theory of Breuil–Kisin modules and étale φ-modules, and explain how it extends to the setting of tame descent data. In Section 3 we define the stacks $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{Z}$, and prove some of their basic properties following [EG19b]. We relate the singularities of $\mathcal{C}$ to those of local models, define the Dieudonné stack, and explain how the morphism from $\mathcal{C}$ to the Dieudonné stack can be thought of in terms of effective Cartier divisors.

In Section 4 we build our families of reducible Galois representations, and show that they are dense in $\mathcal{Z}$. We begin with a thorough study of spaces of extensions of Breuil–Kisin modules, before considering their scheme-theoretic images in $\mathcal{R}$. After some general considerations we specialise to the case of extensions of rank one Breuil–Kisin modules, where we explicitly calculate the dimensions of the extension groups. We also show that the Kisin variety corresponding to an irreducible Galois representation has "small" dimension, by using a base change argument, and proving an upper bound on the Kisin variety for reducible representations via an explicit calculation.

In Section 5 we prove our main results, by combining the hands-on study of $\mathcal{C}$ and $\mathcal{Z}$ of Section 4 with the results on the weight part of Serre's conjecture and the Breuil–Mézard conjecture from [GK14].

We finish with several appendices, summarising results that we use earlier in the paper. Appendix A recalls some properties of formal algebraic stacks from [Eme], and proves a technical result that we use in Section 5. Appendix B recalls some standard facts about Serre weights and the inertial local Langlands correspondence, and finally Appendix C combines the results of [GK14] and [EG14] to prove a geometric Breuil–Mézard result for tamely potentially Barsotti–Tate deformation rings, which we use in Section 5.

1.5. **Final comments.** As explained above, our construction excludes the très ramifiée representations, which are twists of certain extensions of the trivial character by the mod p cyclotomic character. From the point of view of the weight part of Serre's conjecture, they are precisely the representations which admit a twist of the Steinberg representation as their only Serre weight. In accordance with the picture described above, this means that the full moduli stack of 2-dimensional representations of $\mathrm{Gal}(\overline{K}/K)$ can be obtained from our stack by adding in the irreducible components consisting of the très ramifiée representations. This is carried out in [EG19a], and the geometrisation of the weight part of Serre's conjecture described above is extended to this moduli stack, using the results of this paper as an input.

We assume that $p > 2$ in much of the paper; while we expect that our results should also hold if $p = 2$, there are several reasons to exclude this case. We are frequently able to considerably simplify our arguments by assuming that the extension K'/K is not just tamely ramified, but in fact of degree prime to p; this is problematic when $p = 2$, as the consideration of cuspidal types involves a quadratic unramified extension. We also use results on the Breuil–Mézard conjecture which ultimately depend on automorphy lifting theorems that are not available in the case $p = 2$ at present (although it is plausible that the methods of [Tho17] could be used to prove them).

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1.7. **Notation and conventions.**

Topological groups. If M is an abelian topological group with a linear topology, then as in [Sta13, Tag 07E7] we say that M is *complete* if the natural morphism $M \to \varprojlim_i M/U_i$ is an isomorphism, where $\{U_i\}_{i\in I}$ is some (equivalently any) fundamental system of neighbourhoods of 0 consisting of subgroups. Note that in some other references this would be referred to as being *complete and separated*. In particular, any p-adically complete ring A is by definition p-adically separated.

Galois theory and local class field theory. If M is a field, we let G_M denote its absolute Galois group. If M is a global field and v is a place of M, let M_v denote the completion of M at v. If M is a local field, we write I_M for the inertia subgroup of G_M.

Let p be a prime number. Fix a finite extension $K/\mathbf{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. Let e and f be the ramification and inertial degrees of K, respectively, and write $\#k = p^f$ for the cardinality of k. Let K'/K be a finite tamely ramified Galois extension. Let k' be the residue field of K', and let e', f' be the ramification and inertial degrees of K' respectively.

Our representations of G_K will have coefficients in $\overline{\mathbf{Q}}_p$, a fixed algebraic closure of $\mathbf{Q}_p$ whose residue field we denote by $\overline{\mathbf{F}}_p$. Let E be a finite extension of $\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$ and containing the image of every embedding of K' into $\overline{\mathbf{Q}}_p$. Let $\mathcal{O}$ be the ring of integers in E, with uniformiser ϖ and residue field $\mathbf{F} \subset \overline{\mathbf{F}}_p$.

Fix an embedding $\sigma_0 : k' \hookrightarrow \mathbf{F}$, and recursively define $\sigma_i : k' \hookrightarrow \mathbf{F}$ for all $i \in \mathbf{Z}$ so that $\sigma_{i+1}^p = \sigma_i$; of course, we have $\sigma_{i+f'} = \sigma_i$ for all i. We let $e_i \in k' \otimes_{\mathbf{F}_p} \mathbf{F}$ denote the idempotent satisfying $(x \otimes 1)e_i = (1 \otimes \sigma_i(x))e_i$ for all $x \in k'$; note that $\varphi(e_i) = e_{i+1}$. We also denote by e_i the natural lift of e_i to an idempotent in $W(k') \otimes_{\mathbf{Z}_p} \mathcal{O}$. If M is an $W(k') \otimes_{\mathbf{Z}_p} \mathcal{O}$-module, then we write M_i for $e_i M$.

We write $\mathrm{Art}_K \colon K^\times \to W_K^{\mathrm{ab}}$ for the isomorphism of local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements.

Lemma 1.7.1. *Let π be any uniformiser of $\mathcal{O}_K$. The composite $I_K \to \mathcal{O}_K^\times \to k^\times$, where the map $I_K \to \mathcal{O}_K^\times$ is induced by the restriction of Art_K^{-1}, sends an element $g \in I_K$ to the image in $k^\times$ of $g(\pi^{1/(p^f-1)})/\pi^{1/(p^f-1)}$.*

Proof. This follows (for example) from the construction in [Yos08, Prop. 4.4(iii), Prop. 4.7(ii), Cor. 4.9, Def. 4.10]. ☐

For each $\sigma \in \mathrm{Hom}(k, \overline{\mathbf{F}}_p)$ we define the fundamental character ω_σ to σ to be the composite

$$I_K \longrightarrow \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\sigma} \overline{\mathbf{F}}_p^\times,$$

where the map $I_K \to \mathcal{O}_K^\times$ is induced by the restriction of Art_K^{-1}. Let ε denote the p-adic cyclotomic character and $\overline{\varepsilon}$ the mod p cyclotomic character, so that $\prod_{\sigma\in\mathrm{Hom}(k,\overline{\mathbf{F}}_p)} \omega_\sigma^e = \overline{\varepsilon}$. We will often identify characters $I_K \to \overline{\mathbf{F}}_p^\times$ with characters $k^\times \to \overline{\mathbf{F}}_p^\times$ via the Artin map, and similarly for their Teichmüller lifts.

Inertial local Langlands. A two-dimensional *tame inertial type* is (the isomorphism class of) a tamely ramified representation $\tau : I_K \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$ that extends to a representation of G_K and whose kernel is open. Such a representation is of the form $\tau \simeq \eta \oplus \eta'$, and we say that τ is a *tame principal series type* if η, η' both extend to characters of G_K. Otherwise, $\eta' = \eta^q$, and η extends to a character of G_L, where L/K is a quadratic unramified extension. In this case we say that τ is a *tame cuspidal type.*

Henniart's appendix to [BM02] associates a finite dimensional irreducible E-representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$ to each inertial type τ; we refer to this association as the *inertial local Langlands correspondence.* Since we are only working with tame inertial types, this correspondence can be made very explicit as follows.

If $\tau \simeq \eta \oplus \eta'$ is a tame principal series type, then we also write $\eta, \eta' : k^\times \to \mathcal{O}^\times$ for the multiplicative characters determined by $\eta \circ \mathrm{Art}_K|_{\mathcal{O}_K^\times}, \eta' \circ \mathrm{Art}_K|_{\mathcal{O}_K^\times}$ respectively. If $\eta = \eta'$, then we set $\sigma(\tau) = \eta \circ \det$. Otherwise, we write I for the Iwahori subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$ consisting of matrices which are upper triangular modulo a uniformiser ϖ_K of K, and write $\chi = \eta' \otimes \eta : I \to \mathcal{O}^\times$ for the character

$$\begin{pmatrix} a & b \\ \varpi_K c & d \end{pmatrix} \mapsto \eta'(\overline{a})\eta(\overline{d}).$$

Then $\sigma(\tau) := \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$.

If $\tau = \eta \oplus \eta^q$ is a tame cuspidal type, then as above we write L/K for a quadratic unramified extension, and l for the residue field of $\mathcal{O}_L$. We write $\eta : l^\times \to \mathcal{O}^\times$ for the multiplicative character determined by $\eta \circ \mathrm{Art}_L|_{\mathcal{O}_L^\times}$; then $\sigma(\tau)$ is the inflation to $\mathrm{GL}_2(\mathcal{O}_K)$ of the cuspidal representation of $\mathrm{GL}_2(k)$ denoted by $\Theta(\eta)$ in [Dia07].

p-adic Hodge theory. We normalise Hodge–Tate weights so that all Hodge–Tate weights of the cyclotomic character are equal to -1. We say that a potentially crystalline representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has *Hodge type* 0, or is *potentially Barsotti–Tate*, if for each $\varsigma : K \hookrightarrow \overline{\mathbf{Q}}_p$, the Hodge–Tate weights of ρ with respect to ς are 0 and 1. (Note that this is a more restrictive definition of potentially Barsotti–Tate than is sometimes used; however, we will have no reason to deal with representations with non-regular Hodge-Tate weights, and so we exclude them from consideration. Note also that it is more usual in the literature to say that ρ is potentially Barsotti–Tate if it is potentially crystalline, and $\rho^\vee$ has Hodge type 0.)

We say that a potentially crystalline representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has *inertial type* τ if the traces of elements of I_K acting on τ and on

$$\mathrm{D}_{\mathrm{pcris}}(\rho) = \varinjlim_{K'/K} (\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V_\rho)^{G_{K'}}$$

are equal (here V_ρ is the underlying vector space of V_ρ). A representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ *has a potentially Barsotti–Tate lift of type* τ if and only if $\overline{r}$ admits a lift to a representation $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Z}}_p)$ of Hodge type 0 and inertial type τ.

Serre weights. By definition, a *Serre weight* is an irreducible $\mathbf{F}$-representation of $\mathrm{GL}_2(k)$. Concretely, such a representation is of the form

$$\overline{\sigma}_{\vec{t},\vec{s}} := \otimes_{j=0}^{f-1} (\det{}^{t_j} \operatorname{Sym}^{s_j} k^2) \otimes_{k,\sigma_j} \mathbf{F},$$

where $0 \le s_j, t_j \le p-1$ and not all t_j are equal to $p-1$. We say that a Serre weight is *Steinberg* if $s_j = p-1$ for all j, and *non-Steinberg* otherwise.

A remark on normalisations. Given a continuous representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, there is an associated (nonempty) set of Serre weights $W(\overline{r})$ whose precise definition we will recall in Appendix B. There are in fact several different definitions of $W(\overline{r})$ in the literature; as a result of the papers [BLGG13, GK14, GLS15], these definitions are known to be equivalent up to normalisation.

However, the normalisations of Hodge–Tate weights and of inertial local Langlands used in [GK14, GLS15, EGS15] are not all the same, and so for clarity we lay out how they differ, and how they compare to the normalisations of this paper.

Our conventions for Hodge–Tate weights and inertial types agree with those of [GK14, EGS15], but our representation $\sigma(\tau)$ is the representation $\sigma(\tau^\vee)$ of [GK14, EGS15] (where $\tau^\vee = \eta^{-1} \oplus (\eta')^{-1}$); to see this, note the dual in the definition of $\sigma(\tau)$ in [GK14, Thm. 2.1.3] and the discussion in §1.9 of [EGS15].[1]

In all cases one chooses to normalise the set of Serre weights so that the condition of Lemma B.5(1) holds. Consequently, our set of weights $W(\overline{r})$ is the set of duals of the weights $W(\overline{r})$ considered in [GK14]. In turn, the paper [GLS15] has the opposite convention for the signs of Hodge–Tate weights to our convention (and to the convention of [GK14]), so we find that our set of weights $W(\overline{r})$ is the set of duals of the weights $W(\overline{r}^\vee)$ considered in [GLS15].

Stacks. We follow the terminology of [Sta13]; in particular, we write "algebraic stack" rather than "Artin stack". More precisely, an algebraic stack is a stack in groupoids in the *fppf* topology, whose diagonal is representable by algebraic spaces, which admits a smooth surjection from a scheme. See [Sta13, Tag 026N] for a discussion of how this definition relates to others in the literature, and [Sta13, Tag 04XB] for key properties of morphisms representable by algebraic spaces.

For a commutative ring A, an *fppf stack over* A (or *fppf* A-stack) is a stack fibred in groupoids over the big *fppf* site of $\operatorname{Spec} A$.

[1]However, this dual is erroneously omitted when the inertial local Langlands correspondence is made explicit at the end of [EGS15, §3.1]. See Remark B.1.

Scheme-theoretic images. We briefly remind the reader of some definitions from [EG19b, §3.2]. Let $\mathcal{X} \to \mathcal{F}$ be a proper morphism of stacks over a locally Noetherian base-scheme S, where $\mathcal{X}$ is an algebraic stack which is locally of finite presentation over S, and the diagonal of $\mathcal{F}$ is representable by algebraic spaces and locally of finite presentation.

We refer to [EG19b, Defn. 3.2.8] for the definition of the *scheme-theoretic image* $\mathcal{Z}$ of the proper morphism $\mathcal{X} \to \mathcal{F}$. By definition, it is a full subcategory in groupoids of $\mathcal{F}$, and in fact by [EG19b, Lem. 3.2.9] it is a Zariski substack of $\mathcal{F}$. By [EG19b, Lem. 3.2.14], the finite type points of $\mathcal{Z}$ are precisely the finite type points of $\mathcal{F}$ for which the corresponding fibre of $\mathcal{X}$ is nonzero.

The results of [EG19b, §3.2] give criteria for $\mathcal{Z}$ to be an algebraic stack, and prove a number of associated results (such as universal properties of the morphism $\mathcal{Z} \to \mathcal{F}$, and a description of versal deformation rings for $\mathcal{Z}$); rather than recalling these results in detail here, we will refer to them as needed in the body of the paper.

2. Integral p-adic Hodge theory with tame descent data

In this section we introduce various objects in semilinear algebra which arise in the study of potentially Barsotti–Tate Galois representations with tame descent data. Much of this material is standard, and none of it will surprise an expert, but we do not know of a treatment in the literature in the level of generality that we require; in particular, we are not aware of a treatment of the theory of tame descent data for Breuil–Kisin modules. However, the arguments are almost identical to those for strongly divisible modules and Breuil modules, so we will be brief.

The various equivalences of categories between the objects we consider and finite flat group schemes or p-divisible groups will not be relevant to our main arguments, except at a motivational level, so we largely ignore them.

2.1. **Breuil–Kisin modules and φ-modules with descent data.** Recall that we have a finite tamely ramified Galois extension K'/K. Suppose further that there exists a uniformiser π' of $\mathcal{O}_{K'}$ such that $\pi := (\pi')^{e(K'/K)}$ is an element of K, where $e(K'/K)$ is the ramification index of K'/K. Recall that k' is the residue field of K', while e', f' are the ramification and inertial degrees of K' respectively. Let $E(u)$ be the minimal polynomial of π' over $W(k')[1/p]$.

Let φ denote the arithmetic Frobenius automorphism of k', which lifts uniquely to an automorphism of $W(k')$ that we also denote by φ. Define $\mathfrak{S} := W(k')[[u]]$, and extend φ to $\mathfrak{S}$ by

$$\varphi\left(\sum a_i u^i\right) = \sum \varphi(a_i) u^{pi}.$$

By our assumptions that $(\pi')^{e(K'/K)} \in K$ and that K'/K is Galois, for each $g \in \mathrm{Gal}(K'/K)$ we can write $g(\pi')/\pi' = h(g)$ with $h(g) \in \mu_{e(K'/K)}(K') \subset W(k')$, and we let $\mathrm{Gal}(K'/K)$ act on $\mathfrak{S}$ via

$$g\left(\sum a_i u^i\right) = \sum g(a_i) h(g)^i u^i.$$

Let A be a p-adically complete $\mathbf{Z}_p$-algebra, set $\mathfrak{S}_A := (W(k') \otimes_{\mathbf{Z}_p} A)[[u]]$, and extend the actions of φ and $\mathrm{Gal}(K'/K)$ on $\mathfrak{S}$ to actions on $\mathfrak{S}_A$ in the obvious (A-linear) fashion.

Lemma 2.1.1. *An $\mathfrak{S}_A$-module is projective if and only if it is projective as an $A[[u]]$-module.*

Proof. Suppose that $\mathfrak{M}$ is an $\mathfrak{S}_A$-module that is projective as an $A[[u]]$-module. Certainly $W(k')\otimes_{\mathbf{Z}_p}\mathfrak{M}$ is projective over $\mathfrak{S}_A$, and we claim that it has $\mathfrak{M}$ as an $\mathfrak{S}_A$-module direct summand. Indeed, this follows by rewriting $\mathfrak{M}$ as $W(k')\otimes_{W(k')}\mathfrak{M}$ and noting that $W(k')$ is a $W(k')$-module direct summand of $W(k')\otimes_{\mathbf{Z}_p}W(k')$. $\square$

The actions of φ and $\mathrm{Gal}(K'/K)$ on $\mathfrak{S}_A$ extend to actions on $\mathfrak{S}_A[1/u] = (W(k')\otimes_{\mathbf{Z}_p}A)((u))$ in the obvious way. It will sometimes be necessary to consider the subring $\mathfrak{S}^0_A := (W(k)\otimes_{\mathbf{Z}_p}A)[[v]]$ of $\mathfrak{S}_A$ consisting of power series in $v := u^{e(K'/K)}$, on which $\mathrm{Gal}(K'/K)$ acts trivially.

Definition 2.1.2. Fix a p-adically complete $\mathbf{Z}_p$-algebra A. A *weak Breuil–Kisin module with A-coefficients and descent data from K' to K* is a triple $(\mathfrak{M},\varphi_{\mathfrak{M}},\{\hat{g}\}_{g\in\mathrm{Gal}(K'/K)})$ consisting of a $\mathfrak{S}_A$-module $\mathfrak{M}$ and a φ-semilinear map $\varphi_{\mathfrak{M}}:\mathfrak{M}\to\mathfrak{M}$ such that:

- the $\mathfrak{S}_A$-module $\mathfrak{M}$ is finitely generated and u-torsion free, and
- the induced map $\Phi_{\mathfrak{M}} = 1\otimes\varphi_{\mathfrak{M}}:\varphi^*\mathfrak{M}\to\mathfrak{M}$ is an isomorphism after inverting $E(u)$ (here as usual we write $\varphi^*\mathfrak{M} := \mathfrak{S}_A\otimes_{\varphi,\mathfrak{S}_A}\mathfrak{M}$),

together with additive bijections $\hat{g}:\mathfrak{M}\to\mathfrak{M}$, satisfying the further properties that the maps $\hat{g}$ commute with $\varphi_{\mathfrak{M}}$, satisfy $\hat{g}_1\circ\hat{g}_2 = \widehat{g_1\circ g_2}$, and have $\hat{g}(sm) = g(s)\hat{g}(m)$ for all $s\in\mathfrak{S}_A$, $m\in\mathfrak{M}$. We say that $\mathfrak{M}$ is has *height at most h* if the cokernel of $\Phi_{\mathfrak{M}}$ is killed by $E(u)^h$.

If $\mathfrak{M}$ as above is projective as an $\mathfrak{S}_A$-module (equivalently, if the condition that the $\mathfrak{M}$ is u-torsion free is replaced with the condition that $\mathfrak{M}$ is projective) then we say that $\mathfrak{M}$ is a *Breuil–Kisin module with A-coefficients and descent data from K' to K*, or even simply that $\mathfrak{M}$ is a *Breuil–Kisin module.*

The Breuil–Kisin module $\mathfrak{M}$ is said to be of rank d if the underlying finitely generated projective $\mathfrak{S}_A$-module has constant rank d. It is said to be free if the underlying $\mathfrak{S}_A$-module is free.

A morphism of (weak) Breuil–Kisin modules with descent data is a morphism of $\mathfrak{S}_A$-modules that commutes with φ and with the $\hat{g}$. In the case that $K'=K$ the data of the $\hat{g}$ is trivial, so it can be forgotten, giving the category of *(weak) Breuil–Kisin modules with A-coefficients.* In this case it will sometimes be convenient to elide the difference between a Breuil–Kisin module with trivial descent data, and a Breuil–Kisin module without descent data, in order to avoid making separate definitions in the case of Breuil–Kisin modules without descent data; the same convention will apply to the étale φ-modules considered below.

Lemma 2.1.3. *Suppose either that A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a\geq 1$, or that A is p-adically separated and $\mathfrak{M}$ is projective. Then in Definition 2.1.2 the condition that $\Phi_{\mathfrak{M}}$ is an isomorphism after inverting $E(u)$ may equivalently be replaced with the condition that $\Phi_{\mathfrak{M}}$ is injective and its cokernel is killed by a power of $E(u)$.*

Proof. If A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a\geq 0$, then $E(u)^h$ divides $u^{e(a+h-1)}$ in $\mathfrak{S}_A$ (see [EG19b, Lem. 5.2.6] and its proof), so that $\mathfrak{M}[1/u]$ is étale in the sense that the induced map

$$\Phi_{\mathfrak{M}}[1/u]:\varphi^*\mathfrak{M}[1/u]\to\mathfrak{M}[1/u]$$

is an isomorphism. The injectivity of $\Phi_{\mathfrak{M}}$ now follows because $\mathfrak{M}$, and therefore $\varphi^*\mathfrak{M}$, is u-torsion free.

If instead A is p-adically complete, then no Eisenstein polynomial over $W(k')$ is a zero divisor in $\mathfrak{S}_A$: this is plainly true if p is nilpotent in A, from which one deduces the same for p-adically complete A. Assuming that $\mathfrak{M}$ is projective, it follows that the maps $\mathfrak{M} \to \mathfrak{M}[1/E(u)]$ and $\varphi^*\mathfrak{M} \to (\varphi^*\mathfrak{M})[1/E(u)]$ are injective, and we are done. ☐

Remark 2.1.4. We refer the reader to [EG19b, §5.1] for a discussion of foundational results concerning finitely generated modules over the power series ring $A[[u]]$. In particular (using Lemma 2.1.1) we note the following.

(1) An $\mathfrak{S}_A$-module $\mathfrak{M}$ is finitely generated and projective if and only if it is u-torsion free and u-adically complete, and $\mathfrak{M}/u\mathfrak{M}$ is a finitely generated projective A-module ([EG19b, Prop. 5.1.8]).
(2) If the $\mathfrak{S}_A$-module $\mathfrak{M}$ is projective of rank d, then it is Zariski locally free of rank d in the sense that there is a cover of $\operatorname{Spec} A$ by affine opens $\operatorname{Spec} B_i$ such that each of the base-changed modules $\mathfrak{M} \otimes_{\mathfrak{S}_A} \mathfrak{S}_{B_i}$ is free of rank d ([EG19b, Prop. 5.1.9]).
(3) If A is coherent (so in particular, if A is Noetherian), then $A[[u]]$ is faithfully flat over A, and so $\mathfrak{S}_A$ is faithfully flat over A, but this need not hold if A is not coherent.

Definition 2.1.5. If Q is any (not necessarily finitely generated) A-module, and $\mathfrak{M}$ is an $A[[u]]$-module, then we let $\mathfrak{M}\widehat{\otimes}_A Q$ denote the u-adic completion of $\mathfrak{M} \otimes_A Q$.

Lemma 2.1.6. *If $\mathfrak{M}$ is a Breuil–Kisin module and B is an A-algebra, then the base change $\mathfrak{M}\widehat{\otimes}_A B$ is a Breuil–Kisin module.*

Proof. We claim that $\mathfrak{M}\widehat{\otimes}_A B \cong \mathfrak{M} \otimes_{A[[u]]} B[[u]]$ for any finitely generated projective $A[[u]]$-module; the lemma then follows immediately from Definition 2.1.2.

To check the claim, we must see that the finitely generated $B[[u]]$-module $\mathfrak{M} \otimes_{A[[u]]} B[[u]]$ is u-adically complete. But $\mathfrak{M}$ is a direct summand of a free $A[[u]]$-module of finite rank, in which case $\mathfrak{M} \otimes_{A[[u]]} B[[u]]$ is a direct summand of a free $B[[u]]$-module of finite rank and hence is u-adically complete. ☐

Remark 2.1.7. If $I \subset A$ is a finitely generated ideal then $A[[u]] \otimes_A A/I \cong (A/I)[[u]]$, and $\mathfrak{M} \otimes_A A/I \cong \mathfrak{M} \otimes_{A[[u]]} (A/I)[[u]] \cong \mathfrak{M}\widehat{\otimes}_A A/I$; so in this case $\mathfrak{M} \otimes_A A/I$ itself is a Breuil–Kisin module.

Note that the base change (in the sense of Definition 2.1.5) of a weak Breuil–Kisin module may not be a weak Breuil–Kisin module, because the property of being u-torsion free is not always preserved by base change.

We make the following two further remarks concerning base change.

Remark 2.1.8. (1) If A is Noetherian, if Q is finitely generated over A, and if $\mathfrak{N}$ is finitely generated over $A[[u]]$, then $\mathfrak{N} \otimes_A Q$ is finitely generated over $A[[u]]$, and hence (by the Artin–Rees lemma) is automatically u-adically complete. Thus in this case the natural morphism $\mathfrak{N} \otimes_A Q \to \mathfrak{N}\widehat{\otimes}_A Q$ is an isomorphism.

(2) Note that $A[[u]]\widehat{\otimes}_A Q = Q[[u]]$ (the $A[[u]]$-module consisting of power series with coefficients in the A-module Q), and so if $\mathfrak{N}$ is Zariski locally free on $\operatorname{Spec} A$, then $\mathfrak{N}\widehat{\otimes}_A Q$ is Zariski locally isomorphic to a direct sum of copies of $Q[[u]]$, and hence is u-torsion free (as well as being u-adically complete). In particular, by Remark 2.1.4(2), this holds if $\mathfrak{N}$ is projective.

Definition 2.1.9. Let A be a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \geq 1$. A *weak étale φ-module* with A-coefficients and descent data from K' to K is a triple $(M, \varphi_M, \{\hat{g}\})$ consisting of:

- a finitely generated $\mathfrak{S}_A[1/u]$-module M;
- a φ-semilinear map $\varphi_M : M \to M$ with the property that the induced map
$$\Phi_M = 1 \otimes \varphi_M : \varphi^*M := \mathfrak{S}_A[1/u] \otimes_{\varphi, \mathfrak{S}_A[1/u]} M \to M$$
is an isomorphism,

together with additive bijections $\hat{g} : M \to M$ for $g \in \mathrm{Gal}(K'/K)$, satisfying the further properties that the maps $\hat{g}$ commute with φ_M, satisfy $\hat{g_1} \circ \hat{g_2} = \widehat{g_1 \circ g_2}$, and have $\hat{g}(sm) = g(s)\hat{g}(m)$ for all $s \in \mathfrak{S}_A[1/u]$, $m \in M$.

If M as above is projective as an $\mathfrak{S}_A[1/u]$-module then we say simply that M is an étale φ-module. The étale φ-module M is said to be of rank d if the underlying finitely generated projective $\mathfrak{S}_A[1/u]$-module has constant rank d.

Remark 2.1.10. We could also consider étale φ-modules for general p-adically complete $\mathbf{Z}_p$-algebras A, but we would need to replace $\mathfrak{S}_A[1/u]$ by its p-adic completion. As we will not need to consider these modules in this paper, we do not do so here, but we refer the interested reader to [EG19a].

A morphism of weak étale φ-modules with A-coefficients and descent data from K' to K is a morphism of $\mathfrak{S}_A[1/u]$-modules that commutes with φ and with the $\hat{g}$. Again, in the case $K' = K$ the descent data is trivial, and we obtain the usual category of étale φ-modules with A-coefficients.

Note that if A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra, and $\mathfrak{M}$ is a Breuil–Kisin module (resp., weak Breuil–Kisin module) with descent data, then $\mathfrak{M}[1/u]$ naturally has the structure of an étale φ-module (resp., weak étale φ-module) with descent data.

Suppose that A is an $\mathcal{O}$-algebra (where $\mathcal{O}$ is as in Section 1.7). In making calculations, it is often convenient to use the idempotents e_i (again as in Section 1.7). In particular if $\mathfrak{M}$ is a Breuil–Kisin module, then writing as usual $\mathfrak{M}_i := e_i\mathfrak{M}$, we write $\Phi_{\mathfrak{M},i} : \varphi^*(\mathfrak{M}_{i-1}) \to \mathfrak{M}_i$ for the morphism induced by $\Phi_{\mathfrak{M}}$. Similarly if M is an étale φ-module then we write $M_i := e_iM$, and we write $\Phi_{M,i} : \varphi^*(M_{i-1}) \to M_i$ for the morphism induced by Φ_M.

2.2. **Dieudonné modules.** Let A be a $\mathbf{Z}_p$-algebra. We define a *Dieudonné module of rank d with A-coefficients and descent data from K' to K* to be a finitely generated projective $W(k') \otimes_{\mathbf{Z}_p} A$-module D of constant rank d on $\operatorname{Spec} A$, together with:

- A-linear endomorphisms F, V satisfying $FV = VF = p$ such that F is φ-semilinear and V is φ^{-1}-semilinear for the action of $W(k')$, and
- a $W(k') \otimes_{\mathbf{Z}_p} A$-semilinear action of $\mathrm{Gal}(K'/K)$ which commutes with F and V.

Definition 2.2.1. If $\mathfrak{M}$ is a Breuil–Kisin module of height at most 1 and rank d with descent data, then there is a corresponding Dieudonné module $D = D(\mathfrak{M})$ of rank d defined as follows. We set $D := \mathfrak{M}/u\mathfrak{M}$ with the induced action of $\mathrm{Gal}(K'/K)$, and F given by the induced action of φ. The endomorphism V is determined as follows. Write $E(0) = cp$, so that we have $p \equiv c^{-1}E(u) \pmod{u}$. The condition that the cokernel of $\varphi^*\mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)$ allows us to factor the multiplication-by-$E(u)$ map on $\mathfrak{M}$ uniquely as $\mathfrak{V} \circ \varphi$, and V is defined to be $c^{-1}\mathfrak{V}$ modulo u.

2.3. **Galois representations.** The theory of fields of norms [FW79] was used in [Fon90] to relate étale φ-modules with descent data to representations of a certain absolute Galois group; not the group G_K, but rather the group G_{K_∞}, where K_∞ is a certain infinite extension of K (whose definition is recalled below). Breuil–Kisin modules of height $h \leq 1$ are closely related to finite flat group schemes (defined over $\mathcal{O}_{K'}$, but with descent data to K on their generic fibre). Passage from a Breuil–Kisin module to its associated étale φ-module can morally be interpreted as the passage from a finite flat group scheme (with descent data) to its corresponding Galois representation (restricted to G_{K_∞}). Since the generic fibre of a finite flat group scheme over $\mathcal{O}_{K'}$, when equipped with descent data to K, in fact gives rise to a representation of G_K, in the case $h = 1$ we may relate Breuil–Kisin modules with descent data (or, more precisely, their associated étale φ-modules), not only to representations of G_{K_∞}, but to representations of G_K.

In this subsection, we recall some results coming from this connection, and draw some conclusions for Galois deformation rings.

2.3.1. *From étale φ-modules to G_{K_∞}-representations.* We begin by recalling from [Kis09] some constructions arising in p-adic Hodge theory and the theory of fields of norms, which go back to [Fon90]. Following Fontaine, we write $R := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\bar{K}}/p$. Fix a compatible system $(\sqrt[p^n]{\pi})_{n \geq 0}$ of p^nth roots of π in $\bar{K}$ (compatible in the obvious sense that $\left(\sqrt[p^{n+1}]{\pi}\right)^p = \sqrt[p^n]{\pi}$), and let $K_\infty := \cup_n K(\sqrt[p^n]{\pi})$, and also $K'_\infty := \cup_n K'(\sqrt[p^n]{\pi})$. Since $(e(K'/K), p) = 1$, the compatible system $(\sqrt[p^n]{\pi})_{n \geq 0}$ determines a unique compatible system $(\sqrt[p^n]{\pi'})_{n \geq 0}$ of p^nth roots of π' such that $(\sqrt[p^n]{\pi'})^{e(K'/K)} = \sqrt[p^n]{\pi}$. Write $\underline{\pi}' = (\sqrt[p^n]{\pi'})_{n \geq 0} \in R$, and $[\underline{\pi}'] \in W(R)$ for its Teichmüller representative. We have a Frobenius-equivariant inclusion $\mathfrak{S} \hookrightarrow W(R)$ by sending $u \mapsto [\underline{\pi}']$. We can naturally identify $\mathrm{Gal}(K'_\infty/K_\infty)$ with $\mathrm{Gal}(K'/K)$, and doing this we see that the action of $g \in G_{K_\infty}$ on u is via $g(u) = h(g)u$.

We let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}[1/u]$, and let $\mathcal{E}$ be the field of fractions of $\mathcal{O}_{\mathcal{E}}$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to an inclusion $\mathcal{E} \hookrightarrow W(\mathrm{Frac}(R))[1/p]$. Let $\mathcal{E}^{\mathrm{nr}}$ be the maximal unramified extension of $\mathcal{E}$ in $W(\mathrm{Frac}(R))[1/p]$, and let $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}} \subset W(\mathrm{Frac}(R))$ denote its ring of integers. Let $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ be the p-adic completion of $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}$. Note that $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ is stable under the action of G_{K_∞}.

Definition 2.3.2. Suppose that A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra for some $a \geq 1$. If M is a weak étale φ-module with A-coefficients and descent data, set $T_A(M) := \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}[1/u]} M\right)^{\varphi=1}$, an A-module with a G_{K_∞}-action (via the diagonal action on $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ and M, the latter given by the $\hat{g}$). If $\mathfrak{M}$ is a weak Breuil–Kisin module with A-coefficients and descent data, set $T_A(\mathfrak{M}) := T_A(\mathfrak{M}[1/u])$.

Lemma 2.3.3. *Suppose that A is a local $\mathbf{Z}_p$-algebra and that $|A| < \infty$. Then T_A induces an equivalence of categories from the category of weak étale φ-modules with A-coefficients and descent data to the category of continuous representations of G_{K_∞} on finite A-modules. If $A \to A'$ is finite, then there is a natural isomorphism $T_A(M) \otimes_A A' \xrightarrow{\sim} T_{A'}(M \otimes_A A')$. A weak étale φ-module with A-coefficients and descent data M is free of rank d if and only if $T_A(M)$ is a free A-module of rank d.*

Proof. This is due to Fontaine [Fon90], and can be proved in exactly the same way as [Kis09, Lem. 1.2.7]. $\square$

We will frequently simply write T for T_A. Note that if we let M' be the étale φ-module obtained from M by forgetting the descent data, then by definition we have $T(M') = T(M)|_{G_{K'_\infty}}$.

2.3.4. *Relationships between G_K-representations and G_{K_∞}-representations.* We will later need to study deformation rings for representations of G_K in terms of the deformation rings for the restrictions of these representations to G_{K_∞}. Note that the representations of G_{K_∞} coming from Breuil–Kisin modules of height at most 1 admit canonical extensions to G_K by [Kis09, Prop. 1.1.13].

Lemma 2.3.5. *If $\overline{r},\overline{r}' : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ are continuous representations, both of which arise as the reduction mod p of potentially Barsotti–Tate representations of tame inertial type, and there is an isomorphism $\overline{r}|_{G_{K_\infty}} \cong \overline{r}'|_{G_{K_\infty}}$, then $\overline{r} \cong \overline{r}'$.*

Proof. The extension K_∞/K is totally wildly ramified. Since the irreducible $\overline{\mathbf{F}}_p$-representations of G_K are induced from tamely ramified characters, we see that $\overline{r}|_{G_{K_\infty}}$ is irreducible if and only if $\overline{r}$ is irreducible, and if $\overline{r}$ or $\overline{r}'$ is irreducible then we are done. In the reducible case, we see that $\overline{r}$ and $\overline{r}'$ are extensions of the same characters, and the result then follows from [GLS15, Lem. 5.4.2] and Lemma B.5 (2). ☐

Let $\overline{r} : G_K \to \mathrm{GL}_2(\mathbf{F})$ be a continuous representation, let $R_{\overline{r}}$ denote the universal framed deformation $\mathcal{O}$-algebra for $\overline{r}$, and let $R_{\overline{r}}^{[0,1]}$ be the quotient with the property that if A is an Artinian local $\mathcal{O}$-algebra with residue field $\mathbf{F}$, then a local $\mathcal{O}$-morphism $R_{\overline{r}} \to A$ factors through $R_{\overline{r}}^{[0,1]}$ if and only if the corresponding G_K-module (ignoring the A-action) admits a G_K-equivariant surjection from a potentially crystalline $\mathcal{O}$-representation all of whose Hodge–Tate weights are equal to 0 or 1, and whose restriction to $G_{K'}$ is crystalline. (The existence of this quotient follows as in [Kim11, §2.1].)

Let $R_{\overline{r}|_{G_{K_\infty}}}$ be the universal framed deformation $\mathcal{O}$-algebra for $\overline{r}|_{G_{K_\infty}}$, and let $R^{\le 1}_{\overline{r}|_{G_{K_\infty}}}$ denote the quotient with the property that if A is an Artinian local $\mathcal{O}$-algebra with residue field $\mathbf{F}$, then a morphism $R_{\overline{r}|_{G_{K_\infty}}} \to A$ factors through $R^{\le 1}_{\overline{r}|_{G_{K_\infty}}}$ if and only if the corresponding G_{K_∞}-module is isomorphic to $T(\mathfrak{M})$ for some weak Breuil–Kisin module $\mathfrak{M}$ of height at most one with A-coefficients and descent data from K' to K. (The existence of this quotient follows exactly as for [Kim11, Thm. 1.3].)

Proposition 2.3.6. *The natural map induced by restriction from G_K to G_{K_∞} induces an isomorphism* $\operatorname{Spec} R_{\overline{r}}^{[0,1]} \to \operatorname{Spec} R^{\le 1}_{\overline{r}|_{G_{K_\infty}}}$.

Proof. This can be proved in exactly the same way as [Kim11, Cor. 2.2.1] (which is the case that $E = \mathbf{Q}_p$ and $K' = K$). ☐

3. Moduli stacks of Breuil–Kisin modules and φ-modules with descent data

In this section we define moduli stacks of Breuil–Kisin modules with tame descent data, following [PR09, EG19b] (which consider the case without descent data). In particular, we define various stacks $\mathcal{Z}$ in Section 3.9, as scheme-theoretic images of morphisms from moduli stacks of Breuil–Kisin modules to moduli stacks of étale φ-modules; these stacks are the main objects of interest in the rest of the paper. In the rest of the section, we use the theories of local models of Shimura varieties and Dieudonné modules to begin our study of the geometry of these stacks.

3.1. **Moduli stacks of Breuil–Kisin modules.** We begin by defining the moduli stacks of Breuil–Kisin modules, with and without descent data. We will make use of the notion of a ϖ-adic formal algebraic stack, which is recalled in Appendix A.

Definition 3.1.1. For each integer $a \geq 1$, we let $\mathcal{C}^{\mathrm{dd},a}_{d,h,K'}$ be the *fppf* stack over $\mathcal{O}/\varpi^a$ which associates to any $\mathcal{O}/\varpi^a$-algebra A the groupoid $\mathcal{C}^{\mathrm{dd},a}_{d,h,K'}(A)$ of rank d Breuil–Kisin modules of height at most h with A-coefficients and descent data from K' to K.

By [Sta13, Tag 04WV], we may also regard each of the stacks $\mathcal{C}^{\mathrm{dd},a}_{d,h,K'}$ as an *fppf* stack over $\mathcal{O}$, and we then write $\mathcal{C}^{\mathrm{dd}}_{d,h,K'} := \varinjlim_a \mathcal{C}^{\mathrm{dd},a}_{d,h,K'}$; this is again an *fppf* stack over $\mathcal{O}$.

We will frequently omit any (or all) of the subscripts d, h, K' from this notation when doing so will not cause confusion. In the case that $K = K'$, we write $\mathcal{C}^a$ for $\mathcal{C}^{\mathrm{dd},a}$ and $\mathcal{C}$ for $\mathcal{C}^{\mathrm{dd}}$.

The natural morphism $\mathcal{C}^{\mathrm{dd}} \to \operatorname{Spec} \mathcal{O}$ factors through $\operatorname{Spf} \mathcal{O}$, and by construction, there is an isomorphism $\mathcal{C}^{\mathrm{dd},a} \xrightarrow{\sim} \mathcal{C}^{\mathrm{dd}} \times_{\operatorname{Spf} \mathcal{O}} \operatorname{Spec} \mathcal{O}/\varpi^a$, for each $a \geq 1$; in particular, each of the morphisms $\mathcal{C}^{\mathrm{dd},a} \to \mathcal{C}^{\mathrm{dd},a+1}$ is a thickening (in the sense that its pullback under any test morphism $\operatorname{Spec} A \to \mathcal{C}^{\mathrm{dd},a+1}$ becomes a thickening of schemes, as defined in [Sta13, Tag 04EX][2]). In Corollary 3.1.7 below we show that for each integer $a \geq 1$, $\mathcal{C}^{\mathrm{dd},a}$ is in fact an algebraic stack of finite type over $\operatorname{Spec} \mathcal{O}/\varpi^a$, and that $\mathcal{C}^{\mathrm{dd}}$ (which is then *a priori* an Ind-algebraic stack, endowed with a morphism to $\operatorname{Spf} \mathcal{O}$ which is representable by algebraic stacks) is in fact a ϖ-adic formal algebraic stack, in the sense of Definition A.2.

Our approach will be to deduce the statements in the case with descent data from the corresponding statements in the case with no descent data, which follow from the methods of Pappas and Rapoport [PR09]. More precisely, in that reference it is proved that each $\mathcal{C}^a$ is an algebraic stack over $\mathcal{O}/\varpi^a$ [PR09, Thm. 0.1 (i)], and thus that $\mathcal{C} := \varinjlim_a \mathcal{C}^a$ is a ϖ-adic Ind-algebraic stack (in the sense that it is an Ind-algebraic stack with a morphism to $\operatorname{Spf} \mathcal{O}$ that is representable by algebraic stacks). (In [PR09] the stack $\mathcal{C}$ is described as being a p-adic formal algebraic stack. However, in that reference, this term is used synonomously with our notion of a p-adic Ind-algebraic stack; the question of the existence of a smooth cover of $\mathcal{C}$ by a p-adic formal algebraic space is not discussed. As we will see, though, the existence of such a cover is easily deduced from the set-up of [PR09].)

We thank Brandon Levin for pointing out the following result to us. The proof is essentially (but somewhat implicitly) contained in the proof of [CL18, Thm. 3.5], but we take the opportunity to make it explicit. Note that it could also be directly deduced from the results of [PR09] using Lemma A.3, but the proof that we give has the advantage of giving an explicit cover by a formal algebraic space.

Proposition 3.1.2. *For any choice of d, h, $\mathcal{C}$ is a ϖ-adic formal algebraic stack of finite type over* $\operatorname{Spf} \mathcal{O}$ *with affine diagonal.*

Proof. We begin by recalling some results from [PR09, §3.b] (which is where the proof that each $\mathcal{C}^a$ is an algebraic stack of finite type over $\mathcal{O}/\varpi^a$ is given). If A is

[2]Note that for morphisms of algebraic stacks — and we will see below that $\mathcal{C}^{\mathrm{dd},a}$ and $\mathcal{C}^{\mathrm{dd},a+1}$ *are* algebraic stacks — this notion of thickening coincides with the notion defined in [Sta13, Tag 0BPP], by [Sta13, Tag 0CJ7].

an $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, then we set

$$L^+G(A) := GL_d(\mathfrak{S}_A),$$

$$LG^{h,K'}(A) := \{X \in M_d(\mathfrak{S}_A) \mid X^{-1} \in E(u)^{-h} M_d(\mathfrak{S}_A)\},$$

and let $g \in L^+G(A)$ act on the right on $LG^{h,K'}(A)$ by φ-conjugation as $g^{-1} \cdot X \cdot \varphi(g)$. Then we may write

$$\mathcal{C} = [LG^{h,K'} /_{\phi}\, L^+G].$$

For each $n \geq 1$ we have the principal congruence subgroup U_n of L^+G given by $U_n(A) = I + u^n \cdot M_d(\mathfrak{S}_A)$. As in [PR09, §3.b.2], for any integer $n(a) > eah/(p-1)$ we have a natural identification

$$[LG^{h,K'} /_{\phi}\, U_{n(a)}]_{\mathcal{O}/\varpi^a} \cong [LG^{h,K'}/U_{n(a)}]_{\mathcal{O}/\varpi^a} \tag{3.1.3}$$

where the $U_{n(a)}$-action on the right hand side is by left translation by the inverse; moreover this quotient stack is represented by a finite type scheme $(X^{h,K'}_{n(a)})_{\mathcal{O}/\varpi^a}$, and we find that

$$\mathcal{C}^a \cong [(X^{h,K'}_{n(a)})_{\mathcal{O}/\varpi^a} /_{\phi}\, (\mathcal{G}_{n(a)})_{\mathcal{O}/\varpi^a}],$$

where $(\mathcal{G}_{n(a)})_{\mathcal{O}/\varpi^a} = (L^+G/U_{n(a)})_{\mathcal{O}/\varpi^a}$ is a smooth finite type group scheme over $\mathcal{O}/\varpi^a$.

Now define $Y_a := [(X^{h,K'}_{n(a)})_{\mathcal{O}/\varpi^a} /_{\phi}\, (U_{n(1)})_{\mathcal{O}/\varpi^a}]$. If $a \geq b$, then there is a natural isomorphism $(Y_a)_{\mathcal{O}/\varpi^b} \cong Y_b$. Thus we may form the ϖ-adic Ind-algebraic stack $Y := \varinjlim_a Y_a$. Since $Y_1 := (X^{h,K'}_{n(1)})_{\mathbf{F}}$ is a scheme, each Y_a is in fact a scheme [Sta13, Tag 0BPW], and thus Y is a ϖ-adic formal scheme. (In fact, it is easy to check directly that $U_{n(1)}$ acts freely on $X^{h,K'}_{n(a)}$, and thus to see that Y_a is an algebraic space.) The natural morphism $Y \to \mathcal{C}$ is then representable by algebraic spaces; indeed, any morphism from an affine scheme to $\mathcal{C}$ factors through some $\mathcal{C}^a$, and representability by algebraic spaces then follows from the representability by algebraic spaces of $Y_a \to \mathcal{C}^a$, and the Cartesianness of the diagram

$$\begin{tikzcd} Y_a \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{C}^a \arrow[r] & \mathcal{C} \end{tikzcd}$$

Similarly, the morphism $Y \to \mathcal{C}$ is smooth and surjective, and so witnesses the claim that $\mathcal{C}$ is a ϖ-adic formal algebraic stack.

To check that $\mathcal{C}$ has affine diagonal, it suffices to check that each $\mathcal{C}^a$ has affine diagonal, which follows from the fact that $(\mathcal{G}_{n(a)})_{\mathcal{O}/\varpi^a}$ is in fact an affine group scheme over $\mathcal{O}/\varpi^a$ (indeed, as in [PR09, §2.b.1], it is a Weil restriction of GL_d). $\square$

We next introduce the moduli stack of étale φ-modules, again both with and without descent data.

Definition 3.1.4. For each integer $a \geq 1$, we let $\mathcal{R}^{\mathrm{dd},a}_{d,K'}$ be the *fppf* $\mathcal{O}/\varpi^a$-stack which associates to any $\mathcal{O}/\varpi^a$-algebra A the groupoid $\mathcal{R}^{\mathrm{dd},a}_{d,K'}(A)$ of rank d étale φ-modules with A-coefficients and descent data from K' to K.

By [Sta13, Tag 04WV], we may also regard each of the stacks $\mathcal{R}^{\mathrm{dd},a}_{d,K'}$ as an *fppf* $\mathcal{O}$-stack, and we then write $\mathcal{R}^{\mathrm{dd}} := \varinjlim_a \mathcal{R}^{\mathrm{dd},a}$, which is again an *fppf* $\mathcal{O}$-stack.

We will omit d, K' from the notation wherever doing so will not cause confusion, and when $K' = K$, we write $\mathcal{R}$ for $\mathcal{R}^{\mathrm{dd}}$.

Just as in the case of $\mathcal{C}^{\mathrm{dd}}$, the morphism $\mathcal{R}^{\mathrm{dd}} \to \operatorname{Spec} \mathcal{O}$ factors through $\operatorname{Spf} \mathcal{O}$, and for each $a \geq 1$, there is a natural isomorphism $\mathcal{R}^{\mathrm{dd},a} \xrightarrow{\sim} \mathcal{R}^{\mathrm{dd}} \times_{\operatorname{Spf} \mathcal{O}} \operatorname{Spec} \mathcal{O}/\varpi^a$. Thus each of the morphisms $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a+1}$ is a thickening.

There is a natural morphism $\mathcal{C}^{\mathrm{dd}}_{d,h,K'} \to \mathcal{R}^{\mathrm{dd}}_d$, defined via

$$(\mathfrak{M}, \varphi, \{\hat{g}\}_{g \in \operatorname{Gal}(K'/K)}) \mapsto (\mathfrak{M}[1/u], \varphi, \{\hat{g}\}_{g \in \operatorname{Gal}(K'/K)}),$$

and natural morphisms $\mathcal{C}^{\mathrm{dd}} \to \mathcal{C}$ and $\mathcal{R}^{\mathrm{dd}} \to \mathcal{R}$ given by forgetting the descent data. In the optic of Section 2.3, the stack $\mathcal{R}^{\mathrm{dd}}_d$ may morally be thought of as a moduli of G_{K_∞}-representations, and the morphisms $\mathcal{C}^{\mathrm{dd}}_{d,h,K'} \to \mathcal{R}^{\mathrm{dd}}_d$ correspond to passage from a Breuil–Kisin module to its underlying Galois representation.

Proposition 3.1.5. *For each $a \geq 1$, the natural morphism $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^a$ is representable by algebraic spaces, affine, and of finite presentation.*

Proof. To see this, consider the pullback along some morphism $\operatorname{Spec} A \to \mathcal{R}^a$ (where A is a $\mathcal{O}/\varpi^a$-algebra); we must show that given an étale φ-module M of rank d without descent data, the data of giving additive bijections $\hat{g} : M \to M$, satisfying the further property that:

- the maps $\hat{g}$ commute with φ, satisfy $\hat{g_1} \circ \hat{g_2} = \widehat{g_1 \circ g_2}$, and we have $\hat{g}(sm) = g(s)\hat{g}(m)$ for all $s \in \mathfrak{S}_A[1/u]$, $m \in M$

is represented by an affine algebraic space (i.e. an affine scheme!) of finite presentation over A.

To see this, note first that such maps $\hat{g}$ are by definition $\mathfrak{S}^0_A[1/v]$-linear. The data of giving an $\mathfrak{S}^0_A[1/v]$-linear automorphism of M which commutes with φ is representable by an affine scheme of finite presentation over A by [EG19b, Prop. 5.4.8] and so the data of a finite collection of automorphisms is also representable by a finitely presented affine scheme over A. The further commutation and composition conditions on the $\hat{g}$ cut out a closed subscheme, as does the condition of $\mathfrak{S}_A[1/u]$-semi-linearity, so the result follows. $\square$

Corollary 3.1.6. *The diagonal of $\mathcal{R}^{\mathrm{dd}}$ is representable by algebraic spaces, affine, and of finite presentation.*

Proof. Since $\mathcal{R}^{\mathrm{dd}} = \varinjlim_a \mathcal{R}^{\mathrm{dd},a} \xrightarrow{\sim} \varinjlim_a \mathcal{R}^{\mathrm{dd}} \times_{\operatorname{Spf} \mathcal{O}} \operatorname{Spec} \mathcal{O}/\varpi^a$, and since the transition morphisms are closed immersions (and hence monomorphisms), we have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{R}^{\mathrm{dd},a} & \longrightarrow & \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{O}/\varpi^a} \mathcal{R}^{\mathrm{dd},a} \\ \downarrow & & \downarrow \\ \mathcal{R}^{\mathrm{dd}} & \longrightarrow & \mathcal{R}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{R}^{\mathrm{dd}} \end{array}$$

for each $a \geq 1$, and the diagonal morphism of $\mathcal{R}^{\mathrm{dd}}$ is the inductive limit of the diagonal morphisms of the various $\mathcal{R}^{\mathrm{dd},a}$. Any morphism from an affine scheme T to $\mathcal{R}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{R}^{\mathrm{dd}}$ thus factors through one of the $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{O}/\varpi^a} \mathcal{R}^{\mathrm{dd},a}$, and the fibre product $\mathcal{R}^{\mathrm{dd}} \times_{\mathcal{R}^{\mathrm{dd}} \times \mathcal{R}^{\mathrm{dd}}} T$ may be identified with $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^{\mathrm{dd},a} \times \mathcal{R}^{\mathrm{dd},a}} T$. It is thus equivalent to prove that each of the diagonal morphisms $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{O}/\varpi^a} \mathcal{R}^{\mathrm{dd},a}$ is representable by algebraic spaces, affine, and of finite presentation.

The diagonal of $\mathcal{R}^{\mathrm{dd},a}$ may be obtained by composing the pullback over $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{O}} \mathcal{R}^{\mathrm{dd},a}$ of the diagonal $\mathcal{R}^a \to \mathcal{R}^a \times_{\mathcal{O}} \mathcal{R}^a$ with the relative diagonal of the morphism $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^a$. The first of these morphisms is representable by algebraic spaces, affine, and of finite presentation, by [EG19b, Thm. 5.4.11 (2)], and the second is also representable by algebraic spaces, affine, and of finite presentation, since it is the relative diagonal of a morphism which has these properties, by Proposition 3.1.5. ☐

Corollary 3.1.7. *(1) For each $a \geq 1$, $\mathcal{C}^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, with affine diagonal.*
(2) The Ind-algebraic stack $\mathcal{C}^{\mathrm{dd}} := \varinjlim_a \mathcal{C}^{\mathrm{dd},a}$ is furthermore a ϖ-adic formal algebraic stack.
(3) The morphism $\mathcal{C}^{\mathrm{dd}}_h \to \mathcal{R}^{\mathrm{dd}}$ is representable by algebraic spaces and proper.

Proof. By Proposition 3.1.2, $\mathcal{C}^a$ is an algebraic stack of finite type over $\operatorname{Spec} \mathcal{O}/\varpi^a$ with affine diagonal. In particular it has quasi-compact diagonal, and so is quasi-separated. Since $\mathcal{O}/\varpi^a$ is Noetherian, it follows from [Sta13, Tag 0DQJ] that $\mathcal{C}^a$ is in fact of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$.

By Proposition 3.1.5, the morphism $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a \to \mathcal{C}^a$ is representable by algebraic spaces and of finite presentation, so it follows from [Sta13, Tag 05UM] that $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$. In order to show that $\mathcal{C}^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, it therefore suffices to show that the natural monomorphism

$$\mathcal{C}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a \tag{3.1.8}$$

is representable by algebraic spaces and of finite presentation. We will in fact show that it is a closed immersion (in the sense that its pull-back under any morphism from a scheme to its target becomes a closed immersion of schemes); since the target is locally Noetherian, and closed immersions are automatically of finite type and quasi-separated, it follows from [Sta13, Tag 0DQJ] that this closed immersion is of finite presentation, as required.

By [Sta13, Tag 0420], the property of being a closed immersion can be checked after pulling back to an affine scheme, and then working *fpqc*-locally. The claim then follows easily from the proof of [EG19b, Prop. 5.4.8], as *fpqc*-locally the condition that a lattice in an étale φ-module of rank d with descent data is preserved by the action of the $\hat{g}$ is determined by the vanishing of the coefficients of negative powers of u in a matrix.

To complete the proof of (1), it suffices to show that the diagonal of $\mathcal{C}^{\mathrm{dd},a}$ is affine. Since (as we have shown) the morphism (3.1.8) is a closed immersion, and thus a monomorphism, it is equivalent to show that the diagonal of $\mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a$ is affine. To ease notation, we denote this fibre product by $\mathcal{Y}$. We may then factor the diagonal of $\mathcal{Y}$ as the composite of the pull-back over $\mathcal{Y} \times_{\mathcal{O}/\varpi^a} \mathcal{Y}$ of the diagonal morphism $\mathcal{C}^a \to \mathcal{C}^a \times_{\mathcal{O}/\varpi^a} \mathcal{C}^a$ and the relative diagonal $\mathcal{Y} \to \mathcal{Y} \times_{\mathcal{C}^a} \mathcal{Y}$. The former morphism is affine, by [EG19b, Thm. 5.4.9 (1)], and the latter morphism is also affine, since it is the pullback via $\mathcal{C}^a \to \mathcal{R}^a$ of the relative diagonal morphism $\mathcal{R}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{R}^{\mathrm{dd},a}$, which is affine (as already observed in the proof of Corollary 3.1.6).

To prove (2), consider the morphism $\mathcal{C}^{\mathrm{dd}} \to \mathcal{C}$. This is a morphism of ϖ-adic Ind-algebraic stacks, and by what we have already proved, it is representable by algebraic spaces. Since the target is a ϖ-adic formal algebraic stack, it follows from [Eme, Lem. 7.9] that the source is also a ϖ-adic formal algebraic stack, as required.

To prove (3), since each of $\mathcal{C}^{\mathrm{dd},a}$ and $\mathcal{R}^{\mathrm{dd},a}$ is obtained from $\mathcal{C}^{\mathrm{dd}}$ and $\mathcal{R}^{\mathrm{dd}}$ via pull-back over $\mathcal{O}/\varpi^a$, it suffices to prove that each of the morphisms $\mathcal{C}^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$ is representable by algebraic spaces and proper. Each of these morphisms factors as

$$\mathcal{C}^{\mathrm{dd},a} \xrightarrow{(3.1.8)} \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{C}^a \xrightarrow{\text{proj.}} \mathcal{R}^{\mathrm{dd},a}.$$

We have already shown that the first of these morphisms is a closed immersion, and hence representable by algebraic spaces and proper. The second morphism is also representable by algebraic spaces and proper, since it is a base-change of the morphism $\mathcal{C}^a \to \mathcal{R}^a$, which has these properties by [EG19b, Thm. 5.4.11 (1)]. $\square$

The next lemma gives a concrete interpretation of the points of $\mathcal{C}^{\mathrm{dd}}$ over ϖ-adically complete $\mathcal{O}$-algebras, extending the tautological interpretation of the points of each $\mathcal{C}^{\mathrm{dd},a}$ prescribed by Definition 3.1.1.

Lemma 3.1.9. *If A is a ϖ-adically complete $\mathcal{O}$-algebra then the* $\mathrm{Spf}(A)$*-points of* $\mathcal{C}^{\mathrm{dd}}$ *are the Breuil–Kisin modules of rank d and height h with A-coefficients and descent data.*

Proof. Let $\mathfrak{M}$ be a Breuil–Kisin module of rank d and height h with A-coefficients and descent data. Then the sequence $\{\mathfrak{M}/\varpi^a\mathfrak{M}\}_{a\geq 1}$ defines a $\mathrm{Spf}(A)$-point of $\mathcal{C}^{\mathrm{dd}}$ (*cf.* Remark 2.1.7), and since $\mathfrak{M}$ is ϖ-adically complete it is recoverable from the sequence $\{\mathfrak{M}/\varpi^a\mathfrak{M}\}_{a\geq 1}$.

In the other direction, suppose that $\{\mathfrak{M}_a\}$ is a $\mathrm{Spf}(A)$-point of $\mathcal{C}^{\mathrm{dd}}$, so that $\mathfrak{M}_a \in \mathcal{C}^{\mathrm{dd},a}(A/\varpi^a)$. Define $\mathfrak{M} = \varprojlim_a \mathfrak{M}_a$, and similarly define $\varphi_{\mathfrak{M}}$ and $\{\hat{g}\}$ as inverse limits. Observe that $\varphi^*\mathfrak{M} = \varprojlim_a \varphi^*\mathfrak{M}_a$ (since $\varphi : \mathfrak{S}_A \to \mathfrak{S}_A$ makes $\mathfrak{S}_A$ into a free $\mathfrak{S}_A$-module). Since each $\Phi_{\mathfrak{M}_a}$ is injective with cokernel killed by $E(u)^h$ the same holds for $\Phi_{\mathfrak{M}}$.

Since the required properties of the descent data are immediate, to complete the proof it remains to check that $\mathfrak{M}$ is a projective $\mathfrak{S}_A$-module (necessarily of rank d, since its rank will equal that of $\mathfrak{M}_1$), which is a consequence of [GD71, Prop. 0.7.2.10(ii)]. $\square$

We now temporarily reintroduce h to the notation.

Definition 3.1.10. For each $h \geq 0$, write $\mathcal{R}^a_h$ for the scheme-theoretic image of $\mathcal{C}^a_h \to \mathcal{R}^a$ in the sense of [EG19b, Defn. 3.2.8]; then by [EG19b, Thms. 5.4.19, 5.4.20], $\mathcal{R}^a_h$ is an algebraic stack of finite presentation over $\mathrm{Spec}\,\mathcal{O}/\varpi^a$, the morphism $\mathcal{C}^a_h \to \mathcal{R}^a$ factors through $\mathcal{R}^a_h$, and we may write $\mathcal{R}^a \cong \varinjlim_h \mathcal{R}^a_h$ as an inductive limit of closed substacks, the natural transition morphisms being closed immersions.

We similarly write $\mathcal{R}^{\mathrm{dd},a}_h$ for the scheme-theoretic image of the morphism $\mathcal{C}^{\mathrm{dd},a}_h \to \mathcal{R}^{\mathrm{dd},a}$ in the sense of [EG19b, Defn. 3.2.8].

Theorem 3.1.11. *For each $a \geq 1$, $\mathcal{R}^{\mathrm{dd},a}$ is an Ind-algebraic stack. Indeed, we can write $\mathcal{R}^{\mathrm{dd},a} = \varinjlim_h \mathcal{X}_h$ as an inductive limit of algebraic stacks of finite presentation over* $\mathrm{Spec}\,\mathcal{O}/\varpi^a$*, the transition morphisms being closed immersions.*

Proof. As we have just recalled, by [EG19b, Thm. 5.4.20] we can write $\mathcal{R}^a = \varinjlim_h \mathcal{R}^a_h$, so that if we set $\mathcal{X}^{\mathrm{dd},a}_h := \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{R}^a_h$, then $\mathcal{R}^{\mathrm{dd},a} = \varinjlim_h \mathcal{X}^{\mathrm{dd},a}_h$, and the transition morphisms are closed immersions. Since $\mathcal{R}^a_h$ is of finite presentation over $\mathrm{Spec}\,\mathcal{O}/\varpi^a$, and a composite of morphisms of finite presentation is of finite presentation, it follows from Proposition 3.1.5 and [Sta13, Tag 05UM] that $\mathcal{X}_h^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, as required. □

Theorem 3.1.12. *$\mathcal{R}_h^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$. It is a closed substack of $\mathcal{R}^{\mathrm{dd},a}$, and the morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through a morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}_h^{\mathrm{dd},a}$ which is representable by algebraic spaces, scheme-theoretically dominant, and proper.*

Proof. As in the proof of Theorem 3.1.11, if we set $\mathcal{X}_h^{\mathrm{dd},a} := \mathcal{R}^{\mathrm{dd},a} \times_{\mathcal{R}^a} \mathcal{R}_h^a$, then $\mathcal{X}_h^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, and the natural morphism $\mathcal{X}_h^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$ is a closed immersion. The morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through $\mathcal{X}_h^{\mathrm{dd},a}$ (because the morphism $\mathcal{C}_h^a \to \mathcal{R}^a$ factors through its scheme-theoretic image $\mathcal{R}_h^a$), so by [EG19b, Prop. 3.2.31], $\mathcal{R}_h^{\mathrm{dd},a}$ is the scheme-theoretic image of the morphism of algebraic stacks $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{X}_h^{\mathrm{dd},a}$. The required properties now follow from [EG19b, Lem. 3.2.29] (using representability by algebraic spaces and properness of the morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}^{\mathrm{dd},a}$, as proved in Corollary 3.1.7 (3), to see that the induced morphism $\mathcal{C}_h^{\mathrm{dd},a} \to \mathcal{R}_h^{\mathrm{dd},a}$ is representable by algebraic spaces and proper, along with [Sta13, Tag 0DQJ], and the fact that $\mathcal{X}_h^{\mathrm{dd},a}$ is of finite presentation over $\operatorname{Spec} \mathcal{O}/\varpi^a$, to see that $\mathcal{R}_h^{\mathrm{dd},a}$ is of finite presentation). □

3.2. **Representations of tame groups.** Let G be a finite group.

Definition 3.2.1. We let $\operatorname{Rep}_d(G)$ denote the algebraic stack classifying d-dimensional representations of G over $\mathcal{O}$: if X is any $\mathcal{O}$-scheme, then $\operatorname{Rep}_d(G)(X)$ is the groupoid consisting of locally free sheaves of rank d over X endowed with an $\mathcal{O}_X$-linear action of G (rank d locally free G-sheaves, for short); morphisms are G-equivariant isomorphisms of vector bundles.

We now suppose that G is tame, i.e. that it has prime-to-p order. In this case (taking into account the fact that $\mathbf{F}$ has characteristic p, and that $\mathcal{O}$ is Henselian), the isomorphism classes of d-dimensional G-representations of G over E and over $\mathbf{F}$ are in natural bijection. Indeed, any finite-dimensional representation τ of G over E contains a G-invariant $\mathcal{O}$-lattice τ°, and the associated representation of G over $\mathbf{F}$ is given by forming $\overline{\tau} := \mathbf{F} \otimes_{\mathcal{O}} \tau^\circ$.

Lemma 3.2.2. *Suppose that G is tame, and that E is chosen large enough so that each irreducible representation of G over E is absolutely irreducible* (*or, equivalently, so that each irreducible representation of G over $\mathbf{F}$ is absolutely irreducible*)*, and so that each irreducible representation of G over $\overline{\mathbf{Q}}_p$ is defined over E* (*equivalently, so that each irreducible representation of G over $\overline{\mathbf{F}}_p$ is defined over $\mathbf{F}$*)*.*

1. *$\operatorname{Rep}_d(G)$ is the disjoint union of irreducible components $\operatorname{Rep}_d(G)_\tau$, where τ ranges over the finite set of isomorphism classes of d-dimensional representations of G over E.*
2. *A morphism $X \to \operatorname{Rep}_d(G)$ factors through $\operatorname{Rep}_d(G)_\tau$ if and only if the associated locally free G-sheaf on X is Zariski locally isomorphic to $\tau^\circ \otimes_{\mathcal{O}} \mathcal{O}_X$.*
3. *If we write $G_\tau := \operatorname{Aut}_{\mathcal{O}[G]}(\tau^\circ)$, then G_τ is a smooth* (*indeed reductive*) *group scheme over $\mathcal{O}$, and $\operatorname{Rep}_d(G)_\tau$ is isomorphic to the classifying space $[\operatorname{Spec} \mathcal{O}/G_\tau]$.*

Proof. Since G has order prime to p, the representation $P := \oplus_\sigma \sigma^\circ$ is a projective generator of the category of $\mathcal{O}[G]$-modules, where σ runs over a set of representatives for the isomorphism classes of irreducible E-representations of G. (Indeed, each σ° is projective, because the fact that G has order prime to p means that all of the Ext^1s against σ° vanish. To see that $\oplus_\sigma \sigma^\circ$ is a generator, we need to show that every $\mathcal{O}[G]$-module admits a non-zero map from some σ°. We can reduce to the case of a finitely generated module M, and it is then enough (by projectivity) to prove that $M \otimes_{\mathcal{O}} \mathbf{F}$ admits such a map, which is clear.) Our assumption that each σ is absolutely irreducible furthermore shows that $\mathrm{End}_G(\sigma^\circ) = \mathcal{O}$ for each σ, so that $\mathrm{End}_G(P) = \prod_\sigma \mathcal{O}$.

Standard Morita theory then shows that the functor $M \mapsto \mathrm{Hom}_G(P, M)$ induces an equivalence between the category of $\mathcal{O}[G]$-modules and the category of $\prod_\sigma \mathcal{O}$-modules. Of course, a $\prod_\sigma \mathcal{O}$-module is just given by a tuple $(N_\sigma)_\sigma$ of $\mathcal{O}$-modules, and in this optic, the functor $\mathrm{Hom}_G(P, -)$ can be written as $M \mapsto \big(\mathrm{Hom}_G(\sigma^\circ, M)\big)_\sigma$, with a quasi-inverse functor being given by $(N_\sigma) \mapsto \bigoplus_\sigma \sigma^\circ \otimes_{\mathcal{O}} N_\sigma$. It is easily seen (just using the fact that $\mathrm{Hom}_G(P, -)$ induces an equivalence of categories) that M is a finitely generated projective A-module, for some $\mathcal{O}$-algebra A, if and only if each $\mathrm{Hom}_G(\sigma^\circ, M)$ is a finitely generated projective A-module.

The preceding discussion shows that giving a rank d representation of G over an $\mathcal{O}$-algebra A amounts to giving a tuple $(N_\sigma)_\sigma$ of projective A-modules, of ranks n_σ, such that $\sum_\sigma n_\sigma \dim \sigma = d$. For each such tuple of ranks (n_σ), we obtain a corresponding moduli stack $\mathrm{Rep}_{(n_\sigma)}(G)$ classifying rank d representations of G which decompose in this manner, and $\mathrm{Rep}_d(G)$ is isomorphic to the disjoint union of the various stacks $\mathrm{Rep}_{(n_\sigma)}(G)$.

If we write $\tau = \oplus_\sigma \sigma^{n_\sigma}$, then we may relabel $\mathrm{Rep}_{(n_\sigma)}(G)$ as $\mathrm{Rep}_\tau(G)$; statements (1) and (2) are then proved. By construction, there is an isomorphism

$$\mathrm{Rep}_\tau(G) = \mathrm{Rep}_{(n_\sigma)}(G) \xrightarrow{\sim} \prod_\sigma [\operatorname{Spec} \mathcal{O} / \mathrm{GL}_{n_\sigma}].$$

Noting that $G_\tau := \mathrm{Aut}(\tau) = \prod_\sigma \mathrm{GL}_{n_\sigma/\mathcal{O}}$, we find that statement (3) follows as well. □

For each τ, it follows from the identification of $\mathrm{Rep}_d(G)_\tau$ with $[\operatorname{Spec} \mathcal{O}/G_\tau]$ that there is a natural map $\mathrm{Rep}_d(G)_\tau \to \operatorname{Spec} \mathcal{O}$. We let $\pi_0(\mathrm{Rep}_d(G))$ denote the disjoint union of copies of $\operatorname{Spec} \mathcal{O}$, one for each isomorphism class τ; then there is a natural map $\mathrm{Rep}_d(G) \to \pi_0(\mathrm{Rep}_d(G))$. While we do not want to develop a general theory of the étale π_0 groups of algebraic stacks, we note that it is natural to regard $\pi_0(\mathrm{Rep}_d(G))$ as the étale π_0 of $\mathrm{Rep}_d(G)$.

3.3. **Tame inertial types.** Write $I(K'/K)$ for the inertia subgroup of $\mathrm{Gal}(K'/K)$. Since we are assuming that E is large enough that it contains the image of every embedding $K' \hookrightarrow \overline{\mathbf{Q}}_p$, it follows in particular that every $\overline{\mathbf{Q}}_p$-character of $I(K'/K)$ is defined over E.

Recall from Subsection 1.7 that if A is an $\mathcal{O}$-algebra, and $\mathfrak{M}$ is a Breuil–Kisin module with A-coefficients, then we write $\mathfrak{M}_i$ for $e_i\mathfrak{M}$. Since $I(K'/K)$ acts trivially on $W(k')$, the $\hat{g}$ for $g \in I(K'/K)$ stabilise each $\mathfrak{M}_i$, inducing an action of $I(K'/K)$ on $\mathfrak{M}_i/u\mathfrak{M}_i$.

Write

$$\mathrm{Rep}_{d,I(K'/K)} := \prod_{i=0}^{f'-1} \mathrm{Rep}_d\bigl(I(K'/K)\bigr),$$

the fibre product being taken over $\mathcal{O}$. If $\{\tau_i\}$ is an f'-tuple of isomorphism classes of d-dimensional representations of $I(K'/K)$, we write

$$\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}} := \prod_{i=0}^{f'-1} \mathrm{Rep}_d\bigl(I(K'/K)\bigr)_{\tau_i}.$$

Lemma 3.2.2 shows that we may write

$$\mathrm{Rep}_{d,I(K'/K)} = \coprod_{\{\tau_i\}} \mathrm{Rep}_{d,I(K'/K),\tau_i}.$$

Note that since K'/K is tamely ramified, $I(K'/K)$ is abelian of prime-to-p order, and each τ_i is just a sum of characters. If all of the τ_i are equal to some fixed τ, then we write $\mathrm{Rep}_{d,I(K'/K),\tau}$ for $\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}}$. We have corresponding stacks $\pi_0(\mathrm{Rep}_{d,I(K'/K)})$, $\pi_0(\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}})$ and $\pi_0(\mathrm{Rep}_{d,I(K'/K),\tau})$, defined in the obvious way.

If $\mathfrak{M}$ is a Breuil–Kisin module of rank d with descent data and A-coefficients, then $\mathfrak{M}_i/u\mathfrak{M}_i$ is projective A-module of rank d, endowed with an A-linear action of $I(K'/K)$, and so is an A-valued point of $\mathrm{Rep}_d\bigl(I(K'/K)\bigr)$. Thus we obtain a morphism

$$\mathcal{C}_d^{\mathrm{dd}} \to \mathrm{Rep}_{d,I(K'/K)}, \tag{3.3.1}$$

defined via $\mathfrak{M} \mapsto (\mathfrak{M}_0/u\mathfrak{M}_0, \dots, \mathfrak{M}_{f'-1}/u\mathfrak{M}_{f'-1})$.

Definition 3.3.2. Let A be an $\mathcal{O}$-algebra, and let $\mathfrak{M}$ be a Breuil–Kisin module of rank d with A-coefficients. We say that $\mathfrak{M}$ has *mixed type* $(\tau_i)_i$ if the composite $\operatorname{Spec} A \to \mathcal{C}_d^{\mathrm{dd}} \to \mathrm{Rep}_{d,I(K'/K)}$ (the first arrow being the morphism that classifies $\mathfrak{M}$, and the second arrow being (3.3.1)) factors through $\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}}$. Concretely, this is equivalent to requiring that, Zariski locally on $\operatorname{Spec} A$, there is an $I(K'/K)$-equivariant isomorphism $\mathfrak{M}_i/u\mathfrak{M}_i \cong A \otimes_{\mathcal{O}} \tau_i^\circ$ for each i.

If each $\tau_i = \tau$ for some fixed τ, then we say that the type of $\mathfrak{M}$ is unmixed, or simply that $\mathfrak{M}$ has *type* τ.

Remark 3.3.3. If $A = \mathcal{O}$ then a Breuil–Kisin module necessarily has some (unmixed) type τ, since after inverting $E(u)$ and reducing modulo u the map $\Phi_{\mathfrak{M},i}$ gives an $I(K'/K)$-equivariant E-vector space isomorphism $\varphi^*(\mathfrak{M}_{i-1}/u\mathfrak{M}_{i-1})[\frac{1}{p}] \xrightarrow{\sim} (\mathfrak{M}_i/u\mathfrak{M}_i)[\frac{1}{p}]$. However if $A = \mathbf{F}$ there are Breuil–Kisin modules which have a genuinely mixed type; indeed, it is easy to write down examples of free Breuil–Kisin modules of rank one of any mixed type (see also [CL18, Rem. 3.7]), which necessarily cannot lift to characteristic zero. This shows that $\mathcal{C}_d^{\mathrm{dd}}$ is not flat over $\mathbf{Z}_p$. In the following sections, when $d = 2$ and $h = 1$ we define a closed substack $\mathcal{C}_d^{\mathrm{dd,BT}}$ of $\mathcal{C}^{\mathrm{dd}}$ which *is* flat over $\mathbf{Z}_p$, and can be thought of as taking the Zariski closure of $\overline{\mathbf{Q}}_p$-valued Galois representations that become Barsotti–Tate over K' and such that all pairs of labeled Hodge–Tate weights are $\{0, 1\}$ (see Remark 3.5.8 below).

Definition 3.3.4. Let $\mathcal{C}_d^{(\tau_i)}$ be the étale substack of $\mathcal{C}_d^{\mathrm{dd}}$ which associates to each $\mathcal{O}$-algebra A the subgroupoid $\mathcal{C}_d^{(\tau_i)}(A)$ of $\mathcal{C}_d^{\mathrm{dd}}(A)$ consisting of those Breuil–Kisin modules which are of mixed type (τ_i). If each $\tau_i = \tau$ for some fixed τ, we write $\mathcal{C}_d^\tau$ for $\mathcal{C}_d^{(\tau_i)}$.

Proposition 3.3.5. *Each $\mathcal{C}_d^{(\tau_i)}$ is an open and closed substack of $\mathcal{C}_d^{\mathrm{dd}}$, and $\mathcal{C}_d^{\mathrm{dd}}$ is the disjoint union of its substacks $\mathcal{C}_d^{(\tau_i)}$.*

Proof. By Lemma 3.2.2, $\mathrm{Rep}_{d,I(K'/K)}$ is the disjoint union of its open and closed substacks $\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}}$. By definition $\mathcal{C}_d^{(\tau_i)}$ is the preimage of $\mathrm{Rep}_{d,I(K'/K),\{\tau_i\}}$ under the morphism (3.3.1); the lemma follows. ☐

3.4. **Local models: generalities.** Throughout this section we allow d to be arbitrary; in Section 3.5 we specialise to the case $d = 2$, where we relate the local models considered in this section to the local models considered in the theory of Shimura varieties. We will usually omit d from our notation, writing for example $\mathcal{C}$ for $\mathcal{C}_d$, without any further comment. We begin with the following lemma, for which we allow h to be arbitrary.

Lemma 3.4.1. *Let $\mathfrak{M}$ be a rank d Breuil–Kisin module of height h with descent data over an $\mathcal{O}$-algebra A. Assume further either that A is p^n-torsion for some n, or that A is Noetherian and p-adically complete. Then $\operatorname{im}\Phi_{\mathfrak{M}}/E(u)^h\mathfrak{M}$ is a finite projective A-module, and is a direct summand of $\mathfrak{M}/E(u)^h\mathfrak{M}$ as an A-module.*

Proof. We follow the proof of [Kis09, Lem. 1.2.2]. We have a short exact sequence

$$0 \to \operatorname{im}\Phi_{\mathfrak{M}}/E(u)^h\mathfrak{M} \to \mathfrak{M}/E(u)^h\mathfrak{M} \to \mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}} \to 0$$

in which the second term is a finite projective A-module (since it is a finite projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module), so it is enough to show that the third term is a projective A-module. It is therefore enough to show that the finitely generated A-module $\mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}}$ is finitely presented and flat.

To see that it is finitely presented, note that we have the equality

$$\mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}} = (\mathfrak{M}/E(u)^h)/(\operatorname{im}\Phi_{\mathfrak{M}}/E(u)^h),$$

and the right hand side admits a presentation by finitely generated projective A-modules

$$\varphi^*(\mathfrak{M}/E(u)^h) \to \mathfrak{M}/E(u)^h \to (\mathfrak{M}/E(u)^h)/(\operatorname{im}\Phi_{\mathfrak{M}}/E(u)^h) \to 0.$$

To see that it is flat, it is enough to show that for every finitely generated ideal I of A, the map

$$I \otimes_A \mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}} \to \mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}}$$

is injective. It follows easily (for example, from the snake lemma) that it is enough to check that the complex

$$0 \to \varphi^*\mathfrak{M} \to \mathfrak{M} \to \mathfrak{M}/\operatorname{im}\Phi_{\mathfrak{M}} \to 0,$$

which is exact by Lemma 2.1.3, remains exact after tensoring with A/I. Since I is finitely generated we have $\mathfrak{M} \otimes_A A/I \cong \mathfrak{M}\,\widehat{\otimes}_A A/I$ by Remark 2.1.7, and the desired exactness amounts to the injectivity of $\Phi_{\mathfrak{M}\,\widehat{\otimes}_A A/I}$ for the Breuil–Kisin module $\mathfrak{M}\,\widehat{\otimes}_A A/I$. This follows immediately from Lemma 2.1.3 in the case that A is killed by p^n, and otherwise follows from the same lemma once we check that A/I is p-adically complete, which follows from the Artin–Rees lemma (as A is assumed Noetherian and p-adically complete). ☐

We assume from now on that $h = 1$, but we continue to allow arbitrary d. We allow K'/K to be any Galois extension such that $[K' : K]$ is prime to p (so in particular K'/K is tame).

Definition 3.4.2. We let $\mathcal{M}_{\mathrm{loc}}^{K'/K,a}$ be the algebraic stack of finite presentation over $\operatorname{Spec}\mathcal{O}/\varpi^a$ defined as follows: if A is an $\mathcal{O}/\varpi^a$-algebra, then $\mathcal{M}_{\mathrm{loc}}^{K'/K}(A)$ is the groupoid of tuples $(\mathfrak{L}, \mathfrak{L}^+)$, where:

- $\mathfrak{L}$ is a rank d projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module, with a $\mathrm{Gal}(K'/K)$-semilinear, A-linear action of $\mathrm{Gal}(K'/K)$;
- $\mathfrak{L}^+$ is an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-submodule of $\mathfrak{L}$, which is locally on $\operatorname{Spec} A$ a direct summand of $\mathfrak{L}$ as an A-module (or equivalently, for which $\mathfrak{L}/\mathfrak{L}^+$ is projective as an A-module), and is preserved by $\mathrm{Gal}(K'/K)$.

We set $\mathcal{M}_{\mathrm{loc}}^{K'/K} := \varinjlim_a \mathcal{M}_{\mathrm{loc}}^{K'/K,a}$, so that $\mathcal{M}_{\mathrm{loc}}^{K'/K}$ is a ϖ-adic formal algebraic stack, of finite presentation over $\operatorname{Spf}\mathcal{O}$ (indeed, it is easily seen to be the ϖ-adic completion of an algebraic stack of finite presentation over $\operatorname{Spec}\mathcal{O}$).

Definition 3.4.3. By Lemma 3.4.1, we have a natural morphism $\Psi : \mathcal{C}_{1,K'}^{\mathrm{dd}} \to \mathcal{M}_{\mathrm{loc}}^{K'/K}$, which takes a Breuil–Kisin module with descent data $\mathfrak{M}$ of height 1 to the pair

$$(\mathfrak{M}/E(u)\mathfrak{M}, \operatorname{im}\Phi_{\mathfrak{M}}/E(u)\mathfrak{M}).$$

Remark 3.4.4. The definition of the stack $\mathcal{M}_{\mathrm{loc}}^{K'/K,a}$ does not include any condition that mirrors the commutativity between the Frobenius and the descent data on a Breuil–Kisin module, and so in general the morphism $\Psi_A : \mathcal{C}_{1,K'}^{\mathrm{dd}}(A) \to \mathcal{M}_{\mathrm{loc}}^{K'/K}(A)$ cannot be essentially surjective.

It will be convenient to consider the twisted group rings $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$ and $(\mathcal{O}_{K'}\otimes_{\mathbf{Z}_p} A)[\mathrm{Gal}(K'/K)]$, in which the elements $g \in \mathrm{Gal}(K'/K)$ obey the following commutation relation with elements $s \in \mathfrak{S}_A$ (resp. $s \in \mathcal{O}_{K'}$):

$$g \cdot s = g(s) \cdot g.$$

(In the literature these twisted group rings are more often written as $\mathfrak{S}_A*\mathrm{Gal}(K'/K)$, $(\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A) * \mathrm{Gal}(K'/K)$, in order to distinguish them from the usual (untwisted) group rings, but as we will only use the twisted versions in this paper, we prefer to use this notation for them.)

By definition, endowing a finitely generated $\mathfrak{S}_A$-module P with a semilinear $\mathrm{Gal}(K'/K)$-action is equivalent to giving it the structure of a left $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module. If P is projective as an $\mathfrak{S}_A$-module, then it is also projective as an $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module. Indeed, $\mathfrak{S}_A$ is a direct summand of $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$ as a $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module, given by the central idempotent $\frac{1}{|\mathrm{Gal}(K'/K)|}\sum_{g\in\mathrm{Gal}(K'/K)} g$, so P is a direct summand of the projective module $\mathfrak{S}_A[\mathrm{Gal}(K'/K)] \otimes_{\mathfrak{S}_A} P$. Similar remarks apply to the case of $(\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A)[\mathrm{Gal}(K'/K)]$-modules.

Theorem 3.4.5. *The morphism $\Psi : \mathcal{C}_{1,K'}^{\mathrm{dd}} \to \mathcal{M}_{\mathrm{loc}}^{K'/K}$ is representable by algebraic spaces and smooth.*

Proof. We first show that the morphism Ψ is formally smooth, in the sense that it satisfies the infinitesimal lifting criterion for nilpotent thickenings of affine test objects [EG19b, Defn. 2.4.2]. For this, we follow the proof of [Kis09, Prop. 2.2.11] (see also the proof of [CL18, Thm. 4.9]). Let A be an $\mathcal{O}/\varpi^a$-algebra and $I \subset A$ be a nilpotent ideal. Suppose that we are given $\mathfrak{M}_{A/I} \in \mathcal{C}^{\mathrm{dd}}_{1,K'}(A/I)$ and $(\mathfrak{L}_A, \mathfrak{L}_A^+) \in \mathcal{M}^{K'/K}_{\mathrm{loc}}(A)$ together with an isomorphism

$$\Psi(\mathfrak{M}_{A/I}) \xrightarrow{\sim} (\mathfrak{L}_A, \mathfrak{L}_A^+) \otimes_A A/I =: (\mathfrak{L}_{A/I}, \mathfrak{L}_{A/I}^+).$$

We must show that there exists $\mathfrak{M}_A \in \mathcal{C}^{\mathrm{dd}}_{1,K'}(A)$ together with an isomorphism $\Psi(\mathfrak{M}_A) \xrightarrow{\sim} (\mathfrak{L}_A, \mathfrak{L}_A^+)$ lifting the given isomorphism.

As explained above, we can and do think of $\mathfrak{M}_{A/I}$ as a finite projective $\mathfrak{S}_{A/I}[\mathrm{Gal}(K'/K)]$-module, and $\mathfrak{L}_A$ as a finite projective $(\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A)[\mathrm{Gal}(K'/K)]$-module. Since the closed 2-sided ideal of $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$ generated by I consists of nilpotent elements, we may lift $\mathfrak{M}_{A/I}$ to a finite projective $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module $\mathfrak{M}_A$. (This is presumably standard, but for lack of a reference we indicate a proof. In fact the proof of [Sta13, Tag 0D47] goes over unchanged to our setting. Writing $\mathfrak{M}_{A/I}$ as a direct summand of a finite free $\mathfrak{S}_{A/I}[\mathrm{Gal}(K'/K)]$-module F, it is enough to lift the corresponding idempotent in $\mathrm{End}_{\mathfrak{S}_{A/I}[\mathrm{Gal}(K'/K)]}(F)$, which is possible by [Sta13, Tag 05BU]; see also [Lam91, Thm. 21.28] for another proof of the existence of lifts of idempotents in this generality.) Note that since $\mathfrak{M}_{A/I}$ is of rank d as a projective $\mathfrak{S}_{A/I}$-module, $\mathfrak{M}_A$ is of rank d as a projective $\mathfrak{S}_A$-module.

Since $\mathfrak{M}_A/E(u)\mathfrak{M}_A$ is a projective $(\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A)[\mathrm{Gal}(K'/K)]$-module, we may lift the composite

$$\mathfrak{M}_A/E(u)\mathfrak{M}_A \twoheadrightarrow \mathfrak{M}_{A/I}/E(u)\mathfrak{M}_{A/I} \xrightarrow{\sim} \mathfrak{L}_{A/I}$$

to a morphism $\theta : \mathfrak{M}_A/E(u)\mathfrak{M}_A \to \mathfrak{L}_A$. Since the composite of θ with $\mathfrak{L}_A \twoheadrightarrow \mathfrak{L}_{A/I}$ is surjective, it follows by Nakayama's lemma that θ is surjective. But a surjective map of projective modules of the same rank is an isomorphism, and so θ is an isomorphism lifting the given isomorphism $\mathfrak{M}_{A/I}/E(u)\mathfrak{M}_{A/I} \xrightarrow{\sim} \mathfrak{L}_{A/I}$.

We let $\mathfrak{M}_A^+$ denote the preimage in $\mathfrak{M}_A$ of $\theta^{-1}(\mathfrak{L}_A^+)$. The image of the induced map $f : \mathfrak{M}_A^+ \subset \mathfrak{M}_A \twoheadrightarrow \mathfrak{M}_A \widehat{\otimes}_A A/I \cong \mathfrak{M}_{A/I}$ is precisely $\operatorname{im} \Phi_{\mathfrak{M}_{A/I}}$, since the same is true modulo $E(u)$ and because $\mathfrak{M}_A^+$, $\operatorname{im} \Phi_{\mathfrak{M}_{A/I}}$ contain $E(u)\mathfrak{M}_A$, $E(u)\mathfrak{M}_{A/I}$ respectively. Observing that

$$\mathfrak{M}_A/\mathfrak{M}_A^+ \cong (\mathfrak{M}_A/E(u)\mathfrak{M}_A)/(\mathfrak{M}_A^+/E(u)\mathfrak{M}_A) \cong \mathfrak{L}_A/\mathfrak{L}_A^+$$

we deduce that $\mathfrak{M}_A/\mathfrak{M}_A^+$ is projective as an A-module, whence $\mathfrak{M}_A^+$ is an A-module direct summand of $\mathfrak{M}_A$. By the same argument $\operatorname{im} \Phi_{\mathfrak{M}_{A/I}}$ is an A-module direct summand of $\mathfrak{M}_{A/I}$, and we conclude that the map $\mathfrak{M}_A^+ \widehat{\otimes}_A A/I \to \operatorname{im} \Phi_{\mathfrak{M}_{A/I}}$ induced by f is an isomorphism.

Finally, we have the diagram

$$\begin{tikzcd}
\varphi^*\mathfrak{M}_A \arrow[d] & \mathfrak{M}_A^+ \arrow[d] \\
\varphi^*\mathfrak{M}_{A/I} \arrow[r] & \operatorname{im} \Phi_{\mathfrak{M}_{A/I}}
\end{tikzcd}$$

where the horizontal arrow is given by $\Phi_{\mathfrak{M}_{A/I}}$, and the right hand vertical arrow is f. Since $\varphi^*\mathfrak{M}_A$ is a projective $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module, we may find a morphism of $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-modules $\varphi^*\mathfrak{M}_A \to \mathfrak{M}_A^+$ which fills in the commutative square. Since the composite $\varphi^*\mathfrak{M}_A \to \mathfrak{M}_A^+ \to \operatorname{im} \Phi_{\mathfrak{M}_{A/I}} \cong \mathfrak{M}_A^+ \widehat{\otimes}_A A/I$ is surjective, it follows by Nakayama's lemma that $\varphi^*\mathfrak{M}_A \to \mathfrak{M}_A^+$ is also surjective, and the composite $\varphi^*\mathfrak{M}_A \to \mathfrak{M}_A^+ \subset \mathfrak{M}_A$ gives a map $\Phi_{\mathfrak{M}_A}$. Since $\Phi_{\mathfrak{M}_A}[1/E(u)]$ is a surjective map of projective modules of the same rank, it is an isomorphism, and we see that $\mathfrak{M}_A$

together with $\Phi_{\mathfrak{M}_A}$ is our required lifting to a Breuil–Kisin module of rank d with descent data.

Since the source and target of Ψ are of finite presentation over $\operatorname{Spf}\mathcal{O}$, and ϖ-adic, we see that Ψ is representable by algebraic spaces (by [Eme, Lem. 7.10]) and locally of finite presentation (by [EG19b, Cor. 2.1.8] and [Sta13, Tag 06CX]). Thus Ψ is in fact smooth (being formally smooth and locally of finite presentation). $\square$

We now show that the inertial type of a Breuil–Kisin module is visible on the local model.

Lemma 3.4.6. *There is a natural morphism $\mathcal{M}_{\mathrm{loc}}^{K'/K} \to \pi_0(\operatorname{Rep}_{I(K'/K)})$.*

Proof. The morphism $\mathcal{M}_{\mathrm{loc}}^{K'/K} \to \pi_0(\operatorname{Rep}_{I(K'/K)})$ is defined by sending $(\mathfrak{L}, \mathfrak{L}^+) \mapsto \mathfrak{L}/\pi'$. More precisely, $\mathfrak{L}/\pi'$ is a rank d projective $k'\otimes_{\mathbf{Z}_p} A$-module with a linear action of $I(K'/K)$, so determines an A/p-point of $\pi_0(\operatorname{Rep}_{I(K'/K)}) = \prod_{i=0}^{f'-1} \pi_0(\operatorname{Rep}_d(I(K'/K)))$. Since the target is a disjoint union of copies of $\operatorname{Spec}\mathcal{O}$, the morphism $\operatorname{Spec} A/p \to \pi_0(\operatorname{Rep}_{I(K'/K)})$ lifts uniquely to a morphism $\operatorname{Spec} A \to \pi_0(\operatorname{Rep}_{I(K'/K)})$, as required. $\square$

Definition 3.4.7. We let

$$\mathcal{M}_{\mathrm{loc}}^{(\tau_i)} := \mathcal{M}_{\mathrm{loc}}^{K'/K} \otimes_{\pi_0(\operatorname{Rep}_{I(K'/K)})} \pi_0(\operatorname{Rep}_{I(K'/K),\{\tau_i\}}).$$

If each $\tau_i = \tau$ for some fixed τ, we write $\mathcal{M}_{\mathrm{loc}}^{\tau}$ for $\mathcal{M}_{\mathrm{loc}}^{(\tau_i)}$. By Lemma 3.2.2, $\mathcal{M}_{\mathrm{loc}}^{K'/K}$ is the disjoint union of the open and closed substacks $\mathcal{M}_{\mathrm{loc}}^{(\tau_i)}$.

Lemma 3.4.8. *We have $\mathcal{C}^{(\tau_i)} = \mathcal{C}^{\mathrm{dd}} \times_{\mathcal{M}_{\mathrm{loc}}^{K'/K}} \mathcal{M}_{\mathrm{loc}}^{(\tau_i)}$.*

Proof. This is immediate from the definitions. $\square$

In particular, $\mathcal{C}^{(\tau_i)}$ is a closed substack of $\mathcal{C}^{\mathrm{dd}}$.

3.5. **Local models: determinant conditions.** Write $N = K \cdot W(k')[1/p]$, so that K'/N is totally ramified. Since $I(K'/K)$ is cyclic of order prime to p and acts trivially on $\mathcal{O}_N$, we may write

$$(\mathfrak{L}, \mathfrak{L}^+) = \oplus_\xi (\mathfrak{L}_\xi, \mathfrak{L}_\xi^+) \tag{3.5.1}$$

where the sum is over all characters $\xi : I(K'/K) \to \mathcal{O}^\times$, and $\mathfrak{L}_\xi$ (resp. $\mathfrak{L}_\xi^+$) is the $\mathcal{O}_N \otimes A$-submodule of $\mathfrak{L}$ (resp. of $\mathfrak{L}^+$) on which $I(K'/K)$ acts through ξ.

Definition 3.5.2. We say that an object $(\mathfrak{L}, \mathfrak{L}^+)$ of $\mathcal{M}_{\mathrm{loc}}^{K'/K}(A)$ *satisfies the strong determinant condition* if Zariski locally on $\operatorname{Spec} A$ the following condition holds: for all $a \in \mathcal{O}_N$ and all ξ, we have

$$\det{}_A(a|\mathfrak{L}_\xi^+) = \prod_{\psi : N \hookrightarrow E} \psi(a) \tag{3.5.3}$$

as polynomial functions on $\mathcal{O}_N$ in the sense of [Kot92, §5].

Remark 3.5.4. An explicit version of this determinant condition is stated, in this generality, in [Kis09, §2.2], specifically in the proof of [Kis09, Prop. 2.2.5]. We recall this here, with our notation. We have a direct sum decomposition

$$\mathcal{O}_N \otimes_{\mathbf{Z}_p} A \xrightarrow{\sim} \bigoplus_{\sigma_i : k' \hookrightarrow \mathbf{F}} \mathcal{O}_N \otimes_{W(k'),\sigma_i} A$$

Recall that $e_i \in \mathcal{O}_N \otimes_{\mathbf{Z}_p} \mathcal{O}$ denotes the idempotent that identifies $e_i \cdot \mathcal{O}_N \otimes_{\mathbf{Z}_p} A$ with the summand $\mathcal{O}_N \otimes_{W(k'),\sigma_i} A$. For $j = 0, 1, \dots, e-1$, let X_{j,σ_i} be an indeterminate. Then the strong determinant condition on $(\mathfrak{L}, \mathfrak{L}^+)$ is that for all ξ, we have

$$\det_A \left(\sum_{j,\sigma_i} e_i \pi^j X_{j,\sigma_i} \mid \mathfrak{L}^+_\xi \right) = \prod_\psi \sum_{j,\sigma_i} \left(\psi(e_i \pi^j) X_{j,\sigma_i} \right), \tag{3.5.5}$$

where j runs over $0, 1, \dots, e-1$, σ_i over embeddings $k' \hookrightarrow \mathbf{F}$, and ψ over embeddings $\mathcal{O}_N \hookrightarrow \mathcal{O}$. Note that $\psi(e_i) = 1$ if $\psi|_{W(k')}$ lifts σ_i and is equal to 0 otherwise.

Definition 3.5.6. We write $\mathcal{M}_{\mathrm{loc}}^{K'/K,\mathrm{BT}}$ for the substack of $\mathcal{M}_{\mathrm{loc}}^{K'/K}$ given by those $(\mathfrak{L}, \mathfrak{L}^+)$ which satisfy the strong determinant condition. For each (possibly mixed) type (τ_i), we write $\mathcal{M}_{\mathrm{loc}}^{(\tau_i),\mathrm{BT}} := \mathcal{M}_{\mathrm{loc}}^{(\tau_i)} \times_{\mathcal{M}_{\mathrm{loc}}^{K'/K}} \mathcal{M}_{\mathrm{loc}}^{K'/K,\mathrm{BT}}$.

Suppose for the remainder of this section that $d = 2$ and $h = 1$, so that $\mathcal{C}^{\mathrm{dd}}$ consists of Breuil–Kisin modules of rank two and height at most 1. We then set $\mathcal{C}^{\mathrm{dd,BT}} := \mathcal{C}^{\mathrm{dd}} \times_{\mathcal{M}_{\mathrm{loc}}^{K'/K}} \mathcal{M}_{\mathrm{loc}}^{K'/K,\mathrm{BT}}$, and $\mathcal{C}^{(\tau_i),\mathrm{BT}} := \mathcal{C}^{(\tau_i)} \times_{\mathcal{M}_{\mathrm{loc}}^{(\tau_i)}} \mathcal{M}_{\mathrm{loc}}^{(\tau_i),\mathrm{BT}}$.

A Breuil–Kisin module $\mathfrak{M} \in \mathcal{C}^{\mathrm{dd}}(A)$ is said to satisfy the strong determinant condition if and only if its image $\Psi(\mathfrak{M}) \in \mathcal{M}_{\mathrm{loc}}^{K'/K}(A)$ does, i.e. if and only if it lies in $\mathcal{C}^{\mathrm{dd,BT}}$.

Proposition 3.5.7. *$\mathcal{C}^{(\tau_i),\mathrm{BT}}$ (resp. $\mathcal{C}^{\mathrm{dd,BT}}$) is a closed substack of $\mathcal{C}^{(\tau_i)}$ (resp. $\mathcal{C}^{\mathrm{dd}}$); in particular, it is a ϖ-adic formal algebraic stack of finite presentation over $\mathcal{O}$.*

Proof. This is immediate from Corollary 3.1.7 and the definition of the strong determinant condition as an equality of polynomial functions. □

Remark 3.5.8. The motivation for imposing the strong determinant condition is as follows. One can take the flat part (in the sense of [Eme, Ex. 9.11]) of the ϖ-adic formal stack $\mathcal{C}^{\mathrm{dd}}$, and on this flat part, one can impose the condition that the corresponding Galois representations have all pairs of labelled Hodge–Tate weights equal to $\{0, 1\}$; that is, we can consider the substack of $\mathcal{C}^{\mathrm{dd}}$ corresponding to the Zariski closure of the these Galois representations.

We will soon see that $\mathcal{C}^{\mathrm{dd,BT}}$ is flat (Corollary 3.8.3). By Lemma 3.5.16 below, it follows that the substack of the previous paragraph is equal to $\mathcal{C}^{\mathrm{dd,BT}}$; so we may think of the strong determinant condition as being precisely the condition which imposes this condition on the labelled Hodge–Tate weights, and results in a formal stack which is flat over $\operatorname{Spf} \mathcal{O}$. Since the inertial types of p-adic Galois representations are unmixed, it is natural from this perspective to expect that $\mathcal{C}^{\mathrm{dd,BT}}$ should be the disjoint union of the stacks $\mathcal{C}^{\tau,\mathrm{BT}}$ for *unmixed* types, and indeed this will be proved shortly at Corollary 3.5.13.

To compare the strong determinant condition to the condition that the type of a Breuil–Kisin module is unmixed, we make some observations about these conditions in the case of finite field coefficients.

Lemma 3.5.9. *Let $\mathbf{F}'/\mathbf{F}$ be a finite extension, and let $(\mathfrak{L}, \mathfrak{L}^+)$ be an object of $\mathcal{M}_{\mathrm{loc}}^{K'/K}(\mathbf{F}')$. Then $(\mathfrak{L}, \mathfrak{L}^+)$ satisfies the strong determinant condition if and only if the following property holds: for each i and for each $\xi : I(K'/K) \to \mathcal{O}^\times$ we have $\dim_{\mathbf{F}'}(\mathfrak{L}_i^+)_\xi = e$.*

Proof. This is proved in a similar way to [Kis09, Lemma 2.5.1], using the explicit formulation of the strong determinant condition from Remark 3.5.4. In the notation of that remark, we see that the strong determinant condition holds at ξ if and only if for each embedding $\sigma_i : k' \hookrightarrow \mathbf{F}$ we have

$$\det_A \left(\sum_j \pi^j X_{j,\sigma_i} \mid (\mathfrak{L}_i^+)_\xi \right) = \prod_\psi \sum_j \left(\psi(\pi)^j X_{j,\sigma_i} \right), \tag{3.5.10}$$

where the product runs over the embeddings $\psi : \mathcal{O}_N \hookrightarrow \mathcal{O}$ with the property that $\psi|_{W(k')}$ lifts σ_i. Since π induces a nilpotent endomorphism of $(\mathfrak{L}_i^+)_\xi$ the left-hand side of (3.5.10) evaluates to $X_{0,\sigma_i}^{\dim_{\mathbf{F}'}(\mathfrak{L}_i^+)_\xi}$ while the right-hand side, which can be viewed as a norm from $\mathcal{O}_N \otimes_{\mathbf{Z}_p} \mathbf{F}'$ down to $W(k') \otimes_{\mathbf{Z}_p} \mathbf{F}'$, is equal to X_{0,σ_i}^e. $\square$

Lemma 3.5.11. *Let $\mathbf{F}'/\mathbf{F}$ be a finite extension, and let $\mathfrak{M}$ be a Breuil–Kisin module of rank 2 and height at most one with $\mathbf{F}'$-coefficients and descent data.*

(1) $\mathfrak{M}$ satisfies the strong determinant condition if and only if the following property holds: for each i and for each $\xi : I(K'/K) \to \mathcal{O}^\times$ we have $\dim_{\mathbf{F}'}(\operatorname{im}\Phi_{\mathfrak{M},i}/E(u)\mathfrak{M}_i)_\xi = e$.

(2) If $\mathfrak{M}$ satisfies the strong determinant condition, then the determinant of $\Phi_{\mathfrak{M},i}$ with respect to some choice of basis has u-adic valuation e'.

Proof. The first part is immediate from Lemma 3.5.9. For the second part, let $\Phi_{\mathfrak{M},i,\xi}$ be the restriction of $\Phi_{\mathfrak{M},i}$ to $\varphi^*(\mathfrak{M}_{i-1})_\xi$. We think of $\mathfrak{M}_i$ and $\varphi^*(\mathfrak{M}_{i-1})$ as free $\mathbf{F}'[\![v]\!]$-modules of rank $2e(K'/K)$, where $v = u^{e(K'/K)}$. We have

$$\det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i}) = \left(\det_{\mathbf{F}'[\![u]\!]}(\Phi_{\mathfrak{M},i})\right)^{e(K'/K)}.$$

Since $\Phi_{\mathfrak{M},i}$ commutes with the descent datum, we also have

$$\det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i}) = \prod_\xi \det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i,\xi}),$$

where ξ runs over the $e(K'/K)$ characters $I(K'/K) \to \mathcal{O}^\times$.

The proof of the second part of [Kis09, Lemma 2.5.1] implies that, for each ξ, $\det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i,\xi})$ is $v^e = u^{e'}$ times a unit. Indeed, each $\mathfrak{M}_{i,\xi}$ is a free $\mathbf{F}'[\![v]\!]$-module of rank 2. It admits a basis $\{e_{1,\xi}, e_{2,\xi}\}$ such that $\operatorname{im}\Phi_{\mathfrak{M},i,\xi} = \langle v^i e_{1,\xi}, v^j e_{2,\xi}\rangle$ for some non-negative integers i, j. The strong determinant condition on $(\operatorname{im}\Phi_{\mathfrak{M},1,\xi}/v^e\mathfrak{M}_{i,\xi})$ implies that $i + j = 2e - e = e$, and this is precisely the v-adic valuation of $\det_{\mathbf{F}'[\![v]\!]}(\Phi_{\mathfrak{M},i,\xi})$. We deduce that the u-adic valuation of $\left(\det_{\mathbf{F}'[\![u]\!]}(\Phi_{\mathfrak{M},i})\right)^{e(K'/K)}$ is $e(K'/K) \cdot e'$, which implies the second part of the lemma. $\square$

By contrast, we have the following criterion for the type of a Breuil–Kisin module to be unmixed.

Proposition 3.5.12. *Let $\mathbf{F}'/\mathbf{F}$ be a finite extension, and let $\mathfrak{M}$ be a Breuil–Kisin module of rank 2 and height at most one with $\mathbf{F}'$-coefficients and descent data. Then the type of $\mathfrak{M}$ is unmixed if and only if $\dim_{\mathbf{F}'}(\operatorname{im}\Phi_{\mathfrak{M},i}/E(u)\mathfrak{M}_i)_\xi$ is independent of ξ for each fixed i. In particular, if $\mathfrak{M}$ satisfies the strong determinant condition, then the type of $\mathfrak{M}$ is unmixed.*

Proof. We begin the proof of the first part with the following observation. Let Λ be a rank two free $\mathbf{F}'[[u]]$-module with an action of $I(K'/K)$ that is $\mathbf{F}'$-linear and

u-semilinear with respect to a character χ (i.e., such that $g(u\lambda) = \chi(g)ug(\lambda)$ for $\lambda \in \Lambda$). In particular $I(K'/K)$ acts on $\Lambda/u\Lambda$ through a pair of characters which we call η and η'. Let $\Lambda' \subset \Lambda$ be a rank two $I(K'/K)$-sublattice. We claim that there are integers $m, m' \geq 0$ such that the multiset of characters of $I(K'/K)$ occurring in Λ/Λ' has the shape

$$\{\eta\chi^i : 0 \leq i < m\} \cup \{\eta'\chi^i : 0 \leq j < m'\}$$

and the multiset of characters occurring in $\Lambda'/u\Lambda'$ is $\{\eta\chi^m, \eta'\chi^{m'}\}$.

To check the claim we proceed by induction on $\dim_{\mathbf{F}'} \Lambda/\Lambda'$, the case $\Lambda = \Lambda'$ being trivial. Suppose $\dim_{\mathbf{F}'} \Lambda/\Lambda' = 1$, so that Λ' lies between Λ and $u\Lambda$. Consider the chain of containments $\Lambda \supset \Lambda' \supset u\Lambda \supset u\Lambda'$. If without loss of generality $I(K'/K)$ acts on Λ/Λ' via η, then it acts on $\Lambda'/u\Lambda$ by η' and $u\Lambda/u\Lambda'$ by $\chi\eta$, proving the claim with $m = 1$ and $m' = 0$. The general case follows by iterated application of the case $\dim_k \Lambda/\Lambda' = 1$, noting that since $I(K'/K)$ is abelian the quotient Λ/Λ' has a filtration by $I(K/K')$-submodules whose graded pieces have dimension 1.

Now return to the statement of the proposition. Let (τ_i) be the mixed type of $\mathfrak{M}$ and write $\tau_{i-1} = \eta \oplus \eta'$. We apply the preceding observation with $\Lambda = \operatorname{im} \Phi_{\mathfrak{M},i}$ and $\Lambda' = E(u)\mathfrak{M}_i = u^{e'}\mathfrak{M}_i$. Note that χ is a generator of the cyclic group $I(K'/K)$ of order e'/e. Since $\Phi_{\mathfrak{M}}$ commutes with descent data, the group $I(K'/K)$ acts on $\Lambda/u\Lambda$ via η and η'. Then the the multiset

$$\{\eta\chi^i : 0 \leq i < m\} \cup \{\eta'\chi^i : 0 \leq j < m'\}$$

contains each character of $I(K'/K)$ with equal multiplicity if and only if one of η, η' is the successor to $\eta\chi^{m-1}$ in the list $\eta, \eta\chi, \eta\chi^2, \ldots$, and the other is the successor to $\eta'\chi^{m'-1}$ in the list $\eta', \eta'\chi, \eta'\chi^2, \ldots$, i.e., if and only if $\{\eta\chi^m, \eta'\chi^{m'}\} = \{\eta, \eta'\}$. Since $\mathfrak{M}_i/u\mathfrak{M}_i \cong u^{e'}\mathfrak{M}_i/u^{e'+1}\mathfrak{M}_i = \Lambda'/u\Lambda'$, this occurs if and only if that $\tau_i = \tau_{i-1}$.

Finally, the last part of the proposition follows immediately from the first part and Lemma 3.5.11. ☐

Corollary 3.5.13. *$\mathcal{C}^{\mathrm{dd,BT}}$ is the disjoint union of its closed substacks $\mathcal{C}^{\tau,\mathrm{BT}}$.*

Proof. This follows from Propositions 3.3.5 and 3.5.12. Indeed, from Proposition 3.3.5, it suffices to show that if (τ_i) is a mixed type, and $\mathcal{C}^{(\tau_i),\mathrm{BT}}$ is nonzero, then (τ_i) is in fact an unmixed type. Indeed, note that if $\mathcal{C}^{(\tau_i),\mathrm{BT}}$ is nonzero, then it contains a dense set of finite type points, so in particular contains an $\mathbf{F}'$-point for some finite extension $\mathbf{F}'/\mathbf{F}$. It follows from Proposition 3.5.12 that the type is unmixed, as required. ☐

Remark 3.5.14. Since our primary interest is in Breuil–Kisin modules, we will have no further need to consider the stacks $\mathcal{M}_{\mathrm{loc}}^{(\tau_i),\mathrm{BT}}$ or $\mathcal{C}^{(\tau_i),\mathrm{BT}}$ for types that are not unmixed.

Let τ be a tame type; since $I(K'/K)$ is cyclic, we can write $\tau = \eta \oplus \eta'$ for (possibly equal) characters $\eta, \eta' : I(K'/K) \to \mathcal{O}^\times$. Let $(\mathfrak{L}, \mathfrak{L}^+)$ be an object of $\mathcal{M}_{\mathrm{loc}}^{\tau}(A)$. Suppose that $\xi \neq \eta, \eta'$. Then elements of $\mathfrak{L}_\xi$ are divisible by π' in $\mathfrak{L}$, and so multiplication by π' induces an isomorphism of projective $e_i(\mathcal{O}_N \otimes A)$-modules of equal rank

$$p_{i,\xi} : e_i\mathfrak{L}_{\xi\chi_i^{-1}} \xrightarrow{\sim} e_i\mathfrak{L}_\xi$$

where $\chi_i : I(K'/K) \to \mathcal{O}^\times$ is the character sending $g \mapsto \sigma_i(h(g))$. The induced map

$$p^+_{i,\xi} : e_i\mathfrak{L}^+_{\xi\chi_i^{-1}} \longrightarrow e_i\mathfrak{L}^+_\xi$$

is in particular an injection. The following lemma will be useful for checking the strong determinant condition when comparing various different stacks of local model stacks.

Lemma 3.5.15. *Let $(\mathfrak{L}, \mathfrak{L}^+)$ be an object of $\mathcal{M}^{\tau}_{\mathrm{loc}}(A)$. Then $(\mathfrak{L}, \mathfrak{L}^+)$ is an object of $\mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}}(A)$ if and only if both*

(1) the condition (3.5.3) holds for $\xi = \eta, \eta'$, and

(2) the injections $p^+_{i,\xi} : e_i\mathfrak{L}^+_{\xi\chi_i^{-1}} \xrightarrow{\pi'} e_i\mathfrak{L}^+_{\xi}$ are isomorphisms for all $\xi \neq \eta, \eta'$ and for all i.

The second condition is equivalent to

(2′) we have $(\mathfrak{L}^+/\pi'\mathfrak{L}^+)_{\xi} = 0$ for all $\xi \neq \eta, \eta'$.

Proof. The equivalence between (2) and (2′) is straightforward. Suppose now that $\xi \neq \eta, \eta'$. Locally on $\operatorname{Spec} A$ the module $e_i\mathfrak{L}^+_{\xi\chi_i^{-1}}$ is by definition a direct summand of $e_i\mathfrak{L}_{\xi\chi_i^{-1}}$. Since $p_{i,\xi}$ is an isomorphism, the image of $p^+_{i,\xi}$ is locally on $\operatorname{Spec} A$ a direct summand of $e_i\mathfrak{L}^+_{\xi}$. Under the assumption that (3.5.10) holds for i and $\xi\chi_i^{-1}$, the condition (3.5.10) for i and ξ is therefore equivalent to the surjectivity of $p^+_{i,\xi}$. The lemma follows upon noting that χ_i is a generator of the group of characters $I(K'/K) \to \mathcal{O}^{\times}$. ∎

To conclude this section we describe the $\mathcal{O}_{E'}$-points of $\mathcal{C}^{\mathrm{dd,BT}}$, for E'/E a finite extension; recall that our convention is that a two-dimensional Galois representation is Barsotti–Tate if all its labelled pairs of Hodge–Tate weights are equal to $\{0, 1\}$ (and not just that all of the labelled Hodge–Tate weights are equal to 0 or 1).

Lemma 3.5.16. *Let E'/E be a finite extension. Then the $\mathrm{Spf}(\mathcal{O}_{E'})$-points of $\mathcal{C}^{\mathrm{dd,BT}}$ correspond precisely to the potentially Barsotti–Tate Galois representations $G_K \to \mathrm{GL}_2(\mathcal{O}_{E'})$ which become Barsotti–Tate over K'; and the $\mathrm{Spf}(\mathcal{O}_{E'})$-points of $\mathcal{C}^{\tau,\mathrm{BT}}$ correspond to those representations which are potentially Barsotti–Tate of type τ.*

Proof. In light of Lemma 3.1.9 and the first sentence of Remark 3.3.3, we are reduced to checking that a Breuil–Kisin module of rank 2 and height 1 with $\mathcal{O}_{E'}$-coefficients and descent data corresponds to a potentially Barsotti–Tate representation if and only if it satisfies the strong determinant condition, as well as checking that the descent data on the Breuil–Kisin module matches the type of the corresponding Galois representation.

Let $\mathfrak{M}_{\mathcal{O}_{E'}} \in \mathcal{C}^{\mathrm{dd,BT}}(\mathrm{Spf}(\mathcal{O}_{E'}))$. Plainly $\mathfrak{M}_{\mathcal{O}_{E'}}$ satisfies the strong determinant condition if and only if $\mathfrak{M} := \mathfrak{M}_{\mathcal{O}_{E'}}[1/p]$ satisfies the strong determinant condition (with the latter having the obvious meaning). Consider the filtration

$$\mathrm{Fil}^1(\varphi^*(\mathfrak{M}_i)) := \{m \in \varphi^*(\mathfrak{M}_i) \mid \Phi_{\mathfrak{M},i+1}(m) \in E(u)\mathfrak{M}_{i+1}\} \subset \varphi^*\mathfrak{M}_i$$

inducing

$$\mathrm{Fil}^1_i \subset \varphi^*(\mathfrak{M}_i)/E(u)\varphi^*(\mathfrak{M}_i).$$

Note that $\varphi^*(\mathfrak{M}_i)/E(u)\varphi^*(\mathfrak{M}_i)$ is isomorphic to a free $K' \otimes_{W(k'),\sigma_i} E'$-module of rank 2. Then $\mathfrak{M}$ corresponds to a Barsotti–Tate Galois representation

$$G_{K'} \to \mathrm{GL}_2(E')$$

if and only, if for every i, Fil^1_i is isomorphic to $K' \otimes_{W(k'),\sigma_i} E'$ as a $K' \otimes_{W(k'),\sigma_i} E'$-submodule of $\varphi^*\mathfrak{M}_i/E(u)\varphi^*\mathfrak{M}_i$. This follows, for example, by specialising the proof of [Kis08, Cor. 2.6.2] to the Barsotti–Tate case (taking care to note that the conventions of *loc. cit.* for Hodge–Tate weights and for Galois representations associated to Breuil–Kisin modules are both dual to ours).

Let $\xi : I(K'/K) \to \mathcal{O}^\times$ be a character. Consider the filtration

$$\mathrm{Fil}^1_{i,\xi} \subset \varphi^*(\mathfrak{M}_i)_\xi / E(u)\varphi^*(\mathfrak{M}_i)_\xi \simeq N^2 \otimes_{W(k'),\sigma_i} E'$$

induced by Fil^1_i. The strong determinant condition on $(\operatorname{im}\Phi_{\mathfrak{M},i+1}/E(u)\mathfrak{M}_{i+1})_\xi$ holds if and only if $\mathrm{Fil}^1_{i,\xi}$ is isomorphic to $N \otimes_{W(k'),\sigma_i} E'$. By [CL18, Lemma 5.10], we have an isomorphism of $K' \otimes_{W(k'),\sigma_i} E'$-modules

$$\mathrm{Fil}^1_i \simeq K' \otimes_N \mathrm{Fil}^1_{i,\xi}.$$

This, together with the previous paragraph, allows us to conclude. Note that, since u acts invertibly when working with E'-coefficients and after quotienting by $E(u)$, the argument is independent of the choice of character ξ.

For the statement about types, let $S_{K'_0}$ be Breuil's period ring (see e.g. [Bre00, §5.1]) endowed with the evident action of $\mathrm{Gal}(K'/K)$ compatible with the embedding $\mathfrak{S} \hookrightarrow S_{K'_0}$. Here K'_0 is the maximal unramified extension in K'. Recall that by [Liu08, Cor. 3.2.3] there is a canonical (φ, N)-module isomorphism

$$S_{K'_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \cong S_{K'_0} \otimes_{K'_0} D_{\mathrm{pcris}}(T(\mathfrak{M})). \tag{3.5.17}$$

One sees from its construction that the isomorphism (3.5.17) is in fact equivariant for the action of $I(K'/K)$, and the claim follows by reducing modulo u and all its divided powers. □

3.6. **Change of extensions.** We now discuss the behaviour of various of our constructions under change of K'. Let L'/K' be a tame extension such that L'/K is Galois. We suppose that we have fixed a uniformiser π'' of L' such that $(\pi'')^{e(L'/K')} = \pi'$. Let $\mathfrak{S}'_A := (W(l') \otimes_{\mathbf{Z}_p} A)[[u]]$, where l' is the residue field of L', and let $\mathrm{Gal}(L'/K)$ and φ act on $\mathfrak{S}'_A$ via the prescription of Section 2.1 (with π'' in place of π').

There is a natural injective ring homomorphism $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A \to \mathcal{O}_{L'} \otimes_{\mathbf{Z}_p} A$, which is equivariant for the action of $\mathrm{Gal}(L'/K)$ (acting on the source via the natural surjection $\mathrm{Gal}(L'/K) \to \mathrm{Gal}(K'/K)$). There is also an obvious injective ring homomorphism $\mathfrak{S}_A \to \mathfrak{S}'_A$ sending $u \mapsto u^{e(L'/K')}$, which is equivariant for the actions of φ and $\mathrm{Gal}(L'/K)$; we have $(\mathfrak{S}'_A)^{\mathrm{Gal}(L'/K')} = \mathfrak{S}_A$. If τ is an inertial type for $I(K'/K)$, we write τ' for the corresponding type for $I(L'/K)$, obtained by inflation.

For any $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}^{K'/K}_{\mathrm{loc}}$, we define $(\mathfrak{L}', (\mathfrak{L}')^+) \in \mathcal{M}^{L'/K}_{\mathrm{loc}}$ by

$$(\mathfrak{L}', (\mathfrak{L}')^+) := \mathcal{O}_{L'} \otimes_{\mathcal{O}_{K'}} (\mathfrak{L}, \mathfrak{L}^+),$$

with the diagonal action of $\mathrm{Gal}(L'/K)$. Similarly, for any $\mathfrak{M} \in \mathcal{C}^{\mathrm{dd}}(A)$, we let $\mathfrak{M}' := \mathfrak{S}'_A \otimes_{\mathfrak{S}_A} \mathfrak{M}$, with φ and $\mathrm{Gal}(L'/K)$ again acting diagonally.

Proposition 3.6.1.

(1) The assignments $(\mathfrak{L}, \mathfrak{L}^+) \to (\mathfrak{L}', (\mathfrak{L}')^+)$ and $\mathfrak{M} \mapsto \mathfrak{M}'$ induce compatible monomorphisms $\mathcal{M}^{K'/K}_{\mathrm{loc}} \to \mathcal{M}^{L'/K}_{\mathrm{loc}}$ and $\mathcal{C}^{\mathrm{dd}}_{K'} \to \mathcal{C}^{\mathrm{dd}}_{L'}$, i.e., as functors they are fully faithful.

(2) The monomorphism $\mathcal{C}^{\mathrm{dd}}_{K'} \to \mathcal{C}^{\mathrm{dd}}_{L'}$ induces an isomorphism $\mathcal{C}^{\tau} \to \mathcal{C}^{\tau'}$, as well as a monomorphism $\mathcal{C}^{\mathrm{dd,BT}}_{K'} \to \mathcal{C}^{\mathrm{dd,BT}}_{L'}$ and an isomorphism $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{C}^{\tau',\mathrm{BT}}$.

Proof. (1) One checks easily that the assignments $(\mathfrak{L}, \mathfrak{L}^+) \to (\mathfrak{L}', (\mathfrak{L}')^+)$ and $\mathfrak{M} \mapsto \mathfrak{M}'$ are compatible. For the rest of the claim, we consider the case of the functor $\mathfrak{M} \mapsto \mathfrak{M}'$; the arguments for the local models case are similar but slightly easier, and we leave them to the reader. Let A be a ϖ-adically complete $\mathcal{O}$-algebra. If $\mathfrak{N}$ is a rank d Breuil–Kisin module with descent data from L' to K, consider the Galois invariants $\mathfrak{N}^{\mathrm{Gal}(L'/K')}$. Since $(\mathfrak{S}'_A)^{\mathrm{Gal}(L'/K')} = \mathfrak{S}_A$, these invariants are naturally a $\mathfrak{S}_A$-module, and moreover they naturally carry a Frobenius and descent data satisfying the conditions required of a Breuil–Kisin module of height at most h. In general the invariants need not be projective of rank d, and so need not be rank d Breuil–Kisin module with descent data from K' to K. However, in the case $\mathfrak{N} = \mathfrak{M}'$ we have

$$(\mathfrak{M}')^{\mathrm{Gal}(L'/K')} = (\mathfrak{S}'_A \otimes_{\mathfrak{S}_A} \mathfrak{M})^{\mathrm{Gal}(L'/K')} = (\mathfrak{S}'_A)^{\mathrm{Gal}(L'/K')} \otimes_{\mathfrak{S}_A} \mathfrak{M} = \mathfrak{M}.$$

Here the second equality holds e.g. because $\mathrm{Gal}(L'/K')$ has order prime to p, so that taking $\mathrm{Gal}(L'/K')$-invariants is exact, and indeed is given by multiplication by an idempotent $i \in \mathfrak{S}'_A$ (use the decomposition $\mathfrak{S}'_A = i\mathfrak{S}'_A \oplus (1-i)\mathfrak{S}'_A$ and note that i kills the latter summand). It follows immediately that the functor $\mathfrak{M} \to \mathfrak{M}'$ is fully faithful, so $\mathcal{C}^{\mathrm{dd},a}_{K'} \to \mathcal{C}^{\mathrm{dd},a}_{L'}$ is a monomorphism.

(2) Suppose now that $\mathfrak{N}$ has type τ'. In view of what we have proven so far, in order to prove that $\mathcal{C}^{\tau} \to \mathcal{C}^{\tau'}$ is an isomorphism, it is enough to show that $\mathfrak{N}^{\mathrm{Gal}(L'/K')}$ is a rank d Breuil–Kisin module of type τ, and that the natural map of $\mathfrak{S}'_A$-modules

$$\mathfrak{S}'_A \otimes_{\mathfrak{S}_A} \mathfrak{N}^{\mathrm{Gal}(L'/K')} \to \mathfrak{N} \tag{3.6.2}$$

is an isomorphism. For the remainder of this proof, for clarity we write $u_{K'}$, $u_{L'}$ instead of u for the variables of $\mathfrak{S}_A$ and $\mathfrak{S}'_A$ respectively. Since the type τ' of $\mathfrak{N}$ is inflated from τ, the action of $\mathrm{Gal}(L'/K')$ on $\mathfrak{N}/u_{L'}\mathfrak{N}$ factors through $\mathrm{Gal}(l'/k')$; noting that $W(l')$ has a normal basis for $\mathrm{Gal}(l'/k')$ over $W(k')$, we obtain an isomorphism

$$W(l') \otimes_{W(k')} (\mathfrak{N}/u_{L'}\mathfrak{N})^{\mathrm{Gal}(L'/K')} \xrightarrow{\sim} \mathfrak{N}/u_{L'}\mathfrak{N}. \tag{3.6.3}$$

In particular the $W(k') \otimes_{\mathbf{Z}_p} A$-module $(\mathfrak{N}/u_{L'}\mathfrak{N})^{\mathrm{Gal}(L'/K')}$ is projective of rank d.

Observe however that $(\mathfrak{N}/u_{K'}\mathfrak{N})^{\mathrm{Gal}(L'/K')} = (\mathfrak{N}/u_{L'}\mathfrak{N})^{\mathrm{Gal}(L'/K')}$. To see this, by the exactness of taking $\mathrm{Gal}(L'/K')$ invariants it suffices to check that $u^i_{L'}\mathfrak{N}/u^{i+1}_{L'}$ has trivial $\mathrm{Gal}(L'/K')$-invariants for $0 < i < e(L'/K')$. Multiplication by $u^i_{L'}$ gives an isomorphism $\mathfrak{N}/u_{L'}\mathfrak{N} \cong u^i_{L'}\mathfrak{N}/u^{i+1}_{L'}$, so that for i in the above range, the inertia group $I(L'/K')$ acts linearly on $u^i_{L'}\mathfrak{N}/u^{i+1}_{L'}$ through a twist of τ' by a nontrivial character; so there are no $I(L'/K')$-invariants, and thus no $\mathrm{Gal}(L'/K')$-invariants either.

It follows that the isomorphism (3.6.3) is the map (3.6.2) modulo $u_{L'}$. By Nakayama's lemma it follows that (3.6.2) is surjective. Since $\mathfrak{N}$ is projective, the surjection (3.6.2) is split, and is therefore an isomorphism, since it is an isomorphism modulo $u_{L'}$. This isomorphism exhibits $\mathfrak{N}^{\mathrm{Gal}(L'/K')}$ as a direct summand (as an $\mathfrak{S}_A$-module) of the projective module $\mathfrak{N}$, so it is also projective; and it is projective of rank d, since this holds modulo $u_{K'}$.

Finally, we need to check the compatibility of these maps with the strong determinant condition. By Corollary 3.5.13, it is enough to prove this for the case of the morphism $\mathcal{C}^{\tau} \to \mathcal{C}^{\tau'}$ for some τ; by the compatibility in part (1), this comes down to the same for the corresponding map of local model stacks $\mathcal{M}^{\tau}_{\mathrm{loc}} \to \mathcal{M}^{\tau'}_{\mathrm{loc}}$. If $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}^{\tau}_{\mathrm{loc}}$, it therefore suffices to show to show that conditions (1) and (2′) of Lemma 3.5.15 for $(\mathfrak{L}, \mathfrak{L}^+)$ and $(\mathfrak{L}', (\mathfrak{L}')^+) := \mathcal{O}_{L'} \otimes_{\mathcal{O}_{K'}} (\mathfrak{L}, \mathfrak{L}^+)$ are equivalent. This is immediate for condition (2′), since we have $(\mathfrak{L}')^+/\pi''(\mathfrak{L}')^+ \cong l' \otimes_{k'} (\mathfrak{L}^+/\pi'\mathfrak{L}^+)$ as $I(L'/K)$-representations.

Writing $\tau = \eta \oplus \eta'$, it remains to relate the strong determinant conditions on the η, η'-parts over both K' and L'. Unwinding the definitions using Remark 3.5.4, one finds that the condition over L' is a product of $[l' : k']$ copies (with different sets of variables) of the condition over K'. Thus the strong determinant condition over K' implies the condition over L', while the condition over L' implies the condition over K' up to an $[l' : k']$th root of unity. Comparing the terms involving only copies of X_{0,σ_i}'s shows that this root of unity must be 1. $\square$

Remark 3.6.4. The morphism of local model stacks $\mathcal{M}^{\tau}_{\mathrm{loc}} \to \mathcal{M}^{\tau'}_{\mathrm{loc}}$ is not an isomorphism (provided that the extension L'/K' is nontrivial). The issue is that, as we observed in the preceding proof, local models $(\mathfrak{L}', (\mathfrak{L}')^+)$ in the image of the morphism $\mathcal{M}^{\tau}_{\mathrm{loc}} \to \mathcal{M}^{\tau'}_{\mathrm{loc}}$ can have $((\mathfrak{L}')^+/\pi''(\mathfrak{L}')^+)_\xi \neq 0$ only for characters $\xi : I(L'/K) \to \mathcal{O}^\times$ that are inflated from $I(K'/K)$. However, one does obtain an isomorphism from the substack of $\mathcal{M}^{\tau}_{\mathrm{loc}}$ of pairs $(\mathfrak{L}, \mathfrak{L}^+)$ satisfying condition (2) of Lemma 3.5.15 to the analogous substack of $\mathcal{M}^{\tau'}_{\mathrm{loc}}$; therefore the induced map $\mathcal{M}^{\tau,\mathrm{BT}} \to \mathcal{M}^{\tau',\mathrm{BT}}$ *will* also be an isomorphism. Analogous remarks will apply to the maps of local model stacks in §3.7.

3.7. **Explicit local models.** We now explain the connection between the moduli stacks $\mathcal{C}^\tau$ and local models for Shimura varieties at hyperspecial and Iwahori level. This idea has been developed in [CL18] for Breuil–Kisin modules of arbitrary rank with tame principal series descent data, inspired by [Kis09], which analyses the case without descent data.

The results of [CL18] relate the moduli stacks $\mathcal{C}^\tau$(in the case that τ is a principal series type) via a local model diagram to a mixed-characteristic deformation of the affine flag variety, introduced in this generality by Pappas and Zhu [PZ13]. The local models in [PZ13, §6] are defined in terms of Schubert cells in the generic fibre of this mixed-characteristic deformation, by taking the Zariski closure of these Schubert cells. The disadvantage of this approach is that it does not give a direct moduli-theoretic interpretation of the special fibre of the local model. Therefore, it is hard to check directly that our stack $\mathcal{C}^{\tau,\mathrm{BT}}$, which has a moduli-theoretic definition, corresponds to the desired local model under the diagram introduced in [CL18, Cor. 4.11] [3].

In our rank 2 setting, the local models admit a much more explicit condition, using lattice chains and Kottwitz's determinant condition, and in the cases of nonscalar types, we will relate our local models to the naive local model at Iwahori level for the Weil restriction of GL_2, in the sense of [PRS13, §2.4].

We begin with the simpler case of scalar inertial types. Suppose that K'/K is totally ramified, and that τ is a scalar inertial type, say $\tau = \eta \oplus \eta$. In this case we define the local model stack $\mathcal{M}_{\mathrm{loc,hyp}}$ ("hyp" for "hyperspecial") to be the *fppf* stack over $\mathrm{Spf}\,\mathcal{O}$ (in fact, a p-adic formal algebraic stack), which to each

[3]One should be able to check this by adapting the ideas in [HN02, §2.1] and [PZ13, Prop. 6.2] to $\mathrm{Res}_{K/\mathbf{Q}_p}\mathrm{GL}_n$ where $K/\mathbf{Q}_p$ can be ramified.

p-adically complete $\mathcal{O}$-algebra A associates the groupoid $\mathcal{M}_{\mathrm{loc,hyp}}(A)$ consisting of pairs $(\mathfrak{L}, \mathfrak{L}^+)$, where

- $\mathfrak{L}$ is a rank 2 projective $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-module, and
- $\mathfrak{L}^+$ is an $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-submodule of $\mathfrak{L}$, which is locally on $\operatorname{Spec} A$ a direct summand of $\mathfrak{L}$ as an A-module (or equivalently, for which the quotient $\mathfrak{L}/\mathfrak{L}^+$ is projective as an A-module).

We let $\mathcal{M}^{\mathrm{BT}}_{\mathrm{loc,hyp}}$ be the substack of pairs $(\mathfrak{L}, \mathfrak{L}^+)$ with the property that for all $a \in \mathcal{O}_K$, we have

$$\det{}_A(a|\mathfrak{L}^+) = \prod_{\psi: K \hookrightarrow E} \psi(a) \tag{3.7.1}$$

as polynomial functions on $\mathcal{O}_K$.

Lemma 3.7.2. *The functor $(\mathfrak{L}', (\mathfrak{L}')^+) \mapsto ((\mathfrak{L}')_\eta, (\mathfrak{L}')^+_\eta)$ defines a morphism $\mathcal{M}^\tau_{\mathrm{loc}} \to \mathcal{M}_{\mathrm{loc,hyp}}$ which induces an isomorphism $\mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}} \to \mathcal{M}^{\mathrm{BT}}_{\mathrm{loc,hyp}}$. (We remind the reader that K'/K is assumed totally ramified, and that τ is assumed to be a scalar inertial type associated to the character η.)*

Proof. If $(\mathfrak{L}', (\mathfrak{L}')^+)$ is an object of $\mathcal{M}^\tau_{\mathrm{loc}}$, the proof that $((\mathfrak{L}')_\eta, (\mathfrak{L}')^+_\eta)$ is indeed an object of $\mathcal{M}_{\mathrm{loc,hyp}}(A)$ is very similar to the proof of Proposition 3.6.1, and is left to the reader. Similarly, the reader may verify that the functor

$$(\mathfrak{L}, \mathfrak{L}^+) \mapsto (\mathfrak{L}', (\mathfrak{L}')^+) := \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} (\mathfrak{L}, \mathfrak{L}^+),$$

where the action of $\mathrm{Gal}(K'/K)$ is given by the tensor product of the natural action on $\mathcal{O}_{K'}$ with the action on $(\mathfrak{L}, \mathfrak{L}^+)$ given by the character η, defines a morphism $\mathcal{M}_{\mathrm{loc,hyp}} \to \mathcal{M}^\tau_{\mathrm{loc}}$.

The composition $\mathcal{M}_{\mathrm{loc,hyp}} \to \mathcal{M}^\tau_{\mathrm{loc}} \to \mathcal{M}_{\mathrm{loc,hyp}}$ is evidently the identity. The composition in the other order is not, in general, naturally equivalent to the identity morphism, because for $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}^\tau_{\mathrm{loc}}(A)$ one cannot necessarily recover $\mathfrak{L}^+$ from the projection to its η-isotypic part. However, this *will* hold if $\mathfrak{L}^+$ satisfies condition (2) of Lemma 3.5.15 (and so in particular will hold after imposing the strong determinant condition).

Indeed, suppose $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}^\tau_{\mathrm{loc}}(A)$. Then there is a natural $\mathrm{Gal}(K'/K)$-equivariant map of projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-modules

$$\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathfrak{L}_\eta \to \mathfrak{L} \tag{3.7.3}$$

of the same rank (in which $\mathrm{Gal}(K'/K)$ acts by η on $\mathfrak{L}_\eta$). This map is surjective because it is surjective on η-parts and the maps $p_{i,\xi}$ are surjective for all $\xi \neq \eta$; therefore it is an isomorphism. One further has a a natural $\mathrm{Gal}(K'/K)$-equivariant map of $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-modules

$$\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} \mathfrak{L}^+_\eta \to \mathfrak{L}^+ \tag{3.7.4}$$

that is injective because locally on $\mathrm{Spec}(A)$ it is a direct summand of the isomorphism (3.7.3). If one further assumes that $\mathfrak{L}^+$ satisfies condition (2) of Lemma 3.5.15 then (3.7.4) is an isomorphism, as claimed.

It remains to check the compatibility of these maps with the strong determinant condition. If $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc,hyp}}$ then certainly condition (2) of Lemma 3.5.15 holds for $(\mathfrak{L}', (\mathfrak{L}')^+) := \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} (\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}^\tau_{\mathrm{loc}}$. By Lemma 3.5.15 the strong determinant condition holds for $(\mathfrak{L}', (\mathfrak{L}')^+)$ if and only if (3.5.3) holds for $\mathfrak{L}'$ with $\xi = \eta$; but this is exactly the condition (3.7.1) for $\mathfrak{L}$, as required. $\square$

Next, we consider the case of principal series types. We suppose that K'/K is totally ramified. We begin by defining a p-adic formal algebraic stack $\mathcal{M}_{\mathrm{loc},\mathrm{Iw}}$ over $\operatorname{Spf}\mathcal{O}$ ("Iw" for Iwahori). For each complete $\mathcal{O}$-algebra A, we let $\mathcal{M}_{\mathrm{loc},\mathrm{Iw}}(A)$ be the groupoid of tuples $(\mathfrak{L}_1, \mathfrak{L}_1^+, \mathfrak{L}_2, \mathfrak{L}_2^+, f_1, f_2)$, where

- $\mathfrak{L}_1$, $\mathfrak{L}_2$ are rank 2 projective $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-modules,
- $f_1 : \mathfrak{L}_1 \to \mathfrak{L}_2$, $f_2 : \mathfrak{L}_2 \to \mathfrak{L}_1$ are morphisms of $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-modules, satisfying $f_1 \circ f_2 = f_2 \circ f_1 = \pi$,
- both $\operatorname{coker} f_1$ and $\operatorname{coker} f_2$ are rank one projective $k \otimes_{\mathbf{Z}_p} A$-modules,
- $\mathfrak{L}_1^+$, $\mathfrak{L}_2^+$ are $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-submodules of $\mathfrak{L}_1$, $\mathfrak{L}_2$, which are locally on $\operatorname{Spec} A$ direct summands as A-modules (or equivalently, for which the quotients $\mathfrak{L}_i/\mathfrak{L}_i^+$ $(i = 1, 2)$ are projective A-modules), and moreover for which the morphisms f_1, f_2 restrict to morphisms $f_1 : \mathfrak{L}_1^+ \to \mathfrak{L}_2^+$, $f_2 : \mathfrak{L}_2^+ \to \mathfrak{L}_1^+$.

We let $\mathcal{M}_{\mathrm{loc},\mathrm{Iw}}^{\mathrm{BT}}$ be the substack of tuples with the property that for all $a \in \mathcal{O}_K$ and $i = 1, 2$, we have

$$\det{}_A(a|\mathfrak{L}_i^+) = \prod_{\psi : K \hookrightarrow E} \psi(a) \tag{3.7.5}$$

as polynomial functions on $\mathcal{O}_K$.

Write $\tau = \eta \oplus \eta'$ with $\eta \neq \eta'$. Recall that the character $h : \operatorname{Gal}(K'/K) = I(K'/K) \to W(k)^\times$ is given by $h(g) = g(\pi')/\pi'$. Since we are assuming that $\eta \neq \eta'$, for each embedding $\sigma : k \hookrightarrow \mathbf{F}$ (which we also think of as $\sigma : W(k) \hookrightarrow \mathcal{O}$) there are integers $0 < a_\sigma, b_\sigma < e(K'/K)$ with the properties that $\eta'/\eta = \sigma \circ h^{a_\sigma}$, $\eta/\eta' = \sigma \circ h^{b_\sigma}$; in particular, $a_\sigma + b_\sigma = e(K'/K)$. Recalling that $e_\sigma \in W(k) \otimes_{\mathbf{Z}_p} \mathcal{O} \subset \mathcal{O}_K$ is the idempotent corresponding to σ, we set $\pi_1 = \sum_\sigma (\pi')^{a_\sigma} e_\sigma$, $\pi_2 = \sum_\sigma (\pi')^{b_\sigma} e_\sigma$; so we have $\pi_1\pi_2 = \pi$, and $\pi_1 \in (\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} \mathcal{O})^{I(K'/K) = \eta'/\eta}$, $\pi_2 \in (\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} \mathcal{O})^{I(K'/K) = \eta/\eta'}$.

We define a morphism $\mathcal{M}_{\mathrm{loc}}^\tau \to \mathcal{M}_{\mathrm{loc},\mathrm{Iw}}$ as follows. Given a pair $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}_{\mathrm{loc}}^\tau(A)$, we set $(\mathfrak{L}_1, \mathfrak{L}_1^+) = (\mathfrak{L}_\eta, \mathfrak{L}_\eta^+)$, $(\mathfrak{L}_2, \mathfrak{L}_2^+) = (\mathfrak{L}_{\eta'}, \mathfrak{L}_{\eta'}^+)$, and we let f_1, f_2 be given by multiplication by π_1, π_2 respectively. The only point that is perhaps not immediately obvious is to check that $\operatorname{coker} f_1$ and $\operatorname{coker} f_2$ are rank one projective $k \otimes_{\mathbf{Z}_p} A$-modules. To see this, note that by [Sta13, Tag 05BU], we can lift $(\mathfrak{L}/\pi'\mathfrak{L})_{\eta'}$ to a rank one summand U of the projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module $\mathfrak{L}$. Let U_η be the direct summand of $\mathfrak{L}_\eta$ obtained by projection of U to the η-eigenspace $\mathfrak{L}_\eta$; then the projective $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-module U_η has rank one, as can be checked modulo π. Similarly we may lift $(\mathfrak{L}/\pi'\mathfrak{L})_{\eta'}$ to a rank one summand V of $\mathfrak{L}$, and we let $V_{\eta'}$ be the projection to the η'-part.

The natural map

$$\mathcal{O}_{K'} \otimes_{\mathcal{O}_K} (U_\eta \oplus V_{\eta'}) \to \mathfrak{L} \tag{3.7.6}$$

is an isomorphism, since both sides are projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-modules of rank two, and the given map is an isomorphism modulo π'. It follows immediately from (3.7.6) that $\mathfrak{L}_\eta = U_\eta \oplus \pi_2 V_{\eta'}$ and $\mathfrak{L}_{\eta'} = \pi_1 U_\eta \oplus V_{\eta'}$, so that $\operatorname{coker} f_1$ and $\operatorname{coker} f_2$ are projective of rank one, as claimed.

Proposition 3.7.7. *The morphism $\mathcal{M}_{\mathrm{loc}}^\tau \to \mathcal{M}_{\mathrm{loc},\mathrm{Iw}}$ induces an isomorphism $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}} \to \mathcal{M}_{\mathrm{loc},\mathrm{Iw}}^{\mathrm{BT}}$. (We remind the reader that K'/K is assumed totally ramified, and that τ is assumed to be a principal series inertial type.)*

Proof. We begin by constructing a morphism $\mathcal{M}_{\mathrm{loc,Iw}} \to \mathcal{M}^{\tau}_{\mathrm{loc}}$, inspired by the arguments of [RZ96, App. A]. We define an $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-module $\mathfrak{L}$ by

$$\mathfrak{L} = \oplus_{\sigma} \left(\oplus_{i=0}^{a_{\sigma}-1} e_{\sigma}\mathfrak{L}_1^{(i)} \oplus_{j=0}^{b_{\sigma}-1} e_{\sigma}\mathfrak{L}_2^{(j)} \right),$$

where the $\mathfrak{L}_1^{(i)}$'s and $\mathfrak{L}_2^{(j)}$'s are copies of $\mathfrak{L}_1$, $\mathfrak{L}_2$ respectively.

We can upgrade the $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-module structure on $\mathfrak{L}$ to that of an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module by specifying how π' acts. If $i < a_{\sigma}-1$, then we let $\pi' : e_{\sigma}\mathfrak{L}_1^{(i)} \to e_{\sigma}\mathfrak{L}_1^{(i+1)}$ be the map induced by the identity on $\mathfrak{L}_1$, and if $j < b_{\sigma}-1$, then we let $\pi' : e_{\sigma}\mathfrak{L}_2^{(j)} \to e_{\sigma}\mathfrak{L}_2^{(j+1)}$ be the map induced by the identity on $\mathfrak{L}_2$. We let $\pi' : e_{\sigma}\mathfrak{L}_1^{(a_{\sigma}-1)} \to e_{\sigma}\mathfrak{L}_2^{(0)}$ be the map induced by $f_1 : \mathfrak{L}_1 \to \mathfrak{L}_2$, and we let $\pi' : e_{\sigma}\mathfrak{L}_2^{(b_{\sigma}-1)} \to e_{\sigma}\mathfrak{L}_1^{(0)}$ be the map induced by $f_2 : \mathfrak{L}_2 \to \mathfrak{L}_1$. That this indeed gives $\mathfrak{L}$ the structure of an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module follows from our assumption that $f_1 \circ f_2 = f_2 \circ f_1 = \pi$. We give $\mathfrak{L}$ a semilinear action of $\mathrm{Gal}(K'/K) = I(K'/K)$ by letting it act via $(\sigma \circ h)^i \cdot \eta$ on each $e_{\sigma}\mathfrak{L}_1^{(i)}$ and via $(\sigma \circ h)^j \cdot \eta'$ on each $e_{\sigma}\mathfrak{L}_2^{(j)}$.

We claim that $\mathfrak{L}$ is a rank 2 projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module. Since $\operatorname{coker} f_2$ is projective by assumption, we can choose a section to the $k \otimes_{\mathbf{Z}_p} A$-linear morphism $\mathfrak{L}_1/\pi \to \operatorname{coker} f_2$, with image $\overline{U}_{\eta}$, say. Similarly we choose a section to $\mathfrak{L}_2/\pi \to \operatorname{coker} f_1$ with image $\overline{V}_{\eta'}$. We choose lifts U_{η}, $V_{\eta'}$ of $\overline{U}_{\eta}$, $\overline{V}_{\eta'}$ to direct summands of the $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-modules $\mathfrak{L}_1$, $\mathfrak{L}_2$ respectively. There is a map of $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-modules

$$\lambda : \mathcal{O}_{K'} \otimes_{\mathcal{O}_K} (U_{\eta} \oplus V_{\eta'}) \to \mathfrak{L} \tag{3.7.8}$$

induced by the map identifying $U_{\eta}, V_{\eta'}$ with their copies in $\mathfrak{L}_1^{(0)}$ and $\mathfrak{L}_2^{(0)}$ respectively. The map λ is surjective modulo π' by construction, hence surjective by Nakayama's lemma. Regarding λ as a map of projective $\mathcal{O}_K \otimes_{\mathbf{Z}_p} A$-modules of equal rank $2e(K'/K)$, we deduce that λ is an isomorphism. Since the source of λ is a projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module, the claim follows.

We now set

$$\mathfrak{L}^+ = \oplus_{\sigma} \left(\oplus_{i=0}^{a_{\sigma}-1} e_{\sigma}(\mathfrak{L}_1^{(i)})^+ \oplus_{j=0}^{b_{\sigma}-1} e_{\sigma}(\mathfrak{L}_2^{(j)})^+ \right) \subset \mathfrak{L}$$

It is immediate from the construction that $\mathfrak{L}^+$ is preserved by $\mathrm{Gal}(K'/K)$. The hypothesis that f_1, f_2 and preserve $\mathfrak{L}_1^+, \mathfrak{L}_2^+$ implies that $\mathfrak{L}^+$ is an $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-submodule of $\mathfrak{L}$, while the hypothesis that each $\mathfrak{L}_i/\mathfrak{L}_i^+$ is a projective A-module implies the same for $\mathfrak{L}/\mathfrak{L}^+$. This completes the construction of our morphism $\mathcal{M}_{\mathrm{loc,Iw}} \to \mathcal{M}^{\tau}_{\mathrm{loc}}$.

Just as in the proof of Proposition 3.7.2, the morphism $\mathcal{M}_{\mathrm{loc,Iw}} \to \mathcal{M}^{\tau}_{\mathrm{loc}}$ followed by our morphism $\mathcal{M}^{\tau}_{\mathrm{loc}} \to \mathcal{M}_{\mathrm{loc,Iw}}$ is the identity, while the composition in the other order is not, in general, naturally equivalent to the identity morphism. However, it follows immediately from Lemma 3.5.15 and the construction of $\mathfrak{L}^+$ that our morphisms $\mathcal{M}_{\mathrm{loc,Iw}} \to \mathcal{M}^{\tau}_{\mathrm{loc}}$ and $\mathcal{M}^{\tau}_{\mathrm{loc}} \to \mathcal{M}_{\mathrm{loc,Iw}}$ respect the strong determinant condition, and so induce maps $\mathcal{M}^{\mathrm{BT}}_{\mathrm{loc,Iw}} \to \mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}}$ and $\mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}} \to \mathcal{M}^{\mathrm{BT}}_{\mathrm{loc,Iw}}$. To see that the composite $\mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}} \to \mathcal{M}^{\mathrm{BT}}_{\mathrm{loc,Iw}} \to \mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}}$ is naturally equivalent to the identity, suppose that $(\mathfrak{L}, \mathfrak{L}^+) \in \mathcal{M}^{\tau}_{\mathrm{loc}}(A)$ and observe that there is a natural $\mathrm{Gal}(K'/K)$-equivariant isomorphism of $\mathcal{O}_K \otimes_A \mathbf{Z}_p$-modules

$$\oplus_{\sigma} \left(\oplus_{i=0}^{a_{\sigma}-1} e_{\sigma}\mathfrak{L}_{\eta}^{(i)} \oplus_{j=0}^{b_{\sigma}-1} e_{\sigma}\mathfrak{L}_{\eta'}^{(j)} \right) \xrightarrow{\sim} \mathfrak{L} \tag{3.7.9}$$

induced by the maps $\mathfrak{L}_\eta^{(i)} \xrightarrow{(\pi')^i} (\pi')^i\mathfrak{L}_\eta$ and $\mathfrak{L}_{\eta'}^{(j)} \xrightarrow{(\pi')^j} (\pi')^j\mathfrak{L}_{\eta'}$. The commutativity of the diagram

$$\begin{tikzcd} e_\sigma\mathfrak{L}_\eta^{(a_\sigma-1)} \arrow[r, "(\pi')^{a_\sigma-1}"] \arrow[d, "\pi_1"'] & e_\sigma(\pi')^{a_\sigma-1}\mathfrak{L}_\eta \arrow[d, "\pi'"] \\ e_\sigma\mathfrak{L}_{\eta'}^{(0)} \arrow[r, "\mathrm{id}"'] & e_\sigma\mathfrak{L}_{\eta'} \end{tikzcd}$$

implies that the map in (3.7.9) is in fact an $\mathcal{O}_{K'}\otimes_{\mathbf{Z}_p} A$-module isomorphism. The map (3.7.9) induces an inclusion

$$\oplus_\sigma\left(\oplus_{i=0}^{a_\sigma-1}e_\sigma(\mathfrak{L}_\eta^{(i)})^+\oplus_{j=0}^{b_\sigma-1}e_\sigma(\mathfrak{L}_{\eta'}^{(j)})^+\right)\to\mathfrak{L}^+.$$

If furthermore $(\mathfrak{L},\mathfrak{L}^+)\in\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}}$ then this is an isomorphism because $\mathfrak{L}^+$ satisfies condition (2) of Lemma 3.5.15. ☐

Finally, we turn to the case of a cuspidal type. Let L as usual be a quadratic unramified extension of K, and set $K'=L(\pi^{1/(p^{2f}-1)})$. The field N continues to denote the maximal unramified extension of K in K', so that $N=L$. Let τ be a cuspidal type, so that $\tau=\eta\oplus\eta'$, where $\eta\neq\eta'$ but $\eta'=\eta^{p^f}$.

Proposition 3.7.10. *There is a morphism $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}}\to\mathcal{M}_{\mathrm{loc},\mathrm{Iw}}^{\mathrm{BT}}$ which is representable by algebraic spaces and smooth.* (*We remind the reader that τ is now assumed to be a cuspidal inertial type.*)

Proof. Let τ' be the type τ, considered as a (principal series) type for the totally ramified extension K'/N. Let $c\in\mathrm{Gal}(K'/K)$ be the unique element which fixes $\pi^{1/(p^{2f}-1)}$ but acts nontrivially on N. For any map $\alpha: X\to Y$ of $\mathcal{O}_N$-modules we write α^c for the twist $1\otimes\alpha:\mathcal{O}_N\otimes_{\mathcal{O}_N,c}X\to\mathcal{O}_N\otimes_{\mathcal{O}_N,c}Y$.

We may think of an object $(\mathfrak{L},\mathfrak{L}^+)$ of $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}}$ as an object $(\mathfrak{L}',(\mathfrak{L}')^+)$ of $\mathcal{M}_{\mathrm{loc}}^{\tau',\mathrm{BT}}$ equipped with the additional data of an isomorphism of $\mathcal{O}_{K'}\otimes_{\mathbf{Z}_p}A$-modules $\theta:\mathcal{O}_{K'}\otimes_{\mathcal{O}_{K'},c}\mathfrak{L}'\to\mathfrak{L}'$ which is compatible with $(\mathfrak{L}')^+$, which satisfies $\theta\circ\theta^c=\mathrm{id}$, and which is compatible with the action of $\mathrm{Gal}(K'/N)=I(K'/N)$ in the sense that $\theta\circ(1\otimes g)=g^{p^f}\circ\theta$.

Employing the isomorphism of Proposition 3.7.7, we think of $(\mathfrak{L}',(\mathfrak{L}')^+)$ as a tuple $(\mathfrak{L}_1',(\mathfrak{L}_1')^+,\mathfrak{L}_2',(\mathfrak{L}_2')^+,f_1,f_2)$, where the $\mathfrak{L}_i',(\mathfrak{L}_i')^+$ are $\mathcal{O}_N\otimes_{\mathbf{Z}_p}A$-modules; by construction, the map θ induces isomorphisms $\theta_1:\mathcal{O}_N\otimes_{\mathcal{O}_N,c}\mathfrak{L}_1'\xrightarrow{\sim}\mathfrak{L}_2'$, $\theta_2:\mathcal{O}_N\otimes_{\mathcal{O}_N,c}\mathfrak{L}_2'\xrightarrow{\sim}\mathfrak{L}_1'$, which are compatible with $(\mathfrak{L}_1')^+,(\mathfrak{L}_2')^+$ and f_1,f_2, and satisfy $\theta_1\circ\theta_2^c=\mathrm{id}$.

Choose for each embedding $\sigma:k\hookrightarrow\mathbf{F}$ an extension to an embedding $\sigma^{(1)}:k'\hookrightarrow\mathbf{F}$, set $e_1=\sum_\sigma e_{\sigma^{(1)}}$, and write $e_2=1-e_1$. Then the map θ_1 induces isomorphisms $\theta_{11}:e_1\mathfrak{L}_1'\xrightarrow{\sim}e_2\mathfrak{L}_2'$ and $\theta_{12}:e_2\mathfrak{L}_1'\xrightarrow{\sim}e_1\mathfrak{L}_2'$, while θ_2 induces isomorphisms $\theta_{21}:e_1\mathfrak{L}_2'\xrightarrow{\sim}e_2\mathfrak{L}_1'$ and $\theta_{22}:e_2\mathfrak{L}_2'\xrightarrow{\sim}e_1\mathfrak{L}_1'$. The condition that $\theta_1\circ\theta_2^c=\mathrm{id}$ translates to $\theta_{22}=\theta_{11}^{-1}$ and $\theta_{21}=\theta_{12}^{-1}$, and compatibility with $(\mathfrak{L}_1')^+,(\mathfrak{L}_2')^+$ implies that θ_{11},θ_{21} induce isomorphisms $e_1(\mathfrak{L}_1')^+\xrightarrow{\sim}e_2(\mathfrak{L}_2')^+$ and $e_1(\mathfrak{L}_2')^+\xrightarrow{\sim}e_2(\mathfrak{L}_1')^+$ respectively.

Furthermore f_1,f_2 induce maps $e_1f_1:e_1\mathfrak{L}_1'\to e_1\mathfrak{L}_2'$, $e_1g:e_1\mathfrak{L}_2'\to e_1\mathfrak{L}_1'$. It follows that there is a map $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}}\to\mathcal{M}_{\mathrm{loc},\mathrm{Iw}}$, sending $(\mathfrak{L},\mathfrak{L}^+)$ to the tuple $(e_1\mathfrak{L}_1',e_1(\mathfrak{L}_1')^+,e_1\mathfrak{L}_2',e_1(\mathfrak{L}_2')^+,e_1f_1,e_1f_2)$. To see that it respects the strong determinant condition, one has to check that the conditions on $\mathfrak{L}_1^+$, $\mathfrak{L}_2^+$ imply those for $e_1(\mathfrak{L}'_1)^+$, $e_2(\mathfrak{L}'_2)^+$ coincide, and this follows from the definitions (via Remark 3.5.4). We therefore obtain a map $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}} \to \mathcal{M}_{\mathrm{loc},\mathrm{Iw}}^{\mathrm{BT}}$.

Since this morphism is given by forgetting the data of $e_2\mathfrak{L}'_1, e_2\mathfrak{L}'_2$ and the pair of isomorphisms θ_{11}, θ_{21}, it is evidently formally smooth. It is also a morphism between ϖ-adic formal algebraic stacks that are locally of finite presentation, and so is representable by algebraic spaces (by [Eme, Lem. 7.10]) and locally of finite presentation (by [EG19b, Cor. 2.1.8] and [Sta13, Tag 06CX]). Thus this morphism is in fact smooth. ☐

3.8. **Local models: local geometry.** We now deduce our main results on the local structure of our moduli stacks from results in the literature on local models for Shimura varieties.

Proposition 3.8.1. *We can identify $\mathcal{M}_{\mathrm{loc},\mathrm{Iw}}^{\mathrm{BT}}$ with the quotient of* (*the p-adic formal completion of*) *the naive local model for* $\mathrm{Res}_{K/\mathbf{Q}_p}\mathrm{GL}_2$ (*as defined in* [PRS13, §2.4]) *by a smooth group scheme over $\mathcal{O}$.*

Proof. Let $\widetilde{\mathcal{M}}_{\mathrm{loc},\mathrm{Iw}}^{\mathrm{BT}}$ be the p-adic formal completion of the naive local model for $\mathrm{Res}_{K/\mathbf{Q}_p}\mathrm{GL}_2$ corresponding to a standard lattice chain $\mathcal{L}$, as defined in [PRS13, §2.4]. By [RZ96, Prop. A.4], the automorphisms of the standard lattice chain $\mathcal{L}$ are represented by a smooth group scheme $\mathcal{P}_{\mathcal{L}}$ over $\mathcal{O}$. (This is in fact a parahoric subgroup scheme of $\mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}_p}\mathrm{GL}_2$, and in particular it is affine.) Also by *loc. cit.*, every lattice chain of type ($\mathcal{L}$) is Zariski locally isomorphic to $\mathcal{L}$. By comparing the two moduli problems, we see that $\widetilde{\mathcal{M}}_{\mathrm{loc},\mathrm{Iw}}^{\mathrm{BT}}$ is a $\mathcal{P}_{\mathcal{L}}$-torsor over $\mathcal{M}_{\mathrm{loc},\mathrm{Iw}}^{\mathrm{BT}}$ for the Zariski topology and the proposition follows. ☐

The following theorem describes the various regularity properties of local models. Since we are working in the context of formal algebraic stacks, we use the terminology developed in [Eme, §8] (see in particular [Eme, Rem. 8.21] and [Eme, Def. 8.35]).

Theorem 3.8.2. *Suppose that $d = 2$ and that τ is a tame inertial type. Then*

(1) $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}}$ is residually Jacobson and analytically normal, and Cohen–Macaulay.
(2) The special fibre $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT},1}$ is reduced.
(3) $\mathcal{M}_{\mathrm{loc}}^{\tau,\mathrm{BT}}$ is flat over $\mathcal{O}$.

Proof. For scalar types, this follows from [Kis09, Prop. 2.2.2] by Lemma 3.7.2, and so we turn to studying the case of a non-scalar type. The properties in question can be checked smooth locally (see [Eme, §8] for (1), and [Sta13, Tag 04YH] for (2); for (3), note that morphisms that are representable by algebraic spaces and smooth are also flat, and take into account the final statement of [Eme, Lem. 8.34]), and so by Propositions 3.7.7 and 3.7.10 we reduce to checking the assertions of the theorem for $\mathcal{M}_{\mathrm{loc},\mathrm{Iw}}^{\mathrm{BT}}$. Proposition 3.8.1 then reduces us to checking these assertions for the ϖ-adic completion of the naive local model at Iwahori level for $\mathrm{Res}_{K/\mathbf{Q}_p}\mathrm{GL}_2$.

Since this naive local model is a scheme of finite presentation over $\mathcal{O}$, its special fibre (i.e. its base-change to $\mathbf{F}$) is Jacobson, and it is excellent; thus its ϖ-adic completion satisfies the properties of (1) if and only if the naive local model itself is normal and Cohen–Macaulay. The special fibre of its ϖ-adic completion is of course just equal to its own special fibre, and so verifying (2) for the first of these special fibres is equivalent to verifying it for the second. Finally, the ϖ-adic completion of an $\mathcal{O}$-flat Noetherian ring is again $\mathcal{O}$-flat, and so the ϖ-adic completion of the naive local model will be $\mathcal{O}$-flat if the naive local model itself is.

All the properties of the naive local model that are described in the preceding paragraph, other than the Cohen–Macaulay property, are contained in [PRS13, Thm. 2.17], if we can identify the naive local models at Iwahori level with the vertical local models at Iwahori level, in the sense of [PR03, §8] (see also the discussion above *loc. cit.* in [PRS13]).

The vertical local models are obtained by intersecting the preimages at Iwahori level of the flat local models at hyperspecial level. Since the naive local models for the Weil restriction of GL_2 at hyperspecial level are already flat by a special case of [PR03, Cor. 4.3], the naive local models at Iwahori level are identified with the vertical ones and [PRS13, Thm. 2.17] applies to them directly. (To be precise, the results of [PR03] apply to restrictions of scalars $\mathrm{Res}_{F/F_0}\mathrm{GL}_2$ with F/F_0 totally ramified. However, thanks to the decomposition $\mathcal{O}_K\otimes_{\mathbf{Z}_p}A\cong\oplus_{\sigma:W(k)\hookrightarrow\mathcal{O}}\mathcal{O}_K\otimes_{W(k),\sigma}A$ the local model for $\mathrm{Res}_{K/\mathbf{Q}_p}\mathrm{GL}_2$ decomposes as a product of local models for totally ramified extensions.)

Finally, Cohen–Macaulayness can be proved as in [Gör01, Prop. 4.24] and the discussion immediately following. We thank U. Görtz for explaining this argument to us. As in the previous paragraph we reduce to the case of local models at Iwahori level for $\mathrm{Res}_{F/F_0}\mathrm{GL}_2$ with F/F_0 totally ramified of degree e. In this setting the admissible set M for the coweight $\mu=(e,0)$ has precisely one element of length 0 and two elements of each length between 1 and e. Moreover, for elements $x,y\in M$ we have $x<y$ in the Bruhat order if and only if $\ell(x)<\ell(y)$. One checks easily that M is e-Cohen–Macaulay in the sense of [Gör01, Def. 4.23], and we conclude by [Gör01, Prop. 4.24]. Alternatively, this also follows from the much more general recent results of Haines–Richarz [HR19]. ☐

Corollary 3.8.3. *Suppose that $d=2$ and that τ is a tame inertial type. Then*

(1) $\mathcal{C}^{\tau,\mathrm{BT}}$ is analytically normal, and Cohen–Macaulay.
(2) The special fibre $\mathcal{C}^{\tau,\mathrm{BT},1}$ is reduced.
(3) $\mathcal{C}^{\tau,\mathrm{BT}}$ is flat over $\mathcal{O}$.

Proof. This follows from Theorems 3.4.5 and 3.8.2, since all of these properties can be verified smooth locally (as was already noted in the proof of the second of these theorems). ☐

Remark 3.8.4. There is another structural result about vertical local models that is proved in [PR03, §8] but which we haven't incorporated into our preceding results, namely, that the irreducible components of the special fibre are normal. Since the notion of irreducible component is not étale local (and so in particular not smooth local), this statement does not imply the corresponding statement for the special fibres of $\mathcal{M}^{\tau,\mathrm{BT}}_{\mathrm{loc}}$ or $\mathcal{C}^{\tau,\mathrm{BT}}$. Rather, it implies the weaker, and somewhat more technical, statement that each of the analytic branches passing through each closed point of the special fibre of these ϖ-adic formal algebraic stacks is normal. We won't discuss this further here, since we don't need this result.

3.9. **Scheme-theoretic images.** We continue to fix $d=2$, $h=1$, and we set $K'=L(\pi^{1/p^{2f-1}})$, where L/K is the unramified quadratic extension, and π is a uniformiser of K. (This is the choice of K' that we made before in the cuspidal case, and contains the choice of K' that we made in the principal series case; since the category of Breuil–Kisin modules with descent data for the smaller extension is by Proposition 3.6.1 naturally a full subcategory of the category of Breuil–Kisin modules with descent data for the larger extension, we can consider both principal series and cuspidal types in this setting.)

Definition 3.9.1. For each $a \geq 1$ we write $\mathcal{Z}^{\mathrm{dd},a}$ and $\mathcal{Z}^{\tau,a}$ for the scheme-theoretic images (in the sense of [EG19b, Defn. 3.2.8]) of the morphisms $\mathcal{C}^{\mathrm{dd,BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ and $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ respectively. We write $\overline{\mathcal{Z}}$, $\overline{\mathcal{Z}}^{\tau}$ for $\mathcal{Z}^1, \mathcal{Z}^{\tau,1}$ respectively.

The following theorem records some basic properties of these scheme-theoretic images. We refer to Appendix B for the notion of representations admitting a potentially Barsotti–Tate lift of a given type, and for the definition of très ramifiée representations.

Theorem 3.9.2. *(1) For each $a \geq 1$, $\mathcal{Z}^{\mathrm{dd},a}$ is an algebraic stack of finite presentation over $\mathcal{O}/\varpi^a$, and is a closed substack of $\mathcal{R}^{\mathrm{dd},a}$. In turn, each $\mathcal{Z}^{\tau,a}$ is a closed substack of $\mathcal{Z}^{\mathrm{dd},a}$, and thus in particular is an algebraic stack of finite presentation over $\mathcal{O}/\varpi^a$; and $\mathcal{Z}^{\mathrm{dd},a}$ is the union of the $\mathcal{Z}^{\tau,a}$.*

(2) The morphism $\mathcal{C}^{\mathrm{dd,BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through a morphism $\mathcal{C}^{\mathrm{dd,BT},a} \to \mathcal{Z}^{\mathrm{dd},a}$ which is representable by algebraic spaces, scheme-theoretically dominant, and proper. Similarly, the morphism $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through a morphism $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{Z}^{\tau,a}$ which is representable by algebraic spaces, scheme-theoretically dominant, and proper.

(3) The $\overline{\mathbf{F}}_p$-points of $\overline{\mathcal{Z}}$ are naturally in bijection with the continuous representations $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which are not a twist of a très ramifiée extension of the trivial character by the mod p cyclotomic character. Similarly, the $\overline{\mathbf{F}}_p$-points of $\overline{\mathcal{Z}}^{\tau}$ are naturally in bijection with the continuous representations $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which have a potentially Barsotti–Tate lift of type τ.

Proof. Part (1) follows easily from Theorem 3.1.12. Indeed, by [EG19b, Prop. 3.2.31] we may think of $\mathcal{Z}^{\mathrm{dd},a}$ as the scheme-theoretic image of the proper morphism of algebraic stacks $\mathcal{C}^{\mathrm{dd,BT},a} \to \mathcal{R}_1^{\mathrm{dd},a}$ and similarly for each $\mathcal{Z}^{\tau,a}$. The existence of the factorisations in (2) is then formal.

By [EG19b, Lem. 3.2.14], for each finite extension $\mathbf{F}'/\mathbf{F}$, the $\mathbf{F}'$-points of $\overline{\mathcal{Z}}$ (respectively $\overline{\mathcal{Z}}^{\tau}$) correspond to the étale φ-modules with descent data of the form $\mathfrak{M}[1/u]$, where $\mathfrak{M}$ is a Breuil–Kisin module of rank 2 with descent data and $\mathbf{F}$-coefficients which satisfies the strong determinant condition (respectively, which satisfies the strong determinant condition and is of type τ). By Lemma 3.5.16 and Corollary 3.8.3, these precisely correspond to the Galois representations $\overline{r} : G_K \to \mathrm{GL}_2(\mathbf{F})$ which admit potentially Barsotti–Tate lifts of some tame type (respectively, of type τ). The result follows from Lemma B.5. ∎

The thickenings $\mathcal{C}^{\mathrm{dd,BT},a} \hookrightarrow \mathcal{C}^{\mathrm{dd,BT},a+1}$ and $\mathcal{R}^{\mathrm{dd},a} \hookrightarrow \mathcal{R}^{\mathrm{dd},a+1}$ induce closed immersions $\mathcal{Z}^{\mathrm{dd},a} \hookrightarrow \mathcal{Z}^{\mathrm{dd},a+1}$. Similarly, the thickenings $\mathcal{C}^{\tau,\mathrm{BT},a} \hookrightarrow \mathcal{C}^{\tau,\mathrm{BT},a+1}$ give rise to closed immersions $\mathcal{Z}^{\tau,a} \hookrightarrow \mathcal{Z}^{\tau,a+1}$.

Lemma 3.9.3. *Fix $a \geq 1$. Then the morphism $\mathcal{Z}^{\mathrm{dd},a} \hookrightarrow \mathcal{Z}^{\mathrm{dd},a+1}$ is a thickening, and for each tame type τ, the morphism $\mathcal{Z}^{\tau,a} \hookrightarrow \mathcal{Z}^{\tau,a+1}$ is a thickening.*

Proof. In each case, the claim of the lemma follows from the following more general statement: if

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{Y}' \end{tikzcd}$$

is a diagram of morphisms of algebraic stacks in which the upper horizontal arrow is a thickening, the lower horizontal arrow is a closed immersion, and each of the vertical arrows is representable by algebraic spaces, quasi-compact, and scheme-theoretically dominant, then the lower horizontal arrow is also a thickening.

Since the property of being a thickening may be checked smooth locally, and since scheme-theoretic dominance of quasi-compact morphisms is preserved by flat base-change, we may show this after pulling the entire diagram back over a smooth surjective morphism $V' \to \mathcal{Y}$ whose source is a scheme, and thus reduce to the case in which the lower arrow is a morphism of schemes, and the upper arrow is a morphism of algebraic spaces. A surjecive étale morphism is also scheme-theoretically dominant, and so pulling back the top arrow over a surjective étale morphism $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ whose source is a scheme, we finally reduce to considering a diagram of morphisms of schemes

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U' \arrow[d] \\ V \arrow[r] & V' \end{tikzcd}$$

in which the top arrow is a thickening, the vertical arrows are quasi-compact and scheme-theoretically dominant, and the bottom arrow is a closed immersion.

Pulling back over an affine open subscheme of V', and then pulling back the top arrow over the disjoint union of the members of a finite affine open cover of the preimage of this affine open in U' (note that this preimage is quasi-compact), we further reduce to the case when all the schemes involved are affine. That is, we have a diagram of ring morphisms

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

in which the vertical arrows are injective, the horizontal arrows are surjective, and the bottom arrow has nilpotent kernel. One immediately verifies that the top arrow has nilpotent kernel as well. $\square$

We write $\mathcal{C}^{\mathrm{dd,BT}} := \varinjlim_a \mathcal{C}^{\mathrm{dd,BT},a}$ and $\mathcal{Z}^{\mathrm{dd}} := \varinjlim_a \mathcal{Z}^{\mathrm{dd},a}$; we then have evident morphisms of Ind-algebraic stacks

$$\mathcal{C}^{\mathrm{dd,BT}} \to \mathcal{Z}^{\mathrm{dd}} \to \mathcal{R}^{\mathrm{dd}}$$

lying over $\operatorname{Spf} \mathcal{O}$, both representable by algebraic spaces, with the first being furthermore proper and scheme-theoretically dominant in the sense of [Eme, Def. 6.13], and the second being a closed immersion.

Similarly, for each choice of tame type τ, we set $\mathcal{C}^{\tau,\mathrm{BT}} = \varinjlim_a \mathcal{C}^{\tau,a}$, and $\mathcal{Z}^{\tau} := \varinjlim_a \mathcal{Z}^{\tau,a}$. We again have morphisms

$$\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{Z}^{\tau} \to \mathcal{R}^{\mathrm{dd}}$$

of Ind-algebraic stacks over $\operatorname{Spf}\mathcal{O}$, both being representable by algebraic spaces, the first being proper and scheme-theoretically dominant, and the second being a closed immersion. Note that by Corollary 3.5.13, $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ is the disjoint union of the $\mathcal{C}^{\tau,\mathrm{BT}}$, so it follows that $\mathcal{Z}^{\mathrm{dd},\mathrm{BT}}$ is the union (but *not* the disjoint union) of the $\mathcal{Z}^{\tau}$.

Proposition 3.5.7 shows that $\mathcal{C}^{\tau,\mathrm{BT}}$ is a ϖ-adic formal algebraic stack of finite presentation over $\operatorname{Spf}\mathcal{O}$. Each $\mathcal{Z}^{\tau,a}$ is an algebraic stack of finite presentation over $\operatorname{Spec}\mathcal{O}/\varpi^a$ by Theorem 3.9.2. Analogous remarks apply in the case of $\mathcal{C}^{\mathrm{dd},\mathrm{BT}}$ and $\mathcal{Z}^{\mathrm{dd}}$.

Proposition 3.9.4. *$\mathcal{Z}^{\mathrm{dd}}$, and each $\mathcal{Z}^{\tau}$, are ϖ-adic formal algebraic stacks, of finite presentation over* $\operatorname{Spf}\mathcal{O}$.

Proof. We give the argument for $\mathcal{Z}^{\mathrm{dd}}$, the argument for $\mathcal{Z}^{\tau}$ being identical. (Alternatively, this latter case follows from the former and the fact that the canonical morphism $\mathcal{Z}^{\tau} \hookrightarrow \mathcal{Z}^{\mathrm{dd}}$ is a closed immersion.) That $\mathcal{Z}^{\mathrm{dd}}$ is a ϖ-adic formal algebraic stack will follow from Proposition A.6 once we show that the morphism $\mathcal{C}^{\mathrm{dd},\mathrm{BT}} \to \mathcal{Z}^{\mathrm{dd}}$ fits into the framework of Section A.5. It follows from Lemma A.3 that $\mathcal{Z}^{\mathrm{dd}}$ is a formal algebraic stack, and by construction it is locally Ind-finite type over $\operatorname{Spec}\mathcal{O}$. Furthermore, since each $\mathcal{Z}^{\mathrm{dd},a}$ is quasi-compact and quasi-separated (being of finite presentation over $\mathcal{O}/\varpi^a$), we see that $\mathcal{Z}^{\mathrm{dd}}$ is quasi-compact and quasi-separated. Thus Proposition A.6 indeed applies.

The isomorphism $\mathcal{Z}^{\mathrm{dd}} \xrightarrow{\sim} \varinjlim_a \mathcal{Z}^{\mathrm{dd},a}$ induces an isomorphism

$$\mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^b \xrightarrow{\sim} \varinjlim_a \mathcal{Z}^{\mathrm{dd},a} \times_{\mathcal{O}} \mathcal{O}/\varpi^b,$$

for any fixed $b \geq 1$. Since $\mathcal{Z}^{\mathrm{dd}}$ is quasi-compact and quasi-separated, so is $\mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$, and thus this isomorphism factors through $\mathcal{Z}^{\mathrm{dd},a} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$ for some a. Thus the directed system $\mathcal{Z}^{\mathrm{dd},a} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$ in a eventually stabilises, and so we see that

$$\mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^b \xrightarrow{\sim} \mathcal{Z}^{\mathrm{dd},a} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$$

for sufficiently large values of a. Since $\mathcal{Z}^{\mathrm{dd},a}$ is of finite presentation over $\mathcal{O}/\varpi^a$, we find that $\mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^b$ is of finite presentation over $\mathcal{O}/\varpi^b$. Consequently, we conclude that $\mathcal{Z}^{\mathrm{dd}}$ is of finite presentation over $\operatorname{Spf}\mathcal{O}$, as claimed. $\square$

Remark 3.9.5. As observed in the general context of Subsection A.5, the thickening $\mathcal{Z}^{\mathrm{dd},a} \hookrightarrow \mathcal{Z}^{\mathrm{dd}} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ need not be an isomorphism *a priori*, and we have no reason to expect that it is. Nevertheless, in Subsection 5.1 we will prove that this thickening is *generically* an isomorphism for *every* value of $a \geq 1$, and we will furthermore show that each $(\mathcal{Z}^{\mathrm{dd},a})_{/\mathbf{F}}$ is generically reduced; see Proposition 5.1.2 and Remark 5.1.4 below. The proof of this result involves an application of Proposition A.11, and depends on the detailed analysis of the irreducible components of the algebraic stacks $\mathcal{C}^{\tau,a}$ and $\mathcal{Z}^{\tau,a}$ that we will make in Section 4.

We conclude this subsection by establishing some basic lemmas about the reduced substacks underlying each of $\mathcal{C}^{\tau,\mathrm{BT}}$ and $\mathcal{Z}^{\tau}$.

Lemma 3.9.6. *Let $\mathcal{X}$ be an algebraic stack over $\mathcal{O}/\varpi^a$, and let $\mathcal{X}_{\mathrm{red}}$ be the underlying reduced substack of $\mathcal{X}$. Then $\mathcal{X}_{\mathrm{red}}$ is a closed substack of $\mathcal{X}_{/\mathbf{F}} := \mathcal{X} \times_{\mathcal{O}/\varpi^a} \mathbf{F}$.*

Proof. The structural morphism $\mathcal{X} \to \operatorname{Spec} \mathcal{O}/\varpi^a$ induces a natural morphism $\mathcal{X}_{\mathrm{red}} \to (\operatorname{Spec} \mathcal{O}/\varpi^a)_{\mathrm{red}} = \operatorname{Spec} \mathbf{F}$, so the natural morphism $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ factors through $\mathcal{X}_{/\mathbf{F}}$. Since the morphisms $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ and $\mathcal{X}_{/\mathbf{F}} \to \mathcal{X}$ are both closed immersions, so is the morphism $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}_{/\mathbf{F}}$. $\square$

Lemma 3.9.7. *If $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-compact morphism of algebraic stacks, and $\mathcal{W}$ is the scheme-theoretic image of f, then the scheme-theoretic image of the induced morphism of underlying reduced substacks $f_{\mathrm{red}} : \mathcal{X}_{\mathrm{red}} \to \mathcal{Y}_{\mathrm{red}}$ is the underlying reduced substack $\mathcal{W}_{\mathrm{red}}$.*

Proof. Since the definitions of the scheme-theoretic image and of the underlying reduced substack are both smooth local (in the former case see [EG19b, Rem. 3.1.5(3)], and in the latter case it follows immediately from the construction in [Sta13, Tag 0509]), we immediately reduce to the case of schemes, which follows from [Sta13, Tag 056B]. $\square$

Lemma 3.9.8. *For each $a \geq 1$, $\mathcal{C}^{\mathrm{dd,BT},1}$ is the underlying reduced substack of $\mathcal{C}^{\mathrm{dd,BT},a}$, and $\mathcal{Z}^{\mathrm{dd},1}$ is the underlying reduced substack of $\mathcal{Z}^{\mathrm{dd},a}$; consequently, $\mathcal{C}^{\mathrm{dd,BT},1}$ is the underlying reduced substack of $\mathcal{C}^{\mathrm{dd,BT}}$, and $\mathcal{Z}^{\mathrm{dd},1}$ is the underlying reduced substack of $\mathcal{Z}^{\mathrm{dd}}$. Similarly, for each tame type τ, $\mathcal{C}^{\tau,\mathrm{BT},1}$ is the underlying reduced substack of each $\mathcal{C}^{\tau,\mathrm{BT},a}$, and of $\mathcal{C}^{\tau,\mathrm{BT}}$, while $\mathcal{Z}^{\tau,1}$ is the underlying reduced substack of each $\mathcal{Z}^{\tau,a}$, and of $\mathcal{Z}^{\tau}$.*

Proof. The statements for the ϖ-adic formal algebraic stacks follow directly from the corresponding statements for the various algebraic stacks modulo ϖ^a, and so we focus on proving these latter statements, beginning with the case of $\mathcal{C}^{\tau,\mathrm{BT},a}$. Note that $\mathcal{C}^{\tau,\mathrm{BT},1} = \mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{O}/\varpi^a} \mathbf{F}$ is reduced by Corollary 3.8.3, so $\mathcal{C}^{\mathrm{dd,BT},1} = \mathcal{C}^{\mathrm{dd,BT},a} \times_{\mathcal{O}/\varpi^a} \mathbf{F}$ is also reduced by Corollary 3.5.13. The claim follows for $\mathcal{C}^{\mathrm{dd,BT},a}$ and $\mathcal{C}^{\tau,\mathrm{BT},a}$ from Lemma 3.9.6.

The claims for $\mathcal{Z}^{\tau,a}$ and $\mathcal{Z}^{\mathrm{dd},a}$ are then immediate from Lemma 3.9.7, applied to the morphisms $\mathcal{C}^{\tau,\mathrm{BT},a} \to \mathcal{Z}^{\tau,a}$ and $\mathcal{C}^{\mathrm{dd,BT},a} \to \mathcal{Z}^{\mathrm{dd},a}$. $\square$

3.10. **Versal rings and equidimensionality.** We now show that $\mathcal{C}^{\mathrm{dd,BT}}$ and $\mathcal{Z}^{\mathrm{dd,BT}}$ (and their substacks $\mathcal{C}^{\tau,\mathrm{BT}}$, $\mathcal{Z}^{\tau}$) are equidimensional, and compute their dimensions, by making use of their versal rings. In [EG19b, §5] these versal rings were constructed in a more general setting in terms of liftings of étale φ-modules; in our particular setting, we will find it convenient to interpret them as Galois deformation rings.

Fix a finite type point $x : \operatorname{Spec} \mathbf{F}' \to \mathcal{Z}^{\tau,a}$, where $\mathbf{F}'/\mathbf{F}$ is a finite extension; we also denote the induced finite type point of $\mathcal{R}^{\mathrm{dd},a}$ by x. Let $\overline{r} : G_K \to \mathrm{GL}_2(\mathbf{F}')$ be the Galois representation corresponding to x by Theorem 3.9.2 (3). Let E' be the compositum of E and $W(\mathbf{F}')[1/p]$, with ring of integers $\mathcal{O}_{E'}$ and residue field $\mathbf{F}'$.

As in Appendix C, we have the universal framed deformation $\mathcal{O}_{E'}$-algebra $R^{\square}_{\overline{r}}$, and we let $R^{\square}_{\overline{r},0,\tau}$ be the reduced and p-torsion free quotient of $R^{\square}_{\overline{r}}$ whose $\overline{\mathbf{Q}}_p$-points correspond to the potentially Barsotti–Tate lifts of $\overline{r}$ of type τ. In this section we will denote $R^{\square}_{\overline{r},0,\tau}$ by the more suggestive name $R^{\tau,\mathrm{BT}}_{\overline{r}}$. We recall, for instance from [BG19, Thm. 3.3.8], that the ring $R^{\tau,\mathrm{BT}}_{\overline{r}}[1/p]$ is regular.

As in Section 2.3, we write $R_{\overline{r}|_{G_{K_\infty}}}$ for the universal framed deformation $\mathcal{O}_{E'}$-algebra for $\overline{r}|_{G_{K_\infty}}$. By Lemma 2.3.3, we have a natural morphism

$$\operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \to \mathcal{R}^{\mathrm{dd}}. \tag{3.10.1}$$

Lemma 3.10.2. *The morphism* (3.10.1) *is versal* (*at* x).

Proof. By definition, it suffices to show that if $\rho : G_{K_\infty} \to \mathrm{GL}_d(A)$ is a representation with A a finite Artinian $\mathcal{O}_{E'}$-algebra, and if $\rho_B : G_{K_\infty} \to \mathrm{GL}_d(B)$ is a second representation, with B a finite Artinian $\mathcal{O}_{E'}$-algebra admitting a surjection onto A, such that the base change ρ_A of ρ_B to A is isomorphic to ρ (more concretely, so that there exists $M \in \mathrm{GL}_d(A)$ with $\rho = M\rho_A M^{-1}$), then we may find $\rho' : G_{K_\infty} \to \mathrm{GL}_d(B)$ which lifts ρ, and is isomorphic to ρ_B. This is straightforward: the natural morphism $\mathrm{GL}_d(B) \to \mathrm{GL}_d(A)$ is surjective, and so if M' is any lift of M to an element of $\mathrm{GL}_d(B)$, then we may set $\rho' = M'\rho_B(M')^{-1}$. $\square$

Definition 3.10.3. For any pro-Artinian $\mathcal{O}_{E'}$-algebra R with residue field $\mathbf{F}'$ we let $\widehat{\mathrm{GL}}_{2/R}$ denote the completion of $(\mathrm{GL}_2)_{/R}$ along the closed subgroup of its special fibre given by the centraliser of $\overline{r}|_{G_{K_\infty}}$.

Remark 3.10.4. For R as above we have $\widehat{\mathrm{GL}}_{2/R} = \mathrm{Spf}\, R \times_{\mathcal{O}_{E'}} \widehat{\mathrm{GL}}_{2/\mathcal{O}_{E'}}$. Indeed, if $R \cong \varprojlim_i A_i$, then $\mathrm{Spf}\, R \times_{\mathcal{O}_{E'}} \widehat{\mathrm{GL}}_{2/\mathcal{O}_{E'}} \cong \varinjlim_i \mathrm{Spec}\, A_i \times_{\mathcal{O}_{E'}} \widehat{\mathrm{GL}}_{2/\mathcal{O}_{E'}}$, and $\mathrm{Spec}\, A_i \times_{\mathcal{O}_{E'}} \widehat{\mathrm{GL}}_{2/\mathcal{O}_{E'}}$ agrees with the completion of $(\mathrm{GL}_2)_{/A_i}$ because A_i is a finite $\mathcal{O}_{E'}$-module.

It follows from this that $\widehat{\mathrm{GL}}_{2/R}$ has nice base-change properties more generally: if $R \to S$ is a morphism of pro-Artinian $\mathcal{O}_{E'}$-algebras each with residue field $\mathbf{F}'$, then there is an isomorphism $\widehat{\mathrm{GL}}_{2/S} \cong \mathrm{Spf}\, S \times_{\mathrm{Spf}\, R} \widehat{\mathrm{GL}}_{2/R}$. We apply this fact without further comment in various arguments below.

There is a pair of morphisms $\widehat{\mathrm{GL}}_{2/R_{\overline{r}|_{G_{K_\infty}}}} \rightrightarrows \mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}$, the first being simply the projection to $\mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}$, and the second being given by "change of framing". Composing such changes of framing endows $\widehat{\mathrm{GL}}_{2/R_{\overline{r}|_{G_{K_\infty}}}}$ with the structure of a groupoid over $\mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}$. Note that the two morphisms

$$\widehat{\mathrm{GL}}_{2/R_{\overline{r}|_{G_{K_\infty}}}} \rightrightarrows \mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}} \xrightarrow{(3.10.1)} \mathcal{R}^{\mathrm{dd}}$$

coincide, since changing the framing does not change the isomorphism class (as a Galois representation) of a deformation of $\overline{r}|_{G_{K_\infty}}$. Thus there is an induced morphism of groupoids over $\mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}$

$$\widehat{\mathrm{GL}}_{2/R_{\overline{r}|_{G_{K_\infty}}}} \to \mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}} \times_{\mathcal{R}^{\mathrm{dd}}} \mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}. \tag{3.10.5}$$

Lemma 3.10.6. *The morphism* (3.10.5) *is an isomorphism.*

Proof. If A is an Artinian $\mathcal{O}_{E'}$-algebra, with residue field $\mathbf{F}'$, then a pair of A-valued points of $\mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}$ map to the same point of $\mathcal{R}^{\mathrm{dd}}$ if and only if they give rise to isomorphic deformations of $\overline{\rho}$, once we forget the framings. But this precisely means that the second of them is obtained from the first by changing the framing via an A-valued point of $\widehat{\mathrm{GL}}_2$. $\square$

It follows from Lemma 3.10.2 that, for each $a \geq 1$, the quotient $R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$ is a (non-Noetherian) versal ring for $\mathcal{R}^{\mathrm{dd},a}$ at x. By [EG19b, Lem. 3.2.16], for each $a \geq 1$ a versal ring for $\mathcal{Z}^{\tau,a}$ at x is given by the scheme-theoretic image of the morphism

$$\mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a \to \mathrm{Spf}\, R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a, \tag{3.10.7}$$

in the sense that we now explain.

In general, the notion of scheme-theoretic image for morphisms of formal algebraic stacks can be problematic; at the very least it should be handled with care. But in this particular context, a definition is given in [EG19b, Def. 3.2.15]: we write $R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$ as an inverse limit of Artinian local rings A, form the corresponding scheme-theoretic images of the induced morphisms $\mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spec} A \to \operatorname{Spec} A$, and then take the inductive limit of these scheme-theoretic images; this is a formal scheme, which is in fact of the form $\operatorname{Spf} R^{\tau,a}$ for some quotient $R^{\tau,a}$ of $R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$ (where *quotient* should be understood in the sense of topological rings), and is by definition the scheme-theoretic image in question.

The closed immersions $\mathcal{C}^{\tau,\mathrm{BT},a} \hookrightarrow \mathcal{C}^{\tau,\mathrm{BT},a+1}$ induce corresponding closed immersions

$$\mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a \to \mathcal{C}^{\tau,\mathrm{BT},a+1} \times_{\mathcal{R}^{\mathrm{dd},a+1}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}/\varpi^{a+1},$$

and hence closed immersions of scheme-theoretic images $\operatorname{Spf} R^{\tau,a} \to \operatorname{Spf} R^{\tau,a+1}$, corresponding to surjections $R^{\tau,a+1} \to R^{\tau,a}$. (Here we are using the fact that an projective limit of surjections of finite Artin rings is surjective.) Thus we may form the pro-Artinian ring $\varprojlim_a R^{\tau,a}$. This projective limit is a quotient (again in the sense of topological rings) of $R_{\overline{r}|_{G_{K_\infty}}}$, and the closed formal subscheme $\operatorname{Spf}(\varprojlim R^{\tau,a})$ of $\operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$ is the scheme-theoretic image (computed in the sense described above) of the projection

$$\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \to \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \tag{3.10.8}$$

(This is a formal consequence of the construction of the $\operatorname{Spf} R^{\tau,a}$ as scheme-theoretic images, since any discrete Artinian quotient of $R_{\overline{r}|_{G_{K_\infty}}}$ is a discrete Artinian quotient of $R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$, for some $a \geq 1$.) It also follows formally (for example, by the same argument as in the proof of [EG19b, Lem. 4.2.14]) that $\varprojlim R^{\tau,a}$ is a versal ring to $\mathcal{Z}^\tau$ at x. Our next aim is to identify this projective limit with $R_{\overline{r}}^{\tau,\mathrm{BT}}$.

Before we do this, we have to establish some preliminary facts related to the various objects and morphisms we have just introduced.

Lemma 3.10.9.

(1) Each of the rings $R^{\tau,a}$ is a complete local Noetherian ring, endowed with its $\mathfrak{m}$-adic topology, and the same is true of the inverse limit $\varprojlim_a R^{\tau,a}$.

(2) For each $a \geq 1$, the morphism $\operatorname{Spf} R^{\tau,a} \to \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$ induces an isomorphism

$$\mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spf} R^{\tau,a} \xrightarrow{\sim} \mathcal{C}^{\tau,\mathrm{BT},a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a.$$

(3) For each $a \geq 1$, the morphism $\operatorname{Spf} R^{\tau,a} \to \mathcal{R}^{\mathrm{dd},a}$ is effective, i.e. may be promoted (in a unique manner) to a morphism $\operatorname{Spec} R^{\tau,a} \to \mathcal{R}^{\mathrm{dd},a}$, and the induced morphism

$$\mathcal{C}^{\tau,a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spec} R^{\tau,a} \to \operatorname{Spec} R^{\tau,a}$$

is proper and scheme-theoretically dominant.

(4) Each transition morphism $\operatorname{Spec} R^{\tau,a} \hookrightarrow \operatorname{Spec} R^{\tau,a+1}$ is a thickening.

Proof. Recall that in Section 2.3.4 we defined a Noetherian quotient $R^{\leq 1}_{\overline{r}|_{G_{K_\infty}}}$ of $R_{\overline{r}|_{G_{K_\infty}}}$, which is naturally identified with the framed deformation ring $R^{[0,1]}_{\overline{r}}$ by

Proposition 2.3.6. It follows from [EG19b, Lem. 5.4.15] (via an argument almost identical to the one in the proof of [EG19b, Prop. 5.4.17]) that the morphism $\operatorname{Spf} \varprojlim R^{\tau,a} \hookrightarrow \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$ factors through $\operatorname{Spf} R^{\leq 1}_{\overline{r}|_{G_{K_\infty}}} = \operatorname{Spf} R^{[0,1]}_{\overline{r}}$, and indeed that $\varprojlim R^{\tau,a}$ is a quotient of $\operatorname{Spf} R^{[0,1]}_{\overline{r}}$; this proves (1).

It follows by the very construction of the $R^{\tau,a}$ that the morphism (3.10.7) factors through the closed subscheme $\operatorname{Spf} R^{\tau,a}$ of $\operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}/\varpi^a$. The claim of (2) is a formal consequence of this.

We have already observed that the morphism $\operatorname{Spf} R^{\tau,a} \to \mathcal{R}^{\mathrm{dd},a}$ factors through $\mathcal{Z}^{\tau,a}$. This latter stack is algebraic, and of finite type over $\mathcal{O}/\varpi^a$. It follows from [Sta13, Tag 07X8] that the morphism $\operatorname{Spf} R^{\tau,a} \to \mathcal{Z}^{\tau,a}$ is effective. Taking into account part (1) of the present lemma, we deduce from the theorem on formal functions that the formal completion of the scheme-theoretic image of the projection

$$\mathcal{C}^{\tau,a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spec} R^{\tau,a} \to \operatorname{Spec} R^{\tau,a}$$

at the closed point of $\operatorname{Spec} R^{\tau,a}$ coincides with the scheme-theoretic image of the morphism

$$\mathcal{C}^{\tau,a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spf} R^{\tau,a} \to \operatorname{Spf} R^{\tau,a}.$$

Taking into account (2), we see that this latter scheme-theoretic image coincides with $\operatorname{Spf} R^{\tau,a}$ itself. This completes the proof of (3).

The claim of (4) follows from a consideration of the diagram

$$\begin{tikzcd}
\mathcal{C}^{\tau,a} \times_{\mathcal{R}^{\mathrm{dd},a}} \operatorname{Spec} R^{\tau,a} \arrow[r] \arrow[d] & \mathcal{C}^{\tau,a+1} \times_{\mathcal{R}^{\mathrm{dd},a+1}} \operatorname{Spec} R^{\tau,a+1} \arrow[d] \\
\operatorname{Spec} R^{\tau,a} \arrow[r] & \operatorname{Spec} R^{\tau,a+1}
\end{tikzcd}$$

just as in the proof of Lemma 3.9.3. ☐

Lemma 3.10.10.

(1) The projection $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \to \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$ factors through a morphism $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \to \operatorname{Spf}(\varprojlim R^{\tau,a})$, which is scheme-theoretically dominant in the sense that its scheme-theoretic image (computed in the manner described above) is equal to its target.

(2) There is a projective morphism of schemes $X_{\overline{r}} \to \operatorname{Spec}(\varprojlim R^{\tau,a})$, which is uniquely determined, up to unique isomorphism, by the requirement that its $\mathfrak{m}$-adic completion (where $\mathfrak{m}$ denotes the maximal ideal of $\varprojlim R^{\tau,a}$) may be identified with the morphism $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \to \operatorname{Spf}(\varprojlim R^{\tau,a})$ of (1).

Proof. Part (1) follows formally from the various constructions and definitions of the objects involved (just like part (2) of Lemma 3.10.9).

We now consider the morphism

$$\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \to \operatorname{Spf}(\varprojlim R^{\tau,a}).$$

Once we recall that $\varprojlim R^{\tau,a}$ is Noetherian, by Lemma 3.10.9 (1), it follows exactly as in the proof of [Kis09, Prop. 2.1.10] (which treats the case that τ is the trivial type), via an application of formal GAGA [Gro61, Thm. 5.4.5], that this morphism arises as the formal completion along the maximal ideal of $\varprojlim R^{\tau,a}$ of a projective morphism

$X_{\overline{r}} \to \operatorname{Spec}(\varprojlim R^{\tau,a})$ (and $X_{\overline{r}}$ is unique up to unique isomorphism, by [Gro61, Thm. 5.4.1]). $\square$

We next establish various properties of the scheme $X_{\overline{r}}$ constructed in the previous lemma. To ease notation going forward, we write $\widehat{X}_{\overline{r}}$ to denote the fibre product $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}}$ (which is reasonable, since this fibre product is isomorphic to the formal completion of $X_{\overline{r}}$).

Lemma 3.10.11. *The scheme $X_{\overline{r}}$ is Noetherian, normal, and flat over $\mathcal{O}_{E'}$.*

Proof. Since $X_{\overline{r}}$ is projective over the Noetherian ring $\varprojlim R^{\tau,a}$, it is Noetherian. The other claimed properties of $X_{\overline{r}}$ will be deduced from the corresponding properties of $\mathcal{C}^{\tau,\mathrm{BT}}$ that are proved in Corollary 3.8.3.

To this end, we first note that, since the morphism $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{R}^{\mathrm{dd}}$ factors through $\mathcal{Z}^\tau$, it follows (for example as in the proof of [EG19b, Lem. 3.2.16]) that we have isomorphisms

$$\begin{aligned}\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^\tau} \operatorname{Spf}(\varprojlim R^{\tau,a}) &\xrightarrow{\sim} \mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^\tau} \mathcal{Z}^\tau \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} \\ &\xrightarrow{\sim} \mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}|_{G_{K_\infty}}} =: \widehat{X}_{\overline{r}}.\end{aligned}$$

In summary, we may identify $\widehat{X}_{\overline{r}}$ with the fibre product $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^\tau} \operatorname{Spf}(\varprojlim R^{\tau,a})$.

We now show that $\widehat{X}_{\overline{r}}$ is analytically normal. To see this, let $\operatorname{Spf} B \to \widehat{X}_{\overline{r}}$ be a morphism whose source is a Noetherian affine formal algebraic space, which is representable by algebraic spaces and smooth. We must show that the completion $\widehat{B}_{\mathfrak{n}}$ is normal, for each maximal ideal $\mathfrak{n}$ of B. In fact, it suffices to verify this for some collection of such $\operatorname{Spf} B$ which cover $\widehat{X}_{\overline{r}}$, and so without loss of generality we may choose our B as follows: first, choose a collection of morphisms $\operatorname{Spf} A \to \mathcal{C}^{\tau,\mathrm{BT}}$ whose sources are Noetherian affine formal algebraic spaces, and which are representable by algebraic spaces and smooth, which, taken together, cover $\mathcal{C}^{\tau,\mathrm{BT}}$. Next, for each such A, choose a collection of morphisms

$$\operatorname{Spf} B \to \operatorname{Spf}_A \times_{\mathcal{C}^{\tau,\mathrm{BT}}} \widehat{X}_{\overline{r}}$$

whose sources are Noetherian affine formal algebraic spaces, and which are representable by algebraic spaces and smooth, which, taken together, cover the fibre product. Altogether (considering all such B associated to all such A), the composite morphisms

$$\operatorname{Spf} B \to \operatorname{Spf}_A \times_{\mathcal{C}^{\tau,\mathrm{BT}}} \widehat{X}_{\overline{r}} \to \widehat{X}_{\overline{r}}$$

are representable by algebraic spaces and smooth, and cover $\widehat{X}_{\overline{r}}$.

Now, let $\mathfrak{n}$ be a maximal ideal in one of these rings B, lying over a maximal ideal $\mathfrak{m}$ in the corresponding ring A. The extension of residue fields $A/\mathfrak{m} \to B/\mathfrak{n}$ is finite, and each of these fields is finite over $\mathbf{F}'$. Enlarging $\mathbf{F}'$ sufficiently, we may assume that in fact each of these residue fields coincides with $\mathbf{F}'$. (On the level of rings, this amounts to forming various tensor products of the form $-\otimes_{W(\mathbf{F}')} W(\mathbf{F}'')$, which doesn't affect the question of normality.) The morphism $\operatorname{Spf} B_{\mathfrak{n}} \to \operatorname{Spf} A_{\mathfrak{n}}$ is then seen to be smooth in the sense of [Sta13, Tag 06HG], i.e., it satisfies the infinitesimal lifting property for finite Artinian $\mathcal{O}'$-algebras with residue field $\mathbf{F}'$: this follows from the identification of $\widehat{X}_{\overline{r}}$ above as a fibre product, and the fact that $\operatorname{Spf}(\varprojlim R^{\tau,a}) \to \mathcal{Z}^\tau$ is versal at the closed point x. Thus $\operatorname{Spf} B_{\mathfrak{n}}$ is a formal power series ring over $\operatorname{Spf} A_{\mathfrak{m}}$, by [Sta13, Tag 06HL], and hence $\operatorname{Spf} B_{\mathfrak{n}}$ is indeed normal, since $\mathrm{Spf}\, A_{\mathfrak{m}}$ is so, by Corollary 3.8.3. By Lemma 3.10.14 below, this implies that the algebraization $X_{\overline{r}}$ of $\widehat{X}_{\overline{r}}$ is normal.

We next claim that the morphism

$$\mathrm{Spf}(\varprojlim R^{\tau,a}) \to \mathcal{Z}^{\tau} \tag{3.10.12}$$

is a flat morphism of formal algebraic stacks, in the sense of [Eme, Def. 8.35]. Given this, we find that the base-changed morphism $\widehat{X}_{\overline{r}} \to \mathcal{C}^{\tau,\mathrm{BT}}$ is also flat. Since Corollary 3.8.3 shows that $\mathcal{C}^{\tau,\mathrm{BT}}$ is flat over $\mathcal{O}_{E'}$, we conclude that the same is true of $\widehat{X}_{\overline{r}}$. Again, by Lemma 3.10.14, this implies that the algebraization $X_{\overline{r}}$ is also flat over $\mathcal{O}_{E'}$.

It remains to show the claimed flatness. To this end, we note first that for each $a \geq 1$, the morphism

$$\mathrm{Spf}\, R^{\tau,a} \to \mathcal{Z}^{\tau,a} \tag{3.10.13}$$

is a versal morphism from a complete Noetherian local ring to an algebraic stack which is locally of finite type over $\mathcal{O}/\varpi^a$. We already observed in the proof of Lemma 3.10.9 (3) that (3.10.13) is effective, i.e. can be promoted to a morphism $\mathrm{Spec}\, R^{\tau,a} \to \mathcal{Z}^{\tau,a}$. It then follows from [Sta13, Tag 0DR2] that this latter morphism is flat, and thus that (3.10.13) is flat in the sense of [Eme, Def. 8.35]. It follows easily that the morphism (3.10.12) is also flat: use the fact that a morphism of ϖ-adically complete local Noetherian $\mathcal{O}$-algebras which becomes flat upon reduction modulo ϖ^a, for each $a \geq 1$, is itself flat, which follows from (for example) [Sta13, Tag 0523]. □

The following lemma is standard, and is presumably well-known. We sketch the proof, since we don't know a reference.

Lemma 3.10.14. *If S is a complete Noetherian local $\mathcal{O}$-algebra and $Y \to \mathrm{Spec}\, S$ is a proper morphism of schemes, then Y is flat over $\mathrm{Spec}\, \mathcal{O}$ (resp. normal) if and only $\widehat{Y}$ (the $\mathfrak{m}_S$-adic completion of Y) is flat over $\mathrm{Spf}\, \mathcal{O}$ (resp. is analytically normal).*

Proof. The properties of Y that are in question can be tested by considering the various local rings $\mathcal{O}_{Y,y}$, as y runs over the points of Y; namely, we have to consider whether or not these rings are flat over $\mathcal{O}$, or normal. Since any point y specializes to a closed point y_0 of Y, so that $\mathcal{O}_{Y,y}$ is a localization of $\mathcal{O}_{Y,y_0}$, and thus $\mathcal{O}$-flat (resp. normal) if $\mathcal{O}_{Y,y_0}$ is, it suffices to consider the rings $\mathcal{O}_{Y,y_0}$ for closed points y_0 of Y. Note also that since Y is proper over $\mathrm{Spec}\, S$, any closed point of Y lies over the closed point of $\mathrm{Spec}\, S$.

Now let $\mathrm{Spec}\, A$ be an affine neighbourhood of a closed point y_0 of Y; let $\mathfrak{m}$ be the corresponding maximal ideal of A. As we noted, $\mathfrak{m}$ lies over $\mathfrak{m}_S$, and so gives rise to a maximal ideal $\widehat{\mathfrak{m}} := \mathfrak{m}\widehat{A}$ of $\widehat{A}$, the $\mathfrak{m}_S$-adic completion of A; and any maximal ideal of $\widehat{A}$ contains $\mathfrak{m}_S\widehat{A}$, and so arises from a maximal ideal of A in this manner (since $A/\mathfrak{m}_S \xrightarrow{\sim} \widehat{A}/\mathfrak{m}_S$). Write $\widehat{A}_{\mathfrak{m}}$ to denote the $\mathfrak{m}$-adic completion of A (which maps isomorphically to the $\widehat{\mathfrak{m}}$-adic completion of $\widehat{A}$). Then $\widehat{A}$ is faithfully flat over the localization $A_{\mathfrak{m}} = \mathcal{O}_{Y,y_0}$, and hence $A_{\mathfrak{m}}$ is flat over $\mathcal{O}$ if and only if $\widehat{A}_{\mathfrak{m}}$ is. Consequently we see that Y is flat over $\mathcal{O}$ if and only if, for each affine open subset $\mathrm{Spec}\, A$ of Y, the corresponding $\mathfrak{m}_S$-adic completion $\widehat{A}$ becomes flat over $\mathcal{O}$ after completing at each of its maximal ideals. Another application of faithful flatness of completions of Noetherian local rings shows that this holds if and only if each such

$\widehat{A}$ is flat over $\mathcal{O}$ after localizing at each of its maximal ideals, which holds if and only each such $\widehat{A}$ is flat over $\mathcal{O}$. This is precisely what it means for $\widehat{Y}$ to be flat over $\mathcal{O}$.

The proof that analytic normality of $\widehat{Y}$ implies that Y is normal is similar. Indeed, analytic normality by definition means that the completion of $\widehat{A}$ at each of its maximal ideals is normal. This completion is faithfully flat over the localization of $\operatorname{Spec} A$ at its corresponding maximal ideal, and so [Sta13, Tag 033G] implies that this localization is also normal. The discussion of the first paragraph then implies that Y is normal. For the converse direction, we have to deduce normality of the completions $\widehat{A}_{\mathfrak{m}}$ from the normality of the corresponding localizations $A_{\mathfrak{m}}$. This follows from that fact that Y is an excellent scheme (being of finite type over the complete local ring S), so that each A is an excellent ring [Sta13, Tag 0C23]. ☐

Proposition 3.10.15. *The projective morphism $X_{\overline{r}} \to \operatorname{Spec} R_{\overline{r}}^{[0,1]}$ factors through a projective and scheme-theoretically dominant morphism*

$$X_{\overline{r}} \to \operatorname{Spec} R_{\overline{r}}^{\tau,\mathrm{BT}} \tag{3.10.16}$$

which becomes an isomorphism after inverting ϖ.

Proof. We begin by showing the existence of (3.10.16), and that it induces a bijection on closed points after inverting ϖ. Since $X_{\overline{r}}$ is $\mathcal{O}$-flat, by Lemma 3.10.11, it suffices to show that the induced morphism

$$\operatorname{Spec} E \times_{\mathcal{O}} X_{\overline{r}} \to \operatorname{Spec} R_{\overline{r}}^{[0,1]}[1/\varpi]$$

factors through a morphism

$$\operatorname{Spec} E \times_{\mathcal{O}} X_{\overline{r}} \to \operatorname{Spec} R_{\overline{r}}^{\tau,\mathrm{BT}}[1/\varpi], \tag{3.10.17}$$

which induces a bijection on closed points.

This can be proved in exactly the same way as [Kis09, Prop. 2.4.8], which treats the case that τ is trivial. Indeed, the computation of the D_{cris} of a Galois representation in the proof of [Kis09, Prop. 2.4.8] goes over essentially unchanged to the case of a Galois representation coming from $\mathcal{C}^{\tau,\mathrm{BT}}$, and finite type points of $\operatorname{Spec} R_{\overline{r}}^{\tau,\mathrm{BT}}[1/\varpi]$ yield p-divisible groups and thus Breuil–Kisin modules exactly as in the proof of [Kis09, Prop. 2.4.8] (bearing in mind Lemma 3.5.16 above). The tame descent data comes along for the ride.

The morphism (3.10.17) is a projective morphism whose target is Jacobson, and which induces a bijection on closed points. It is thus proper and quasi-finite, and hence finite. Its source is reduced (being even normal, by Lemma 3.10.11), and its target is normal (as it is even regular, as we noted above). A finite morphism whose source is reduced, whose target is normal and Noetherian, and which induces a bijection on finite type points, is indeed an isomorphism. (The connected components of a normal scheme are integral, and so base-changing over the connected components of the target, we may assume that the target is integral. The source is a union of finitely many irreducible components, each of which has closed image in the target. Since the morphism is surjective on finite type points, it is surjective, and thus one of these closed images coincides with the target. The injectivity on finite type points then shows that the source is also irreducible, and thus integral, as it is reduced. It follows from [Sta13, Tag 0AB1] that the morphism is an isomorphism.) Thus (3.10.17) is an isomorphism. Finally, since $R_{\overline{r}}^{\tau,\mathrm{BT}}$ is also flat over $\mathcal{O}$ (by its definition), this implies that (3.10.16) is scheme-theoretically dominant. ☐

Corollary 3.10.18. $\varprojlim R^{\tau,a} = R_{\overline{r}}^{\tau,\mathrm{BT}}$*; thus* $R_{\overline{r}}^{\tau,\mathrm{BT}}$ *is a versal ring to* $\mathcal{Z}^\tau$ *at* x*.*

Proof. The theorem on formal functions shows that if we write the scheme-theoretic image of (3.10.16) in the form $\operatorname{Spec} B$, for some quotient B of $R_{\overline{r}_{|G_{K_\infty}}}$, then the scheme-theoretic image of the morphism (3.10.8) coincides with $\operatorname{Spf} B$. The corollary then follows from Proposition 3.10.15, which shows that (3.10.16) is scheme-theoretically dominant. □

Proposition 3.10.19. *The algebraic stacks* $\mathcal{Z}^{\mathrm{dd},a}$ *and* $\mathcal{Z}^{\tau,a}$ *are equidimensional of dimension* $[K:\mathbf{Q}_p]$*.*

Proof. Let x be a finite type point of $\mathcal{Z}^{\tau,a}$, defined over some finite extension $\mathbf{F}'$ of $\mathbf{F}$, and corresponding to a Galois representation $\overline{r}$ with coefficients in $\mathbf{F}'$. By Corollary 3.10.18 the ring $R_{\overline{r}}^{\tau,\mathrm{BT}}$ coincides with the versal ring $\varprojlim_a R^{\tau,a}$ at x of the ϖ-adic formal algebraic stack $\mathcal{Z}^\tau$, and so $\operatorname{Spf} R^{\tau,a} \xrightarrow{\sim} \operatorname{Spf} R_{\overline{r}}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^\tau} \mathcal{Z}^{\tau,a}$. Since $\mathcal{Z}^\tau$ is a ϖ-adic formal algebraic stack, the natural morphism $\mathcal{Z}^{\tau,1} \to \mathcal{Z}^\tau \times_{\operatorname{Spf}\mathcal{O}} \mathbf{F}$ is a thickening, and thus the same is true of the morphism $\operatorname{Spf} R^{\tau,1} \to \operatorname{Spf} R_{\overline{r}}^{\tau,\mathrm{BT}}/\varpi$ obtained by pulling the former morphism back over $\operatorname{Spf} R_{\overline{r}}^{\tau,\mathrm{BT}}/\varpi$.

Since $R_{\overline{r}}^{\tau,\mathrm{BT}}$ is flat over $\mathcal{O}_{E'}$ and equidimensional of dimension $5+[K:\mathbf{Q}_p]$, it follows that $R^{\tau,1}$ is equidimensional of dimension $4 + [K:\mathbf{Q}_p]$. The same is then true of each $R^{\tau,a}$, since these are thickenings of $R^{\tau,1}$, by Lemma 3.10.9 (4).

We have a versal morphism $\operatorname{Spf} R^{\tau,a} \to \mathcal{Z}^{\tau,a}$ at the finite type point x of $\mathcal{Z}^{\tau,a}$. It follows from Lemma 3.10.6 that

$$\widehat{\mathrm{GL}_2}_{/\operatorname{Spf} R^{\tau,a}} \xrightarrow{\sim} \operatorname{Spf} R^{\tau,a} \times_{\mathcal{Z}^{\tau,a}} \operatorname{Spf} R^{\tau,a}.$$

To find the dimension of $\mathcal{Z}^{\tau,a}$ it suffices to compute its dimension at finite type points (*cf.* [Sta13, Tag 0DRX], recalling the definition of the dimension of an algebraic stack, [Sta13, Tag 0AFP]). It follows from [EG17, Lem. 2.40] applied to the presentation $[\operatorname{Spf} R^{\tau,a}/\widehat{\mathrm{GL}_2}_{/\operatorname{Spf} R^{\tau,a}}]$ of $\widehat{\mathcal{Z}}_x^{\tau,a}$, together with Remark 3.10.4, that $\mathcal{Z}^{\tau,a}$ is equidimensional of dimension $[K:\mathbf{Q}_p]$. Since $\mathcal{Z}^{\mathrm{dd},a}$ is the union of the $\mathcal{Z}^{\tau,a}$ by Theorem 3.9.2, $\mathcal{Z}^{\mathrm{dd},a}$ is also equidimensional of dimension $[K:\mathbf{Q}_p]$ by [Sta13, Tag 0DRZ]. □

Proposition 3.10.20. *The algebraic stacks* $\mathcal{C}^{\tau,\mathrm{BT},a}$ *are equidimensional of dimension* $[K:\mathbf{Q}_p]$*.*

Proof. Let x' be a finite type point of $\mathcal{C}^{\tau,\mathrm{BT},a}$, defined over some finite extension $\mathbf{F}'$ of $\mathbf{F}$, lying over the finite type point x of $\mathcal{Z}^{\tau,a}$. Let $\overline{r}$ be the Galois representation with coefficients in $\mathbf{F}'$ corresponding to x, and recall that $X_{\overline{r}}$ denotes a projective $\operatorname{Spec} R_{\overline{r}}^{\tau,\mathrm{BT}}$-scheme whose pull-back $\widehat{X}_{\overline{r}}$ over $\operatorname{Spf} R_{\overline{r}_{|G_{K_\infty}}}$ is isomorphic to $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{R}^{\mathrm{dd}}} \operatorname{Spf} R_{\overline{r}_{|G_{K_\infty}}}$. The point x' gives rise to a closed point $\tilde{x}$ of $X_{\overline{r}}$ (of which x' is the image under the morphism $X_{\overline{r}} \to \mathcal{C}^{\tau,\mathrm{BT}}$). Let $\widehat{\mathcal{O}}_{X_{\overline{r}},\tilde{x}}$ denote the complete local ring to $X_{\overline{r}}$ at the point $\tilde{x}$; then the natural morphism $\operatorname{Spf}\widehat{\mathcal{O}}_{X_{\overline{r}},\tilde{x}} \to \mathcal{C}^{\tau,\mathrm{BT}}$ is versal at $\tilde{x}$, so that $\widehat{\mathcal{O}}_{X_{\overline{r}},\tilde{x}}/\varpi^a$ is a versal ring for the point x' of $\mathcal{C}^{\tau,\mathrm{BT},a}$.

The isomorphism (3.10.5) induces (after pulling back over $\mathcal{C}^{\tau,\mathrm{BT}}$) an isomorphism

$$\widehat{\mathrm{GL}_2}_{/\widehat{X}_{\overline{r}}} \xrightarrow{\sim} \widehat{X}_{\overline{r}} \times_{\mathcal{C}^{\tau,\mathrm{BT}}} \widehat{X}_{\overline{r}},$$

and thence an isomorphism

$$\widehat{\mathrm{GL}_2}_{/\widehat{\mathcal{O}}_{X_{\overline{r}},\tilde{x}}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X_{\overline{r}},\tilde{x}} \times_{\mathcal{C}^{\tau,\mathrm{BT}}} \widehat{\mathcal{O}}_{X_{\overline{r}},\tilde{x}}.$$

Since $R^{\tau,\mathrm{BT}}$ is equidimensional of dimension $5+[K:\mathbf{Q}_p]$, it follows from Proposition 3.10.15 that $X_{\overline{r}}$ is equidimensional of dimension $5+[K:\mathbf{Q}_p]$, and thus (taking into account the flatness statement of Lemma 3.10.11) that $\widehat{\mathcal{O}}_{X_{\overline{r}},\tilde{x}}/\varpi^a$ is equidimensional of dimension $4+[K:\mathbf{Q}_p]$. As in the proof of Proposition 3.10.19, an application of [EG17, Lem. 2.40] shows that $\dim_{x'}\mathcal{C}^{\tau,\mathrm{BT},a}$ is equal to $[K:\mathbf{Q}_p]$. Since x' was an arbitrary finite type point, the result follows. $\square$

3.11. **The Dieudonné stack.** We now specialise the choice of K' in the following way. Choose a tame inertial type $\tau=\eta\oplus\eta'$. Fix a uniformiser π of K. If τ is a tame principal series type, we take $K'=K(\pi^{1/(p^f-1)})$, while if τ is a tame cuspidal type, we let L be an unramified quadratic extension of K, and set $K'=L(\pi^{1/(p^{2f}-1)})$. Let N be the maximal unramified extension of K in K'. In either case K'/K is a Galois extension; in the principal series case, we have $e'=(p^f-1)e$, $f'=f$, and in the cuspidal case we have $e'=(p^{2f}-1)e$, $f'=2f$. We refer to this choice of extension as the *standard choice* (for the fixed type τ and uniformiser π).

For the rest of this section we assume that $\eta\neq\eta'$ (we will not need to consider Dieudonné modules for scalar types).

Let $\mathfrak{M}$ be a Breuil–Kisin module with A-coefficients and descent data of type τ and height at most 1, and let $D:=\mathfrak{M}/u\mathfrak{M}$ be its corresponding Dieudonné module as in Definition 2.2.1. If we write $D_i:=e_iD$, then this Dieudonné module is given by rank two projective modules D_j over A ($j=0,\dots,f'-1$) with linear maps $F:D_j\to D_{j+1}$ and $V:D_j\to D_{j-1}$ (subscripts understood modulo f') such that $FV=VF=p$.

Now, $I(K'/K)$ is abelian of order prime to p, so we can write $D=D_\eta\oplus D_{\eta'}$, where D_η is the submodule on which $I(K'/K)$ acts via η. Since $\mathfrak{M}_\eta$ is obtained from the projective $\mathfrak{S}_A$-module $\mathfrak{M}$ by applying a projector, each $D_{\eta,j}$ is an invertible A-module, and F,V induce linear maps $F:D_{\eta,j}\to D_{\eta,j+1}$ and $V:D_{\eta,j+1}\to D_{\eta,j}$ such that $FV=VF=p$.

We can of course apply the same construction with η' in the place of η, obtaining a Dieudonné module $D_{\eta'}$. We now prove some lemmas relating these various Dieudonné modules. We will need to make use of a variant of the strong determinant condition, so we begin by discussing this and its relationship to the strong determinant condition of Subsection 3.5.

Definition 3.11.1. Let $(\mathfrak{L},\mathfrak{L}^+)$ be a pair consisting of a rank two projective $\mathcal{O}_{K'}\otimes_{\mathbf{Z}_p}A$-module $\mathfrak{L}$, and an $\mathcal{O}_{K'}\otimes_{\mathbf{Z}_p}A$-submodule $\mathfrak{L}^+\subset\mathfrak{L}$, such that Zariski locally on $\operatorname{Spec}A$, $\mathfrak{L}^+$ is a direct summand of $\mathfrak{L}$ as an A-module.

Then we say that the pair $(\mathfrak{L},\mathfrak{L}^+)$ *satisfies the Kottwitz determinant condition over* K' if for all $a\in\mathcal{O}_{K'}$, we have

$$\det{}_A(a|\mathfrak{L}^+)=\prod_{\psi:K'\hookrightarrow E}\psi(a)$$

as polynomial functions on $\mathcal{O}_{K'}$ in the sense of [Kot92, §5].

There is a finite type stack $\mathcal{M}_{K',\det}$ over $\operatorname{Spec}\mathcal{O}$, with $\mathcal{M}_{K',\det}(\operatorname{Spec}A)$ being the groupoid of pairs $(\mathfrak{L},\mathfrak{L}^+)$ as above which satisfy the Kottwitz determinant condition over K'. As we have seen above, by a result of Pappas–Rapoport, this stack is flat over $\operatorname{Spec}\mathcal{O}$ (see [Kis09, Prop. 2.2.2]).

Lemma 3.11.2. *If A is an E-algebra, then a pair $(\mathfrak{L}, \mathfrak{L}^+)$ as in Definition 3.11.1 satisfies the Kottwitz determinant condition over K' if and only if $\mathfrak{L}^+$ is a rank one projective $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$-module.*

Proof. We may write $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A = K' \otimes_{\mathbf{Q}_p} A \cong \prod_{\psi: K' \hookrightarrow E} A$, where the embedding $\psi : K' \hookrightarrow E$ corresponds to an idempotent $e_\psi \in K' \otimes_{\mathbf{Q}_p} A$. Decomposing $\mathfrak{L}^+$ as $\oplus_\psi e_\psi \mathfrak{L}^+$, the left-hand side of the Kottwitz determinant condition becomes $\prod_\psi \det_A(a|e_\psi \mathfrak{L}^+) = \prod_\psi \psi(a)^{\mathrm{rk}_A e_\psi \mathfrak{L}^+}$. It follows that the Kottwitz determinant condition is satisfied if and only if the projective A-module $e_\psi \mathfrak{L}^+$ has rank one for all ψ, which is equivalent to $\mathfrak{L}^+$ being a rank one projective $K' \otimes_{\mathbf{Q}_p} A$-module, as required. $\square$

Proposition 3.11.3. *If $\mathfrak{M}$ is an object of $\mathcal{C}^{\tau,\mathrm{BT}}(A)$, then the pair*

$$(\mathfrak{M}/E(u)\mathfrak{M}, \operatorname{im} \Phi_{\mathfrak{M}}/E(u)\mathfrak{M})$$

satisfies the Kottwitz determinant condition for K'.

Proof. Let $\mathcal{C}^{\tau,\mathrm{BT}'}$ be the closed substack of $\mathcal{C}^\tau$ consisting of those $\mathfrak{M}$ for which the pair $(\mathfrak{M}/E(u)\mathfrak{M}, \operatorname{im} \Phi_{\mathfrak{M}}/E(u)\mathfrak{M})$ satisfies the Kottwitz determinant condition for K'. We need to show that $\mathcal{C}^{\tau,\mathrm{BT}}$ is a closed substack of $\mathcal{C}^{\tau,\mathrm{BT}'}$. Since $\mathcal{C}^{\tau,\mathrm{BT}}$ is flat over $\operatorname{Spf} \mathcal{O}$ by Corollary 3.8.3, it is enough to show that if A is an E-algebra, then $\mathcal{C}^{\tau,\mathrm{BT}}(A) = \mathcal{C}^{\tau,\mathrm{BT}'}(A)$.

To see this, let $\mathfrak{M}$ be an object of $\mathcal{C}^\tau(A)$. By Lemma 3.11.2, $\mathfrak{M}$ is an object of $\mathcal{C}^{\tau,\mathrm{BT}'}(A)$ if and only if $\operatorname{im} \Phi_{\mathfrak{M}}/E(u)\mathfrak{M}$ is a rank one projective $K' \otimes_{\mathbf{Q}_p} A$-module. Similarly, $\mathfrak{M}$ is an object of $\mathcal{C}^{\tau,\mathrm{BT}}(A)$ if and only if for each ξ, $(\operatorname{im} \Phi_{\mathfrak{M}})_\xi/E(u)\mathfrak{M}_\xi$ is a rank one projective $N \otimes_{\mathbf{Q}_p} A$-module. Since

$$\operatorname{im} \Phi_{\mathfrak{M}}/E(u)\mathfrak{M} = \oplus_\xi (\operatorname{im} \Phi_{\mathfrak{M}})_\xi/E(u)\mathfrak{M}_\xi,$$

the equivalence of these two conditions is clear. $\square$

Lemma 3.11.4. *If $(\mathfrak{L}, \mathfrak{L}^+)$ is an object of $\mathcal{M}_{K',\det}(A)$ (i.e. satisfies the Kottwitz determinant condition over K'), then the morphism $\bigwedge^2_{\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A} \mathfrak{L}^+ \to \bigwedge^2_{\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A} \mathfrak{L}$ induced by the inclusion $\mathfrak{L}^+ \subset \mathfrak{L}$ is identically zero.*

Remark 3.11.5. Note that, although $\mathfrak{L}^+$ need not be locally free over $\mathcal{O}_{K'} \otimes_{\mathbf{Z}_p} A$, its exterior square is nevertheless defined, so that the statement of the lemma makes sense.

Proof of Lemma 3.11.4. Since $\mathcal{M}_{K',\det}$ is $\mathcal{O}$-flat, it is enough to treat the case that A is $\mathcal{O}$-flat. In this case $\mathfrak{L}$, and thus also $\bigwedge^2 \mathfrak{L}$, are $\mathcal{O}$-flat. Given this additional assumption, it suffices to prove that the morphism of the lemma becomes zero after tensoring with $\mathbf{Q}_p$ over $\mathbf{Z}_p$. This morphism may naturally be identified with the morphism $\bigwedge^2_{K' \otimes_{\mathbf{Z}_p} A} \mathfrak{L}^+ \to \bigwedge^2_{K' \otimes_{\mathbf{Z}_p} A} \mathfrak{L}$ induced by the injection $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathfrak{L}^+ \hookrightarrow \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathfrak{L}$. Locally on $\operatorname{Spec} A$, this is the embedding of a free $K' \otimes_{\mathbf{Z}_p} A$-module of rank one as a direct summand of a free $K' \otimes_{\mathbf{Z}_p} A$-module of rank two. Thus $\bigwedge^2$ of the source in fact vanishes, and hence so does $\bigwedge^2$ of the embedding. $\square$

Lemma 3.11.6. *If $\mathfrak{M}$ is an object of $\mathcal{C}^{\tau,\mathrm{BT}}(A)$, then $\bigwedge^2 \Phi_{\mathfrak{M}} : \bigwedge^2 \varphi^* \mathfrak{M} \hookrightarrow \bigwedge^2 \mathfrak{M}$ is exactly divisible by $E(u)$, i.e. can be written as $E(u)$ times an isomorphism of $\mathfrak{S}_A$-modules.*

Proof. It follows from Proposition 3.11.3 and Lemma 3.11.4 that the reduction of $\bigwedge^2 \Phi_{\mathfrak{M}}$ modulo $E(u)$ vanishes, so we can think of $\bigwedge^2 \Phi_{\mathfrak{M}}$ as a morphism $\bigwedge^2 \varphi^*\mathfrak{M} \to E(u) \bigwedge^2 \mathfrak{M}$. We need to show that the cokernel X of this morphism vanishes. Since $\operatorname{im} \Phi_{\mathfrak{M}} \supseteq E(u)\mathfrak{M}$, X is a finitely generated A-module, so that in order to prove that it vanishes, it is enough to prove that $X/pX = 0$.

Since the formation of cokernels is compatible with base change, this means that we can (and do) assume that A is an $\mathbf{F}$-algebra. Since the special fibre $\mathcal{C}^{\tau,\mathrm{BT}}$ is of finite type over $\mathbf{F}$, we can and do assume that A is furthermore of finite type over $\mathbf{F}$. The special fibre of $\mathcal{C}^{\tau,\mathrm{BT}}$ is reduced by Corollary 3.8.3, so we may assume that A is reduced, and it is therefore enough to prove that X vanishes modulo each maximal ideal of A. Since the residue fields at such maximal ideals are finite, we are reduced to the case that A is a finite field, when the result follows from [Kis09, Lem. 2.5.1]. □

Lemma 3.11.7. *There is a canonical isomorphism*

$$\text{“}(F \otimes F)/p\text{”} : D_{\eta,j} \otimes_A D_{\eta',j} \xrightarrow{\sim} D_{\eta,j+1} \otimes_A D_{\eta',j+1},$$

characterised by the fact that it is compatible with change of scalars, and that

$$p \cdot \text{“}(F \otimes F)/p\text{”} = F \otimes F.$$

Proof. Since $\mathcal{C}^{\mathrm{BT}}$ is flat over $\mathcal{O}$, we see that in the universal case, the formula

$$p \cdot \text{“}(F \otimes F)/p\text{”} = F \otimes F$$

uniquely determines the isomorphism “$(F \otimes F)/p$” (if it exists). Since any Breuil–Kisin module with descent data is obtained from the universal case by change of scalars, we see that the isomorphism “$(F \otimes F)/p$” is indeed characterised by the properties stated in the lemma, provided that it exists.

To check that the isomorphism exists, we can again consider the universal case, and hence assume that A is a flat $\mathcal{O}$-algebra. In this case, it suffices to check that the morphism $F \otimes F : D_{\eta,j} \otimes_A D_{\eta',j} \to D_{\eta,j+1} \otimes_A D_{\eta,j+1}$ is divisible by p, and that the formula $(F \otimes F)/p$ is indeed an isomorphism. Noting that the direct sum over $j = 0, \dots, f'-1$ of these morphisms may be identified with the reduction modulo u of the morphism $\bigwedge^2 \Phi_{\mathfrak{M}} : \bigwedge^2 \varphi^*\mathfrak{M} \to \bigwedge^2 \mathfrak{M}$, this follows from Lemma 3.11.6. □

The isomorphism “$(F \otimes F)/p$” of the preceding lemma may be rewritten as an isomorphism of invertible A-modules

$$\operatorname{Hom}_A(D_{\eta,j}, D_{\eta,j+1}) \xrightarrow{\sim} \operatorname{Hom}_A(D_{\eta',j+1}, D_{\eta',j}). \tag{3.11.8}$$

Lemma 3.11.9. *The isomorphism* (3.11.8) *takes F to V.*

Proof. The claim of the lemma is equivalent to showing that the composite

$$D_{\eta,j} \otimes_A D_{\eta',j+1} \xrightarrow{\mathrm{id} \otimes V} D_{\eta,j} \otimes_A D_{\eta',j} \xrightarrow{\text{“}(F\otimes F)/p\text{”}} D_{\eta,j+1} \otimes_A D_{\eta',j+1}$$

coincides with the morphism $F \otimes \mathrm{id}$. It suffices to check this in the universal case, and thus we may assume that p is a non-zero divisor in A, and hence verify the required identity of morphisms after multiplying each of them by p. The identity to be verified then becomes

$$(F \otimes F) \circ (\mathrm{id} \otimes V) \stackrel{?}{=} p(F \otimes \mathrm{id}),$$

which follows immediately from the formula $FV = p$. □

We now consider the moduli stacks classifying the Dieudonné modules with the properties we have just established, and the maps from the moduli stacks of Breuil–Kisin modules to these stacks.

Suppose first that we are in the principal series case. Then there is a moduli stack classifying the data of the $D_{\eta,j}$ together with the F and V, namely the stack

$$\mathcal{D}_\eta := \Big[\big(\operatorname{Spec} W(k)[X_0, Y_0, \dots, X_{f-1}, Y_{f-1}]/(X_j Y_j - p)_{j=0,\dots,f-1}\big)/\mathbf{G}_m^f\Big],$$

where the f copies of $\mathbf{G}_m$ act as follows:

$$(u_0, \dots, u_{f-1}) \cdot (X_j, Y_j) \mapsto (u_j u_{j+1}^{-1} X_j, u_{j+1} u_j^{-1} Y_j).$$

To see this, recall that the stack

$$[\mathrm{point}/\mathbf{G}_m]$$

classifies line bundles, so the f copies of $\mathbf{G}_m$ in $\mathcal{D}_\eta$ correspond to f line bundles, which are the line bundles $D_{\eta,j}$ ($j = 0, \dots, f-1$). If we locally trivialise these line bundles, then the maps $F : D_{\eta,j} \to D_{\eta,j+1}$ and $V : D_{\eta,j+1} \to D_{\eta,j}$ act by scalars, which we denote by X_j and Y_j respectively. The f copies of $\mathbf{G}_m$ are then encoding possible changes of trivialisation, by units u_j, which induce the indicated changes on the X_j's and Y_j's.

There is then a natural map

$$\mathcal{C}^\tau \to \mathcal{D}_\eta,$$

classifying the Dieudonné modules underlying the Breuil–Kisin modules with descent data.

There is a more geometric way to think about what $\mathcal{D}_\eta$ classifies. To begin with, we just rephrase what we've already indicated: it represents the functor which associates to a $W(k)$-scheme the groupoid whose objects are f-tuples of line bundles $(D_{\eta,j})_{j=0,\dots,f-1}$ equipped with morphisms $X_j : D_{\eta,j} \to D_{\eta,j+1}$ and $Y_j : D_{\eta,j+1} \to D_{\eta,j}$ such that $Y_j X_j = p$. (Morphisms in the groupoid are just isomorphisms between collections of such data.) Equivalently, we can think of this as giving the line bundle $D_{\eta,0}$, and then the f line bundles $\mathcal{D}_j := D_{\eta,j+1} \otimes D_{\eta,j}^{-1}$, equipped with sections $X_j \in \mathcal{D}_j$ and $Y_j \in \mathcal{D}_j^{-1}$ whose product in $\mathcal{D}_j \otimes \mathcal{D}_j^{-1} = \mathcal{O}$ (the trivial line bundle) is equal to the element p. Note that it superficially looks like we are remembering $f+1$ line bundles, rather than f, but this is illusory, since in fact $\mathcal{D}_0 \otimes \cdots \otimes \mathcal{D}_{f-1}$ is trivial; indeed, the isomorphism $\mathcal{D}_0 \otimes \cdots \otimes \mathcal{D}_{f-1} \xrightarrow{\sim} \mathcal{O}$ is part of the data we should remember.

It will be helpful to introduce another stack, the stack $\mathcal{G}_\eta$ of η-gauges. This classifies f-tuples of line bundles $\mathcal{D}_j$ ($j = 0, \dots, f-1$) equipped with sections $X_j \in \mathcal{D}_j$ and $Y_j \in \mathcal{D}_j^{-1}$. Explicitly, it can be written as the quotient stack

$$\mathcal{G}_\eta := \Big[\big(\operatorname{Spec} W(k)[X_0, Y_0, \dots, X_{f-1}, Y_{f-1}]/(X_j Y_j - p)_{j=0,\dots,f-1}\big)/\mathbf{G}_m^f\Big],$$

where the f copies of $\mathbf{G}_m$ act as follows:

$$(v_0, \dots, v_{f-1}) \cdot (X_j, Y_j) \mapsto (v_j X_j, v_j^{-1} Y_j).$$

There is a natural morphism of stacks $\mathcal{D}_\eta \to \mathcal{G}_\eta$ given by forgetting forgetting D_0 and the isomorphism $\mathcal{D}_0 \otimes \mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_{f-1} \xrightarrow{\sim} \mathcal{O}$. In terms of the explicit descriptions via quotient stacks, we have a morphism $\mathbf{G}_m^f \to \mathbf{G}_m^f$ given by $(u_j)_{j=0,\dots,f-1} \mapsto (u_j u_{j+1}^{-1})_{j=0,\dots,f-1}$, which is compatible with the actions of these two groups on

$\operatorname{Spec} W(k)[(X_j, Y_j)_{j=0,\dots,f-1}]/(X_jY_j - p)_{j=0,\dots,f-1}$, and we are just considering the map from the quotient by the first $\mathbf{G}_m^f$ to the quotient by the second $\mathbf{G}_m^f$.

Composing our morphism $\mathcal{C}^\tau \to \mathcal{D}_\eta$ with the forgetful morphism $\mathcal{D}_\eta \to \mathcal{G}_\eta$, we obtain a morphism $\mathcal{C}^\tau \to \mathcal{G}_\eta$.

We now turn to the case that τ is a cuspidal type. In this case our Dieudonné modules have unramified as well as inertial descent data; accordingly, we let φ^f denote the element of $\operatorname{Gal}(K'/K)$ which acts trivially on $\pi^{1/(p^{2f}-1)}$ and non-trivially on L. Then the descent data of φ^f induces isomorphisms $D_j \xrightarrow{\sim} D_{j+f}$, which are compatible with the F, V, and which identify $D_{\eta,j}$ with $D_{\eta',f+j}$.

If we choose local trivialisations of the line bundles $D_{\eta,0}, \dots, D_{\eta,f}$, then the maps $F : D_{\eta,j} \to D_{\eta,j+1}$ and $V : D_{\eta,j+1} \to D_{\eta,j}$ for $0 \le j \le f-1$ are given by scalars X_j and Y_j respectively. The identification of $D_{\eta,j}$ and $D_{\eta',f+j}$ given by φ^f identifies $D_{\eta,j} \otimes D_{\eta,j+1}^{-1}$ with $D_{\eta',f+j} \otimes D_{\eta',f+j+1}^{-1}$, which via the isomorphsim (3.11.8) is identified with $D_{\eta,f+j+1} \otimes D_{\eta,f+j}^{-1}$. It follows that for $0 \le j \le f-2$ the data of $D_{\eta,j}$, $D_{\eta,j+1}$ and $D_{\eta,f+j}$ recursively determines $D_{\eta,f+j+1}$. From Lemma 3.11.9 we see, again recursively for $0 \le j \le f-2$, that there are unique trivialisations of $D_{\eta,f+1}, \dots, D_{\eta,2f-1}$ such that $F : D_{\eta,f+j} \to D_{\eta,f+j+1}$ is given by Y_j, and $V : D_{\eta,f+j+1} \to D_{\eta,f+j}$ is given by X_j. Furthermore, there is some unit α such that $F : D_{\eta,2f-1} \to D_{\eta,0}$ is given by αY_{f-1}, and $V : D_{\eta,0} \to D_{\eta,2f-1}$ is given by $\alpha^{-1}X_{f-1}$. Note that the map $F^{2f} : D_{\eta,0} \to D_{\eta,0}$ is precisely $p^f\alpha$.

Consequently, we see that the data of the $D_{\eta,j}$ (together with the F, V) is classified by the stack

$$\mathcal{D}_\eta := \Big[\big(\operatorname{Spec} W(k)[X_0, Y_0, \dots, X_{f-1}, Y_{f-1}]/(X_jY_j - p)_{j=0,\dots,f-1}) \times \mathbf{G}_m\big)/\mathbf{G}_m^{f+1}\Big],$$

where the $f+1$ copies of $\mathbf{G}_m$ act as follows:

$$(u_0, \dots, u_{f-1}, u_f) \cdot ((X_j, Y_j), \alpha) \mapsto ((u_j u_{j+1}^{-1} X_j, u_{j+1} u_j^{-1} Y_j), \alpha).$$

We again define

$$\mathcal{G}_\eta := \Big[\big(\operatorname{Spec} W(k)[X_0, Y_0, \dots, X_{f-1}, Y_{f-1}]/(X_jY_j - p)_{j=0,\dots,f-1})\big)/\mathbf{G}_m^f\Big],$$

where the f copies of $\mathbf{G}_m$ act as

$$(v_0, \dots, v_{f-1}) \cdot (X_j, Y_j) \mapsto (v_j X_j, v_j^{-1} Y_j).$$

There are again natural morphisms of stacks $\mathcal{C}^\tau \to \mathcal{D}_\eta \to \mathcal{G}_\eta$, where the second morphism is given in terms of the explicit descriptions via quotient stacks as follows: we have a morphism $\mathbf{G}_m^{f+1} \to \mathbf{G}_m^f$ given by $(u_j)_{j=0,\dots,f} \mapsto (u_j u_{j+1}^{-1})_{j=0,\dots,f-1}$, and the morphism $\mathcal{D}_\eta \to \mathcal{G}_\eta$ is the obvious one which forgets the factor of $\mathbf{G}_m$ coming from α.

For our analysis of the irreducible components of the stacks $\mathcal{C}^{\tau,\mathrm{BT},1}$ at the end of Section 4, it will be useful to have a more directly geometric interpretation of a morphism $S \to \mathcal{G}_\eta$, in the case that the source is a *flat* $W(k)$-scheme, or, more generally, a flat p-adic formal algebraic stack over $\operatorname{Spf} W(k)$. In order to do this we will need some basic material on effective Cartier divisors for (formal) algebraic stacks; while it is presumably possible to develop this theory in considerable generality, we only need a very special case, and we limit ourselves to this setting.

The property of a closed subscheme being an effective Cartier divisor is not preserved under arbitrary pull-back, but it is preserved under flat pull-back. More precisely, we have the following result.

Lemma 3.11.10. *If X is a scheme, and Z is a closed subscheme of X, then the following are equivalent:*

(1) Z is an effective Cartier divisor on X.
(2) For any flat morphism of schemes $U \to X$, the pull-back $Z \times_X U$ is an effective Cartier divisor on U.
(3) For some fpqc covering $\{X_i \to X\}$ of X, each of the pull-backs $Z \times_X X_i$ is an effective Cartier divisor on X_i.

Proof. Since Z is an effective Cartier divisor if and only if its ideal sheaf $\mathcal{I}_Z$ is an invertible sheaf on X, this follows from the fact that the invertibility of a quasi-coherent sheaf is a local property in the *fpqc* topology. □

Lemma 3.11.11. *If A is a Noetherian adic topological ring, then pull-back under the natural morphism* $\operatorname{Spf} A \to \operatorname{Spec} A$ *induces a bijection between the closed subschemes of* $\operatorname{Spec} A$ *and the closed subspaces of* $\operatorname{Spf} A$.

Proof. It follows from [Sta13, Tag 0ANQ] that closed immersions $Z \to \operatorname{Spf} A$ are necessarily of the form $\operatorname{Spf} B \to \operatorname{Spf} A$, and correspond to continuous morphisms $A \to B$, for some complete linearly topologized ring B, which are taut (in the sense of [Sta13, Tag 0AMX]), have closed kernel, and dense image. Since A is adic, it admits a countable basis of neighbourhoods of the origin, and so it follows from [Sta13, Tag 0APT] (recalling also [Sta13, Tag 0AMV]) that $A \to B$ is surjective. Because any ideal of definition I of A is finitely generated, it follows from [Sta13, Tag 0APU] that B is endowed with the I-adic topology. Finally, since A is Noetherian, any ideal in A is I-adically closed. Thus closed immersions $\operatorname{Spf} B \to \operatorname{Spf} A$ are determined by giving the kernel of the corresponding morphism $A \to B$, which can be arbitrary. The same is true of closed immersions $\operatorname{Spec} B \to \operatorname{Spec} A$, and so the lemma follows. □

Definition 3.11.12. If A is a Noetherian adic topological ring, then we say that a closed subspace of $\operatorname{Spf} A$ is an *effective Cartier divisor* on $\operatorname{Spf} A$ if the corresponding closed subscheme of $\operatorname{Spec} A$ is an effective Cartier divisor on $\operatorname{Spec} A$.

Lemma 3.11.13. *Let* $\operatorname{Spf} B \to \operatorname{Spf} A$ *be a flat adic morphism of Noetherian affine formal algebraic spaces. If* $Z \hookrightarrow \operatorname{Spf} A$ *is a Cartier divisor, then* $Z \times_{\operatorname{Spf} A} \operatorname{Spf} B \hookrightarrow \operatorname{Spf} B$ *is a Cartier divisor. Conversely, if* $\operatorname{Spf} B \to \operatorname{Spf} A$ *is furthermore surjective, and if* $Z \hookrightarrow \operatorname{Spf} A$ *is a closed subspace for which the base-change* $Z \times_{\operatorname{Spf} A} \operatorname{Spf} B \hookrightarrow \operatorname{Spf} B$ *is a Cartier divisor, then Z is a Cartier divisor on* $\operatorname{Spf} A$.

Proof. The morphism $\operatorname{Spf} B \to \operatorname{Spf} A$ corresponds to an adic flat morphism $A \to B$ ([Sta13, Tag 0AN0] and [Eme, Lem. 8.18]) and hence is induced by a flat morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, which is furthermore faithfully flat if and only if $\operatorname{Spf} B \to \operatorname{Spf} A$ is surjective (again by [Eme, Lem. 8.18]). The present lemma thus follows from Lemma 3.11.10. □

The preceding lemma justifies the following definition.

Definition 3.11.14. We say that a closed substack $\mathcal{Z}$ of a locally Noetherian formal algebraic stack $\mathcal{X}$ is an *effective Cartier divisor* on $\mathcal{X}$ if for any morphism $U \to \mathcal{X}$ whose source is a Noetherian affine formal algebraic space, and which is representable by algebraic spaces and flat, the pull-back $\mathcal{Z} \times_{\mathcal{X}} U$ is an effective Cartier divisor on U.

We consider the $W(k)$-scheme $\operatorname{Spec} W(k)[X,Y]/(XY-p)$, which we endow with a $\mathbf{G}_m$-action via $u\cdot(X,Y) := (uX, u^{-1}Y)$. There is an obvious morphism

$$\operatorname{Spec} W(k)[X,Y]/(XY-p) \to \operatorname{Spec} W(k)[X] = \mathbf{A}^1$$

given by $(X,Y)\to X$, which is $\mathbf{G}_m$-equivariant (for the action of $\mathbf{G}_m$ on $\mathbf{A}^1$ given by $u\cdot X := uX$), and so induces a morphism

$$[\big(\operatorname{Spec} W(k)[X,Y]/(XY-p)\big)/\mathbf{G}_m] \to [\mathbf{A}^1/\mathbf{G}_m]. \tag{3.11.15}$$

Lemma 3.11.16. *If $\mathcal{X}$ is a locally Noetherian p-adic formal algebraic stack which is furthermore flat over* $\operatorname{Spf} W(k)$*, then the groupoid of morphisms*

$$\mathcal{X} \to [\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m]$$

is in fact a setoid, and is equivalent to the set of effective Cartier divisors on $\mathcal{X}$ that are contained in the effective Cartier divisor $(\operatorname{Spec} k)\times_{\operatorname{Spf} W(k)}\mathcal{X}$ *on $\mathcal{X}$.*

Proof. Essentially by definition (and taking into account [Eme, Lem. 8.18]), it suffices to prove this in the case when $\mathcal{X} = \operatorname{Spf} B$, where B is a flat Noetherian adic $W(k)$-algebra admitting (p) as an ideal of definition. In this case, the restriction map

$$[\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m](\operatorname{Spec} B) \to [\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m](\operatorname{Spf} B)$$

is an equivalence of groupoids. Indeed, the essential surjectivity follows from the (standard and easily verified) fact that if $\{M_i\}$ is a compatible family of locally free B/p^iB-modules of rank one, then $M := \varprojlim M_i$ is a locally free B-module of rank one, for which each of the natural morphisms $M/p^iM \to M_i$ is an isomorphism. The full faithfulness follows from the fact that a locally free B-module of rank one is p-adically complete, and so is recovered as the inverse limit of its compatible family of quotients $\{M/p^iM\}$.

We are therefore reduced to the same statement with $\mathcal{X} = \operatorname{Spec} B$. The composite morphism $\operatorname{Spec} B \to [\mathbf{A}^1/\mathbf{G}_m]$ induced by (3.11.15) corresponds to giving a pair $(\mathcal{D}, X)$ where $\mathcal{D}$ is a line bundle on $\operatorname{Spec} B$, and X is a global section of $\mathcal{D}^{-1}$. Indeed, giving a morphism $\operatorname{Spec} B \to [\mathbf{A}^1/\mathbf{G}_m]$ is equivalent to giving a $\mathbf{G}_m$-torsor $P \to \operatorname{Spec} B$, together with a $\mathbf{G}_m$-equivariant morphism $P \to \mathbf{A}^1$. Giving a $\mathbf{G}_m$-torsor P over $\operatorname{Spec} B$ is equivalent to giving an invertible sheaf $\mathcal{D}$ on $\operatorname{Spec} B$ (the associated $\mathbf{G}_m$-torsor is then obtained by deleting the zero section from the line bundle $D \to X$ corresponding to $\mathcal{D}$), and giving a $\mathbf{G}_m$-equivariant morphism $P \to \mathbf{A}^1$ is equivalent to giving a global section of $\mathcal{D}^{-1}$.

It follows that giving a morphism $\operatorname{Spec} B \to [\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m]$ corresponds to giving a line bundle $\mathcal{D}$ and sections $X \in \mathcal{D}^{-1}$, $Y \in \mathcal{D}$ satisfying $XY = p$. To say that B is flat over $W(k)$ is just to say that p is a regular element on B, and so we see that X (resp. Y) is a regular section of $\mathcal{D}^{-1}$ (resp. $\mathcal{D}$). Again, since p is a regular element on B, we see that Y is uniquely determined by X and the equation $XY = p$, and so giving a morphism $\operatorname{Spec} B \to [\operatorname{Spec} W(k)[X,Y]/(XY-p)/\mathbf{G}_m]$ is equivalent to giving a line bundle $\mathcal{D}$ and a regular section X of $\mathcal{D}^{-1}$, such that $pB \subset X\otimes_B \mathcal{D} \subset \mathcal{D}^{-1}\otimes_B \mathcal{D} \xrightarrow{\sim} B$; this last condition guarantees the existence of the (then uniquely determined) Y.

Now giving a line bundle $\mathcal{D}$ on $\operatorname{Spec} B$ and a regular section $X \in \mathcal{D}^{-1}$ is the same as giving the zero locus D of X, which is a Cartier divisor on $\operatorname{Spec} B$. (There is a canonical isomorphism $(\mathcal{D}, X) \cong \big(\mathcal{I}_D, 1\big)$, where $\mathcal{I}_D$ denotes the ideal sheaf of D.)

The condition that $pB \subset X \otimes_B \mathcal{D}$ is equivalent to the condition that $p \in \mathcal{I}_D$, i.e. that D be contained in $\operatorname{Spec} B/pB$, and we are done. $\square$

Lemma 3.11.17. *If $\mathcal{S}$ is a locally Noetherian p-adic formal algebraic stack which is flat over $W(k)$, then giving a morphism $\mathcal{S} \to \mathcal{G}_\eta$ over $W(k)$ is equivalent to giving a collection of effective Cartier divisors $\mathcal{D}_j$ on $\mathcal{S}$ $(j = 0, \dots, f-1)$, with each $\mathcal{D}_j$ contained in the Cartier divisor $\overline{\mathcal{S}}$ cut out by the equation $p = 0$ on $\mathcal{S}$ (i.e. the* special fibre *of $\mathcal{S}$).*

Proof. This follows immediately from Lemma 3.11.16, by the definition of $\mathcal{G}_\eta$. $\square$

4. Extensions of rank one Breuil–Kisin modules with descent data

The goal of this section is to construct certain universal families of extensions of rank one Breuil–Kisin modules over $\mathbf{F}$ with descent data, and to use these to describe the generic behaviour of the various irreducible components of the special fibres of $\mathcal{C}^{\tau,\mathrm{BT}}$ and $\mathcal{Z}^\tau$.

In Subsection 4.1 we present some generalities on extensions of Breuil–Kisin modules. In Subsection 4.3 we explain how to construct our desired families of extensions. In Subsection 4.4 we recall the fundamental computations related to extensions of rank one Breuil–Kisin modules from [DS15], to which the results of Subsection 4.3 will be applied.

We assume throughout this section that $[K' : K]$ is not divisible by p; since we are assuming throughout the paper that K'/K is tamely ramified, this is equivalent to assuming that K' does not contain an unramified extension of K of degree p. In our final applications K'/K will contain unramified extensions of degree at most 2, and p will be odd, so this assumption will be satisfied. (In fact, we specialize to such a context begining in Subsection 4.7.)

4.1. **Extensions of Breuil–Kisin modules with descent data.** When discussing the general theory of extensions of Breuil–Kisin modules, it is convenient to embed the category of Breuil–Kisin modules in a larger category which is abelian, contains enough injectives and projectives, and is closed under passing to arbitrary limits and colimits. The simplest way to obtain such a category is as the category of modules over some ring, and so we briefly recall how a Breuil–Kisin module with A-coefficients and descent data can be interpreted as a module over a certain A-algebra.

Let $\mathfrak{S}_A[F]$ denote the twisted polynomial ring over $\mathfrak{S}_A$, in which the variable F obeys the following commutation relation with respect to elements $s \in \mathfrak{S}_A$:

$$F \cdot s = \varphi(s) \cdot F.$$

Let $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]$ denote the twisted group ring over $\mathfrak{S}_A[F]$, in which the elements $g \in \mathrm{Gal}(K'/K)$ commute with F, and obey the following commutation relations with elements $s \in \mathfrak{S}_A$:

$$g \cdot s = g(s) \cdot g.$$

One immediately confirms that giving a left $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]$-module $\mathfrak{M}$ is equivalent to equipping the underlying $\mathfrak{S}_A$-module $\mathfrak{M}$ with a φ-linear morphism $\varphi : \mathfrak{M} \to \mathfrak{M}$ and a semi-linear action of $\mathrm{Gal}(K'/K)$ which commutes with φ.

In particular, if we let $\mathcal{K}(A)$ denote the category of left $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]$-modules, then a Breuil–Kisin module with descent data from K' to K may naturally be regarded as an object of $\mathcal{K}(A)$. In the following lemma, we record the fact that extensions of Breuil–Kisin modules with descent data may be computed as extensions in the category $\mathcal{K}(A)$.

Lemma 4.1.1. *If $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ is a short exact sequence in $\mathcal{K}(A)$, such that $\mathfrak{M}'$ (resp. $\mathfrak{M}''$) is a Breuil–Kisin module with descent data of rank d' and height at most h' (resp. of rank d'' and height at most h''), then $\mathfrak{M}$ is a Breuil–Kisin module with descent data of rank $d' + d''$ and height at most $h' + h''$.*

More generally, if $E(u)^h \in \mathrm{Ann}_{\mathfrak{S}_A}(\mathrm{coker}\, \Phi_{\mathfrak{M}'}) \mathrm{Ann}_{\mathfrak{S}_A}(\mathrm{coker}\, \Phi_{\mathfrak{M}''})$, then $\mathfrak{M}$ is a Breuil–Kisin module with descent data of height at most h.

Proof. Note that since $\Phi_{\mathfrak{M}'}[1/E(u)]$ and $\Phi_{\mathfrak{M}''}[1/E(u)]$ are both isomorphisms by assumption, it follows from the snake lemma that $\Phi_{\mathfrak{M}}[1/E(u)]$ is isomorphism. Similarly we have a short exact sequence of $\mathfrak{S}_A$-modules

$$0 \to \mathrm{coker}\, \Phi_{\mathfrak{M}'} \to \mathrm{coker}\, \Phi_{\mathfrak{M}} \to \mathrm{coker}\, \Phi_{\mathfrak{M}''} \to 0.$$

The claims about the height and rank of $\mathfrak{M}$ follow immediately. □

We now turn to giving an explicit description of the functors $\mathrm{Ext}^i(\mathfrak{M}, -)$ for a Breuil–Kisin module with descent data $\mathfrak{M}$.

Definition 4.1.2. Let $\mathfrak{M}$ be a Breuil–Kisin module with A-coefficients and descent data (of some height). If $\mathfrak{N}$ is any object of $\mathcal{K}(A)$, then we let $C^\bullet_{\mathfrak{M}}(\mathfrak{N})$ denote the complex

$$\mathrm{Hom}_{\mathfrak{S}_A[\mathrm{Gal}(K'/K)]}(\mathfrak{M}, \mathfrak{N}) \to \mathrm{Hom}_{\mathfrak{S}_A[\mathrm{Gal}(K'/K)]}(\varphi^*\mathfrak{M}, \mathfrak{N}),$$

with differential being given by

$$\alpha \mapsto \Phi_{\mathfrak{N}} \circ \varphi^*\alpha - \alpha \circ \Phi_{\mathfrak{M}}.$$

Also let $\Phi^*_{\mathfrak{M}}$ denote the map $C^0_{\mathfrak{M}}(\mathfrak{N}) \to C^1_{\mathfrak{M}}(\mathfrak{N})$ given by $\alpha \mapsto \alpha \circ \Phi_{\mathfrak{M}}$. When $\mathfrak{M}$ is clear from the context we will usually suppress it from the notation and write simply $C^\bullet(\mathfrak{N})$.

Each $C^i(\mathfrak{N})$ is naturally an $\mathfrak{S}^0_A$-module. The formation of $C^\bullet(\mathfrak{N})$ is evidently functorial in $\mathfrak{N}$, and is also exact in $\mathfrak{N}$, since $\mathfrak{M}$, and hence also $\varphi^*\mathfrak{M}$, is projective over $\mathfrak{S}_A$, and since $\mathrm{Gal}(K'/K)$ has prime-to-p order. Thus the cohomology functors $H^0\big(C^\bullet(-)\big)$ and $H^1\big(C^\bullet(-)\big)$ form a δ-functor on $\mathcal{K}(A)$.

Lemma 4.1.3. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, -) \cong H^0\big(C^\bullet(-)\big).$$

Proof. This is immediate. □

It follows from this lemma and a standard dimension shifting argument (or, equivalently, the theory of δ-functors) that there is an embedding of functors

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, -) \hookrightarrow H^1\big(C^\bullet(-)\big). \tag{4.1.4}$$

Lemma 4.1.5. *The embedding of functors (4.1.4) is an isomorphism.*

Proof. We first describe the embedding (4.1.4) explicitly. Suppose that

$$0 \to \mathfrak{N} \to \mathfrak{E} \to \mathfrak{M} \to 0$$

is an extension in $\mathcal{K}(A)$. Since $\mathfrak{M}$ is projective over $\mathfrak{S}_A$, and since $\mathrm{Gal}(K'/K)$ is of prime-to-p order, we split this short exact sequence over the twisted group ring

$\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$, say via some element $\sigma \in \mathrm{Hom}_{\mathfrak{S}_A[\mathrm{Gal}(K'/K)]}(\mathfrak{M}, \mathfrak{E})$. This splitting is well-defined up to the addition of an element $\alpha \in \mathrm{Hom}_{\mathfrak{S}_A[\mathrm{Gal}(K'/K)]}(\mathfrak{M}, \mathfrak{N})$.

This splitting is a homomorphism in $\mathcal{K}(A)$ if and only if the element

$$\Phi_{\mathfrak{E}} \circ \varphi^*\sigma - \sigma \circ \Phi_{\mathfrak{M}} \in \mathrm{Hom}_{\mathfrak{S}_A[\mathrm{Gal}(K'/K)]}(\varphi^*\mathfrak{M}, \mathfrak{N})$$

vanishes. If we replace σ by $\sigma + \alpha$, then this element is replaced by

$$(\Phi_{\mathfrak{E}} \circ \varphi^*\sigma - \sigma \circ \Phi_{\mathfrak{M}}) + (\Phi_{\mathfrak{N}} \circ \varphi^*\alpha - \alpha \circ \Phi_{\mathfrak{M}}).$$

Thus the coset of $\Phi_{\mathfrak{E}} \circ \varphi^*\sigma - \sigma \circ \Phi_{\mathfrak{M}}$ in $H^1\big(C^\bullet(\mathfrak{N})\big)$ is well-defined, independent of the choice of σ, and this coset is the image of the class of the extension $\mathfrak{E}$ under the embedding

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \hookrightarrow H^1\big(C^\bullet(\mathfrak{N})\big) \tag{4.1.6}$$

(up to a possible overall sign, which we ignore, since it doesn't affect the claim of the lemma).

Now, given any element $\nu \in \mathrm{Hom}_{\mathfrak{S}_A[\mathrm{Gal}(K'/K)]}(\varphi^*\mathfrak{M}, \mathfrak{N})$, we may give the $\mathfrak{S}_A[\mathrm{Gal}(K'/K)]$-module $\mathfrak{E} := \mathfrak{N} \oplus \mathfrak{M}$ the structure of a $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]$-module as follows: we need to define a φ-linear morphism $\mathfrak{E} \to \mathfrak{E}$, or equivalently a linear morphism $\Phi_{\mathfrak{E}} : \varphi^*\mathfrak{E} \to \mathfrak{E}$. We do this by setting

$$\Phi_{\mathfrak{E}} := \begin{pmatrix} \Phi_{\mathfrak{N}} & \nu \\ 0 & \Phi_{\mathfrak{M}} \end{pmatrix}.$$

Then $\mathfrak{E}$ is an extension of $\mathfrak{M}$ by $\mathfrak{N}$, and if we let σ denote the obvious embedding of $\mathfrak{M}$ into $\mathfrak{E}$, then one computes that

$$\nu = \Phi_{\mathfrak{E}} \circ \varphi^*\sigma - \sigma \circ \Phi_{\mathfrak{M}}.$$

This shows that (4.1.6) is an isomorphism, as claimed. $\square$

Another dimension shifting argument, taking into account the preceding lemma, shows that $\mathrm{Ext}^2_{\mathcal{K}(A)}(\mathfrak{M}, -)$ embeds into $H^2\big(C^\bullet(-)\big)$. Since the target of this embedding vanishes, we find that the same is true of the source. This yields the following corollary.

Corollary 4.1.7. *If $\mathfrak{M}$ is a Breuil–Kisin module with A-coefficients and descent data, then* $\mathrm{Ext}^2_{\mathcal{K}(A)}(\mathfrak{M}, -) = 0$.

We summarise the above discussion in the following corollary.

Corollary 4.1.8. *If $\mathfrak{M}$ is a Breuil–Kisin module with A-coefficients and descent data, and $\mathfrak{N}$ is an object of $\mathcal{K}(A)$, then we have a natural short exact sequence*

$$0 \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to C^0(\mathfrak{N}) \to C^1(\mathfrak{N}) \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to 0.$$

The following lemma records the behaviour of these complexes with respect to base change.

Lemma 4.1.9. *Suppose that $\mathfrak{M}$, $\mathfrak{N}$ are Breuil–Kisin modules with descent data and A-coefficients, that B is an A-algebra, and that Q is a B-module. Then the complexes $C^\bullet_{\mathfrak{M}}(\mathfrak{N} \widehat{\otimes}_A Q)$ and $C^\bullet_{\mathfrak{M} \widehat{\otimes}_A B}(\mathfrak{N} \widehat{\otimes}_A Q)$ coincide, the former complex formed with respect to $\mathcal{K}(A)$ and the latter with respect to $\mathcal{K}(B)$.*

Proof. Indeed, there is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{S}_A[\mathrm{Gal}(K'/K)]}(\mathfrak{M}, \mathfrak{N} \widehat{\otimes}_A Q) \cong \mathrm{Hom}_{\mathfrak{S}_B[\mathrm{Gal}(K'/K)]}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A Q),$$

and similarly with $\varphi^*\mathfrak{M}$ in place of $\mathfrak{M}$. $\square$

The following slightly technical lemma is crucial for establishing finiteness properties, and also base-change properties, of Exts of Breuil–Kisin modules.

Lemma 4.1.10. *Let A be a $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, suppose that $\mathfrak{M}$ is a Breuil–Kisin module with descent data and A-coefficients, of height at most h, and suppose that $\mathfrak{N}$ is a u-adically complete, u-torsion free object of $\mathcal{K}(A)$.*

Let $C^\bullet$ be the complex defined in Definition 4.1.2*, and write δ for its differential. Suppose that Q is an A-module with the property that $C^i \otimes_A Q$ is v-torsion free for $i = 0, 1$ and v-adically separated for $i = 0$.*

Then:

(1) For any integer $M \geq (eah+1)/(p-1)$, $\ker(\delta \otimes \mathrm{id}_Q) \cap v^M C^0 \otimes_A Q = 0$.
(2) For any integer $N \geq (peah+1)/(p-1)$, $\delta \otimes \mathrm{id}_Q$ induces an isomorphism

$$(\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1 \otimes_A Q) \xrightarrow{\sim} v^N (C^1 \otimes_A Q).$$

Consequently, for N as in (2) *the natural morphism of complexes of A-modules*

$$[C^0 \otimes_A Q \xrightarrow{\delta \otimes \mathrm{id}_Q} C^1 \otimes_A Q] \to [C^0 \otimes_A Q/\big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1 \otimes_A Q)\big) \xrightarrow{\delta \otimes \mathrm{id}_Q} C^1 \otimes_A Q/v^N C^1 \otimes_A Q]$$

is a quasi-isomorphism.

Since we are assuming that the $C^i \otimes_A Q$ are v-torsion free, the expression $v^r C^i(\mathfrak{N}) \otimes_A Q$ may be interpreted as denoting either $v^r\big(C^i(\mathfrak{N}) \otimes_A Q\big)$ or $\big(v^r C^i(\mathfrak{N})\big) \otimes_A Q$, the two being naturally isomorphic.

Remark 4.1.11. Before giving the proof of Lemma 4.1.10, we observe that the hypotheses on the $C^i \otimes_A Q$ are satisfied if either $Q = A$, or else $\mathfrak{N}$ is a projective $\mathfrak{S}_A$-module and Q is a finitely generated B-module for some finitely generated A-algebra B. (Indeed $C^1 \otimes_A Q$ is v-adically separated as well in these cases.)

(1) Since $\mathfrak{M}$ is projective of finite rank over $A[[u]]$, and since $\mathfrak{N}$ is u-adically complete and u-torsion free, each C^i is v-adically separated and v-torsion free. In particular the hypothesis on Q is always satisfied by $Q = A$. (In fact since $\mathfrak{N}$ is u-adically complete it also follows that the C^i are v-adically complete. Here we use that $\mathrm{Gal}(K'/K)$ has order prime to p to see that C^0 is an $\mathfrak{S}^0_A$-module direct summand of $\mathrm{Hom}_{\mathfrak{S}_A}(\mathfrak{M}, \mathfrak{N})$, and similarly for C^1.)

(2) Suppose $\mathfrak{N}$ is a projective $\mathfrak{S}_A$-module. Then the C^i are projective $\mathfrak{S}^0_A$-modules, again using that $\mathrm{Gal}(K'/K)$ has order prime to p. Since each $C^i(\mathfrak{N})/vC^i(\mathfrak{N})$ is A-flat, it follows that $C^i(\mathfrak{N}) \otimes_A Q$ is v-torsion free. If furthermore B is a finitely generated A-algebra, and Q is a finitely generated B-module, then the $C^i(\mathfrak{N}) \otimes_A Q$ are v-adically separated (being finitely generated modules over the ring $A[[v]] \otimes_A B$, which is a finitely generated algebra over the Noetherian ring $A[[v]]$, and hence is itself Noetherian).

Proof of Lemma 4.1.10. Since $p^a = 0$ in A, there exists $H(u) \in \mathfrak{S}_A$ with $u^{e'ah} = E(u)^h H(u)$ in $\mathfrak{S}_A$. Thus the image of $\Phi_{\mathfrak{M}}$ contains $u^{e'ah}\mathfrak{M} = v^{eah}\mathfrak{M}$, and there exists a map $\Upsilon : \mathfrak{M} \to \varphi^*\mathfrak{M}$ such that $\Phi_{\mathfrak{M}} \circ \Upsilon$ is multiplication by v^{eah}.

We begin with (1). Suppose that $f \in \ker(\delta \otimes \mathrm{id}_Q) \cap v^M C^0 \otimes_A Q$. Since $C^0 \otimes_A Q$ is v-adically separated, it is enough, applying induction on M, to show that $f \in v^{M+1} C^0 \otimes_A Q$. Since $f \in \ker(\delta \otimes \mathrm{id}_Q)$, we have $f \circ \Phi_{\mathfrak{M}} = \Phi_{\mathfrak{N}} \circ \varphi^* f$. Since $f \in v^M C^0 \otimes_A Q$, we have $f \circ \Phi_{\mathfrak{M}} = \Phi_{\mathfrak{N}} \circ \varphi^* f \in v^{pM} C^1 \otimes_A Q$. Precomposing with Υ gives $v^{eah} f \in v^{pM} C^0 \otimes_A Q$. Since $C^0 \otimes_A Q$ is v-torsion free, it follows that $f \in v^{pM-eah} C^0 \otimes_A Q \subseteq u^{M+1} C^0 \otimes_A Q$, as required.

We now move on to (2). Set $M = N - eah$. By precomposing with Υ we see that $\alpha \circ \Phi_{\mathfrak{M}} \in v^N C^1 \otimes_A Q$ implies $\alpha \in v^M C^0 \otimes_A Q$; from this, together with the inequality $pM \geq N$, it is straightforward to check that

$$(\Phi_{\mathfrak{M}}^*)^{-1}(v^N C^1 \otimes_A Q) = (\delta \otimes \mathrm{id}_Q)^{-1}(v^N C^1 \otimes_A Q) \cap v^M C^0 \otimes_A Q.$$

Note that M satisfies the condition in (1). To complete the proof we will show that for any M as in (1) and any $N \geq M + eah$ the map δ induces an isomorphism

$$(\delta \otimes \mathrm{id}_Q)^{-1}(v^N C^1 \otimes_A Q) \cap v^M C^0 \otimes_A Q \xrightarrow{\sim} v^N C^1 \otimes_A Q.$$

By (1), $\delta \otimes \mathrm{id}_Q$ induces an injection $(\delta \otimes \mathrm{id}_Q)^{-1}(v^N C^1 \otimes_A Q) \cap v^M C^0 \otimes_A Q \hookrightarrow v^N C^1 \otimes_A Q$, so it is enough to show that $(\delta \otimes \mathrm{id}_Q)(v^M C^0 \otimes_A Q) \supseteq v^N C^1 \otimes_A Q$. Equivalently, we need to show that

$$v^N C^1 \otimes_A Q \to (C^1 \otimes_A Q)/(\delta \otimes \mathrm{id}_Q)\big(v^M C^0 \otimes_A Q\big)$$

is identically zero. Since the formation of cokernels is compatible with tensor products, we see that this morphism is obtained by tensoring the corresponding morphism

$$v^N C^1 \to C^1/\delta\big(v^M C^0\big)$$

with Q over A, so we are reduced to the case $Q = A$. (Recall from Remark 4.1.11(1) that the hypotheses of the Lemma are satisfied in this case, and that C^1 is v-adically separated.)

We claim that for any $g \in v^N C^1$, we can find an $f \in v^{N-eah} C^0$ such that $\delta(f) - g \in v^{p(N-eah)} C^1$. Admitting the claim, given any $g \in v^N C^1$, we may find $h \in v^M C^0$ with $\delta(h) = g$ by successive approximation in the following way: Set $h_0 = f$ for f as in the claim; then $h_0 \in v^{N-eah} C^0 \subseteq v^M C^0$, and $\delta(h_0) - g \in v^{p(N-eah)} C^1 \subseteq v^{N+1} C^1$. Applying the claim again with N replaced by $N+1$, and g replaced by $g - \delta(h_0)$, we find $f \in v^{N+1-eah} C^0 \subseteq v^{M+1} C^0$ with $\delta(f) - g + \delta(h_0) \in v^{p(N+1-eah)} C^1 \subseteq v^{N+1} C^1$. Setting $h_1 = h_0 + f$, and proceeding inductively, we obtain a Cauchy sequence converging (in the v-adically complete $A[[v]]$-module C^0) to the required element h.

It remains to prove the claim. Since $\delta(f) = \Phi_{\mathfrak{N}} \circ \varphi^* f - f \circ \Phi_{\mathfrak{M}}$, and since if $f \in v^{N-eah} C^0$ then $\Phi_{\mathfrak{N}} \circ \varphi^* f \in v^{p(N-eah)} C^1$, it is enough to show that we can find an $f \in v^{N-eah} C^0$ with $f \circ \Phi_{\mathfrak{M}} = -g$. Since $\Phi_{\mathfrak{M}}$ is injective, the map $\Upsilon \circ \Phi_{\mathfrak{M}}$ is also multiplication by v^{eah}, and so it suffices to take f with $v^{eah} f = -g \circ \Upsilon \in v^N C^0$. $\square$

Corollary 4.1.12. *Let A be a Noetherian $\mathcal{O}/\varpi^a$-algebra, and let $\mathfrak{M}$, $\mathfrak{N}$ be Breuil–Kisin modules with descent data and A-coefficients. If B is a finitely generated A-algebra, and Q is a finitely generated B-module, then the natural morphism of complexes of B-modules*

$$[C^0(\mathfrak{N}) \otimes_A Q \xrightarrow{\delta \otimes \mathrm{id}_Q} C^1(\mathfrak{N}) \otimes_A Q] \to [C^0(\mathfrak{N} \widehat{\otimes}_A Q) \xrightarrow{\delta} C^1(\mathfrak{N} \widehat{\otimes}_A Q)]$$

is a quasi-isomorphism.

Proof. By Remarks 4.1.11 and 2.1.8(2) we can apply Lemma 4.1.10 to both $C^i(\mathfrak{N} \widehat{\otimes}_A Q)$ and $C^i(\mathfrak{N}) \otimes_A Q$, and we see that it is enough to show that the natural morphism of complexes

$$\begin{array}{c} [\big(C^0(\mathfrak{N}) \otimes_A Q\big)/(\Phi_{\mathfrak{M}}^* \otimes \mathrm{id}_Q)^{-1}\big(v^N C^1(\mathfrak{N}) \otimes_A Q\big) \xrightarrow{\delta} \big(C^1(\mathfrak{N}) \otimes_A Q\big)/\big(v^N C^1(\mathfrak{N}) \otimes_A Q\big)] \\ \big\downarrow \\ [C^0(\mathfrak{N} \widehat{\otimes}_A Q)/\big(\Phi_{\mathfrak{M}}^*\big)^{-1}\big(v^N C^1(\mathfrak{N} \widehat{\otimes}_A Q)\big) \xrightarrow{\delta} C^1(\mathfrak{N} \widehat{\otimes}_A Q)/v^N C^1(\mathfrak{N} \widehat{\otimes}_A Q)] \end{array}$$

is a quasi-isomorphism. In fact, it is even an isomorphism. □

Proposition 4.1.13. *Let A be a $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let $\mathfrak{M}$, $\mathfrak{N}$ be Breuil–Kisin modules with descent data and A-coefficients. Then $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ and $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}/u^i\mathfrak{N})$ for $i \geq 1$ are finitely presented A-modules.*

If furthermore A is Noetherian, then $\operatorname{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ and $\operatorname{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}/u^i\mathfrak{N})$ for $i \geq 1$ are also finitely presented (equivalently, finitely generated) A-modules.

Proof. The statements for $\mathfrak{N}/u^i\mathfrak{N}$ follow easily from those for $\mathfrak{N}$, by considering the short exact sequence $0 \to u^i\mathfrak{N} \to \mathfrak{N} \to \mathfrak{N}/u^i\mathfrak{N} \to 0$ in $\mathcal{K}(A)$ and applying Corollary 4.1.7. By Corollary 4.1.8, it is enough to consider the cohomology of the complex $C^\bullet$. By Lemma 4.1.10 with $Q = A$, the cohomology of $C^\bullet$ agrees with the cohomology of the induced complex

$$C^0/\big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1)\big) \to C^1/v^N C^1,$$

for an appropriately chosen value of N. It follows that for an appropriately chosen value of N, $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ can be computed as the cokernel of the induced morphism $C^0/v^N C^0 \to C^1/v^N C^1$.

Under our hypothesis on $\mathfrak{N}$, $C^0/v^N C^0$ and $C^1/v^N C^1$ are finitely generated projective A-modules, and thus finitely presented. It follows that $\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is finitely presented.

In the case that A is furthermore assumed to be Noetherian, it is enough to note that since $v^N C^0 \subseteq (\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1)$, the quotient $C^0/\big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1)\big)$ is a finitely generated A-module. □

Proposition 4.1.14. *Let A be a $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let $\mathfrak{M}$ and $\mathfrak{N}$ be Breuil–Kisin modules with descent data and A-coefficients. Let B be an A-algebra, and let $f_B : \mathfrak{M} \widehat{\otimes}_A B \to \mathfrak{N} \widehat{\otimes}_A B$ be a morphism of Breuil–Kisin modules with B-coefficients.*

Then there is a finite type A-subalgebra B' of B and a morphism of Breuil–Kisin modules $f_{B'} : \mathfrak{M} \widehat{\otimes}_A B' \to \mathfrak{N} \widehat{\otimes}_A B'$ such that f_B is the base change of $f_{B'}$.

Proof. By Lemmas 4.1.3 and 4.1.9 (the latter applied with $Q = B$) we can and do think of f_B as being an element of the kernel of $\delta : C^0(\mathfrak{N} \widehat{\otimes}_A B) \to C^1(\mathfrak{N} \widehat{\otimes}_A B)$, the complex $C^\bullet$ here and throughout this proof denoting $C^\bullet_{\mathfrak{M}}$ as usual.

Fix N as in Lemma 4.1.10, and write $\overline{f}_B$ for the corresponding element of $C^0(\mathfrak{N} \widehat{\otimes}_A B)/v^N = (C^0(\mathfrak{N})/v^N) \otimes_A B$ (this equality following easily from the assumption that $\mathfrak{M}$ and $\mathfrak{N}$ are projective $\mathfrak{S}_A$-modules of finite rank). Since $C^0(\mathfrak{N})/v^N$ is a projective A-module of finite rank, it follows that for some finite type A-subalgebra B' of B, there is an element $\overline{f}_{B'} \in (C^0(\mathfrak{N})/v^N) \otimes_A B' = C^0(\mathfrak{N} \widehat{\otimes}_A B')/v^N$ such that $\overline{f}_{B'} \otimes_{B'} B = \overline{f}_B$. Denote also by $\overline{f}_{B'}$ the induced element of

$$C^0(\mathfrak{N} \widehat{\otimes}_A B')/\big(\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1(\mathfrak{N} \widehat{\otimes}_A B')\big).$$

By Lemma 4.1.10 (and Lemma 4.1.3) we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^0(C^\bullet(\mathfrak{N} \widehat{\otimes}_A B')) \arrow[r] \arrow[d] & C^0(\mathfrak{N} \widehat{\otimes}_A B')/\big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1(\mathfrak{N} \widehat{\otimes}_A B'))\big) \arrow[r, "\delta"] \arrow[d] & C^1(\mathfrak{N} \widehat{\otimes}_A B')/v^N \arrow[d] \\
0 \arrow[r] & H^0(C^\bullet(\mathfrak{N} \widehat{\otimes}_A B)) \arrow[r] & C^0(\mathfrak{N} \widehat{\otimes}_A B)/\big((\Phi^*_{\mathfrak{M}})^{-1}(v^N C^1(\mathfrak{N} \widehat{\otimes}_A B))\big) \arrow[r, "\delta"] & C^1(\mathfrak{N} \widehat{\otimes}_A B)/v^N
\end{tikzcd}$$

in which the vertical arrows are induced by $\widehat{\otimes}_{B'}B$. By a diagram chase we only need to show that $\delta(\overline{f}_{B'}) = 0$. Since $\delta(f_B) = 0$, it is enough to show that the right hand vertical arrow is an injection. This arrow can be rewritten as the tensor product of the injection of A-algebras $B' \hookrightarrow B$ with the flat (even projective of finite rank) A-module $C^1(\mathfrak{N})/v^N$, so the result follows. $\square$

We have the following key base-change result for Ext^1's of Breuil–Kisin modules with descent data.

Proposition 4.1.15. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a $\mathcal{O}/\varpi^a$-algebra A. Then for any A-algebra B, and for any B-module Q, there are natural isomorphisms* $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B, \mathfrak{N}\widehat{\otimes}_A B) \otimes_B Q \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B, \mathfrak{N}\widehat{\otimes}_A Q)$.

Proof. We first prove the lemma in the case of an A-module Q. It follows from Lemmas 4.1.5 and 4.1.10 that we may compute $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})$ as the cokernel of the morphism

$$C^0(\mathfrak{N})/v^N C^0(\mathfrak{N}) \xrightarrow{\delta} C^1(\mathfrak{N})/v^N C^1(\mathfrak{N}),$$

for some sufficiently large value of N (not depending on $\mathfrak{N}$), and hence that we may compute $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q$ as the cokernel of the morphism

$$\big(C^0(\mathfrak{N})/v^N C^0(\mathfrak{N})\big) \otimes_A Q \xrightarrow{\delta} \big(C^1(\mathfrak{N})/v^N C^1(\mathfrak{N})\big) \otimes_A Q.$$

We may similarly compute $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}\widehat{\otimes}_A Q)$ as the cokernel of the morphism

$$C^0(\mathfrak{N}\widehat{\otimes}_A Q)/v^N C^0(\mathfrak{N}\widehat{\otimes}_A Q) \xrightarrow{\delta} C^1(\mathfrak{N}\widehat{\otimes}_A Q)/v^N C^1(\mathfrak{N}\widehat{\otimes}_A Q).$$

(Remark 2.1.8 (2) shows that $\mathfrak{N}\widehat{\otimes}_A Q$ satisfies the necessary hypotheses for Lemma 4.1.10 to apply.) Once we note that the natural morphism

$$\big(C^i(\mathfrak{N})/v^N C^i(\mathfrak{N})\big) \otimes_A Q \to C^i(\mathfrak{N}\widehat{\otimes}_A Q)/v^N C^i(\mathfrak{N}\widehat{\otimes}_A Q)$$

is an isomorphism for $i = 0$ and 1 (because $\mathfrak{M}$ is a finitely generated projective $\mathfrak{S}_A$-module), we obtain the desired isomorphism

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}\widehat{\otimes}_A Q).$$

If B is an A-algebra, and Q is a B-module, then by Lemma 4.1.9 there is a natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}\widehat{\otimes}_A Q) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B,\mathfrak{N}\widehat{\otimes}_A Q);$$

combined with the preceding base-change result, this yields one of our claimed isomorphisms, namely

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B,\mathfrak{N}\widehat{\otimes}_A Q).$$

Taking Q to be B itself, we then obtain an isomorphism

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A B \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B,\mathfrak{N}\widehat{\otimes}_A B).$$

This allows us to identify $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A Q$, which is naturally isomorphic to $\big(\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}) \otimes_A B\big) \otimes_B Q$, with $\mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B,\mathfrak{N}\widehat{\otimes}_A B) \otimes_B Q$, yielding the second claimed isomorphism. $\square$

In contrast to the situation for extensions (*cf.* Proposition 4.1.15), the formation of homomorphisms between Breuil–Kisin modules is in general not compatible with arbitrary base-change, as the following example shows.

Example 4.1.16. Take $A = (\mathbf{Z}/p\mathbf{Z})[x^{\pm 1}, y^{\pm 1}]$, and let $\mathfrak{M}_x$ be the free Breuil–Kisin module of rank one and A-coefficients with $\varphi(e) = xe$ for some generator e of $\mathfrak{M}_x$. Similarly define $\mathfrak{M}_y$ with $\varphi(e') = ye'$ for some generator e' of $\mathfrak{M}_y$. Then $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}_x, \mathfrak{M}_y) = 0$. On the other hand, if $B = A/(x-y)$ then $\mathfrak{M}_x \widehat{\otimes}_A B$ and $\mathfrak{M}_y \widehat{\otimes}_A B$ are isomorphic, so that $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_x \widehat{\otimes} B, \mathfrak{M}_y \widehat{\otimes} B) \not\cong \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}_x, \mathfrak{M}_y) \otimes_A B$.

However, it is possible to establish such a compatibility in some settings. Corollary 4.1.18, which gives a criterion for the vanishing of $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A B)$ for any A-algebra B, is a first example of a result in this direction. Lemma 4.1.20 deals with flat base change, and Lemma 4.1.21, which will be important in Section 4.3, proves that formation of homomorphisms is compatible with base-change over a dense open subscheme of $\operatorname{Spec} A$.

Proposition 4.1.17. *Suppose that A is a Noetherian $\mathcal{O}/\varpi^a$-algebra, and that $\mathfrak{M}$ and $\mathfrak{N}$ are objects of $\mathcal{K}(A)$ that are finitely generated over $\mathfrak{S}_A$ (or, equivalently, over $A[[u]]$). Consider the following conditions:*

(1) $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A B) = 0$ *for any finite type A-algebra B.*
(2) $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big) = 0$ *for each maximal ideal $\mathfrak{m}$ of A.*
(3) $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q) = 0$ *for any finitely generated A-module Q.*

Then we have (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3). If A is furthermore Jacobson, then all three conditions are equivalent.

Proof. If $\mathfrak{m}$ is a maximal ideal of A, then $\kappa(\mathfrak{m})$ is certainly a finite type A-algebra, and so evidently (1) implies (2). It is even a finitely generated A-module, and so also (2) follows from (3).

We next prove that (2) implies (3). To this end, recall that if A is any ring, and M is any A-module, then M injects into the product of its localizations at all maximal ideals. If A is Noetherian, and M is finitely generated, then, by combining this fact with the Artin–Rees Lemma, we see that M embeds into the product of its completions at all maximal ideals. Another way to express this is that, if I runs over all cofinite length ideals in A (i.e. all ideals for which A/I is finite length), then M embeds into the projective limit of the quotients M/IM (the point being that this projective limit is the same as the product over all $\mathfrak{m}$-adic completions). We are going to apply this observation with A replaced by $\mathfrak{S}_A$, and with M taken to be $\mathfrak{N} \otimes_A Q$ for some finitely generated A-module Q.

In $A[[u]]$, one sees that u lies in the Jacobson radical (because $1 + fu$ is invertible in $A[[u]]$ for every $f \in A[[u]]$), and thus in every maximal ideal, and so the maximal ideals of $A[[u]]$ are of the form $(\mathfrak{m}, u)$, where $\mathfrak{m}$ runs over the maximal ideals of A. Thus the ideals of the form (I, u^n), where I is a cofinite length ideal in A, are cofinal in all cofinite length ideals in $A[[u]]$. Since $\mathfrak{S}_A$ is finite over $A[[u]]$, we see that the ideals (I, u^n) in $\mathfrak{S}_A$ are also cofinal in all cofinite length ideals in $A[[u]]$. Since $A[[u]]$, and hence $\mathfrak{S}_A$, is furthermore Noetherian when A is, we see that if Q is a finitely generated A-module, and $\mathfrak{N}$ is a finitely generated $\mathfrak{S}_A$-module, then $\mathfrak{N} \otimes_A (Q/IQ)$ is u-adically complete, for any cofinite length ideal I in A, and hence equal to the limit over n of $\mathfrak{N} \otimes_A Q/(I, u^n)$. Putting this together with the observation of the preceding paragraph, we see that the natural morphism

$$\mathfrak{N} \otimes_A Q \to \varprojlim_I \mathfrak{N} \otimes_A (Q/IQ)$$

(where I runs over all cofinite length ideals of A) is an embedding. The induced morphism

$$\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q) \to \varprojlim_I \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A (Q/IQ))$$

is then evidently also an embedding.

Thus, to conclude that $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q)$ vanishes, it suffices to show that $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A (Q/IQ))$ vanishes for each cofinite length ideal I in A. An easy induction on the length of A/I reduces this to showing that $\mathrm{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M}, \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big)$, or, equivalently, $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big)$, vanishes for each maximal ideal $\mathfrak{m}$. Since this is the hypothesis of (2), we see that indeed (2) implies (3).

It remains to show that (3) implies (1) when A is Jacobson. Applying the result "(2) implies (3)" (with A replaced by B, and taking Q in (3) to be B itself as a B-module) to $\mathfrak{M} \widehat{\otimes}_A B$ and $\mathfrak{N} \widehat{\otimes}_A B$, we see that it suffices to prove the vanishing of

$$\mathrm{Hom}_{\mathcal{K}(B)}\big((\mathfrak{M} \widehat{\otimes}_A B) \otimes_B \kappa(\mathfrak{n}), (\mathfrak{N} \widehat{\otimes}_A B) \otimes_B \kappa(\mathfrak{n})\big) = \mathrm{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M}, \mathfrak{N} \widehat{\otimes}_A \kappa(\mathfrak{n})\big)$$

for each maximal ideal $\mathfrak{n}$ of B. Since A is Jacobson, the field $\kappa(\mathfrak{n})$ is in fact a finitely generated A-module, hence $\mathfrak{N} \widehat{\otimes} \kappa(\mathfrak{n}) = \mathfrak{N} \otimes_A \kappa(\mathfrak{n})$, and so the desired vanishing is a special case of (3). ∎

Corollary 4.1.18. *If A is a Noetherian and Jacobson $\mathcal{O}/\varpi^a$-algebra, and if $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data and A-coefficients, then the following three conditions are equivalent:*

1. $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A B) = 0$ *for any A-algebra B.*
2. $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big) = 0$ *for each maximal ideal $\mathfrak{m}$ of A.*
3. $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q) = 0$ *for any finitely generated A-module Q.*

Proof. By Proposition 4.1.17, we need only prove that if $\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M} \widehat{\otimes}_A B, \mathfrak{N} \widehat{\otimes}_A B)$ vanishes for all finitely generated A-algebras B, then it vanishes for all A-algebras B. This is immediate from Proposition 4.1.14. ∎

Corollary 4.1.19. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A, and that furthermore $\mathrm{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big)$ vanishes for each maximal ideal $\mathfrak{m}$ of A. Then the A-module $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is projective of finite rank.*

Proof. By Proposition 4.1.13, in order to prove that $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is projective of finite rank over A, it suffices to prove that it is flat over A. For this, it suffices to show that $Q \mapsto \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \otimes_A Q$ is exact when applied to finitely generated A-modules Q. Proposition 4.1.15 (together with Remark 2.1.8 (1)) allows us to identify this functor with the functor $Q \mapsto \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q)$. Note that the functor $Q \mapsto \mathfrak{N} \otimes_A Q$ is an exact functor of Q, since $\mathfrak{S}_A$ is a flat A-module (as A is Noetherian; see Remark 2.1.4(3)). Thus, taking into account Corollary 4.1.7, we see that it suffices to show that $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A Q) = 0$ for each finitely generated A-module Q, under the hypothesis that $\mathrm{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big) = 0$ for each maximal ideal $\mathfrak{m}$ of A. This is the implication (2) $\implies$ (3) of Proposition 4.1.17. ∎

Lemma 4.1.20. *Suppose that $\mathfrak{M}$ is a Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A. Suppose that $\mathfrak{N}$ is either a Breuil–Kisin module with A-coefficients, or that $\mathfrak{N} = \mathfrak{N}'/u^N\mathfrak{N}'$, where $\mathfrak{N}'$ a Breuil–Kisin module with A-coefficients and $N \geq 1$. Then, if B is a finitely generated flat A-algebra, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}\widehat{\otimes}_A B, \mathfrak{N}\widehat{\otimes}_A B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \otimes_A B.$$

Proof. By Corollary 4.1.8 and the flatness of B, we have a left exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \otimes_A B \to C^0(\mathfrak{N}) \otimes_A B \to C^1(\mathfrak{N}) \otimes_A B$$

and therefore (applying Corollary 4.1.12 to treat the case that $\mathfrak{N}$ is projective) a left exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \otimes_A B \to C^0(\mathfrak{N}\widehat{\otimes}_A B) \to C^1(\mathfrak{N}\widehat{\otimes}_A B).$$

The result follows from Corollary 4.1.8 and Lemma 4.1.9. ∎

Lemma 4.1.21. *Suppose that $\mathfrak{M}$ is a Breuil–Kisin module with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A which is furthermore a domain. Suppose also that $\mathfrak{N}$ is either a Breuil–Kisin module with A-coefficients, or that $\mathfrak{N} = \mathfrak{N}'/u^N\mathfrak{N}'$, where $\mathfrak{N}'$ is a Breuil–Kisin module with A-coefficients and $N \geq 1$. Then there is some nonzero $f \in A$ with the following property: writing $\mathfrak{M}_{A_f} = \mathfrak{M}\widehat{\otimes}_A A_f$ and $\mathfrak{N}_{A_f} = \mathfrak{N}\widehat{\otimes}_A A_f$, then for any finitely generated A_f-algebra B, and any finitely generated B-module Q, there are natural isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \otimes_{A_f} Q &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} B) \otimes_B Q \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} Q).\end{aligned}$$

Proof of Lemma 4.1.21. Note that since A is Noetherian, by Remark 2.1.4(3) we see that $\mathfrak{N}$ is A-flat. By Corollary 4.1.8 we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to C^0(\mathfrak{N}) \to C^1(\mathfrak{N}) \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to 0.$$

Since by assumption $\mathfrak{M}$ is a projective $\mathfrak{S}_A$-module, and $\mathfrak{N}$ is a flat A-module, the $C^i(\mathfrak{N})$ are also flat A-modules.

By Proposition 4.1.13, $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is a finitely generated A-module, so by the generic freeness theorem [Sta13, Tag 051R] there is some nonzero $f \in A$ such that $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})_f$ is free over A_f.

Since localisation is exact, we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}, \mathfrak{N})_f \to C^0(\mathfrak{N})_f \to C^1(\mathfrak{N})_f \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})_f \to 0$$

and therefore (applying Corollary 4.1.12 to treat the case that $\mathfrak{N}$ is a Breuil–Kisin module) an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \to C^0(\mathfrak{N}_{A_f}) \to C^1(\mathfrak{N}_{A_f}) \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})_f \to 0.$$

Since the last three terms are flat over A_f, this sequence remains exact upon tensoring over A_f with Q. Applying Corollary 4.1.12 again to treat the case that $\mathfrak{N}$ is a Breuil–Kisin module, we see that in particular we have a left exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \otimes_{A_f} Q \to C^0(\mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} Q) \to C^1(\mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} Q),$$

and Corollary 4.1.8 together with Lemma 4.1.9 yield one of the desired isomorphisms, namely

$$\mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \otimes_{A_f} Q \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} Q).$$

If we consider the case when $Q = B$, we obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}, \mathfrak{N}_{A_f}) \otimes_{A_f} B \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} B).$$

Rewriting the tensor product $-\otimes_{A_f} Q$ as $-\otimes_{A_f} B \otimes_B Q$, we then find that

$$\mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} B) \otimes_B Q \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(B)}(\mathfrak{M}_{A_f} \widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f} \widehat{\otimes}_{A_f} Q),$$

which gives the second desired isomorphism. □

Variants on the preceding result may be proved using other versions of the generic freeness theorem.

Example 4.1.22. Returning to the setting of Example 4.1.16, one can check using Corollary 4.1.18 that the conclusion of Lemma 4.1.21 (for $\mathfrak{M} = \mathfrak{M}_x$ and $\mathfrak{N} = \mathfrak{M}_y$) holds with $f = x - y$. In this case all of the resulting Hom groups vanish (*cf.* also the proof of Lemma 4.3.7). It then follows from Corollary 4.1.19 that $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})_f$ is projective over A_f, so that the proof of Lemma 4.1.21 even goes through with this choice of f.

As well as considering homomorphisms and extensions of Breuil–Kisin modules, we need to consider the homomorphisms and extensions of their associated étale φ-modules; recall that the passage to associated étale φ-modules amounts to inverting u, and so we briefly discuss this process in the general context of the category $\mathcal{K}(A)$.

We let $\mathcal{K}(A)[1/u]$ denote the full subcategory of $\mathcal{K}(A)$ consisting of objects on which multiplication by u is invertible. We may equally well regard it as the category of left $\mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)]$-modules (this notation being interpreted in the evident manner). There are natural isomorphisms (of bi-modules)

$$\mathfrak{S}_A[1/u] \otimes_{\mathfrak{S}_A} \mathfrak{S}_A[F, \mathrm{Gal}(K'/K)] \xrightarrow{\sim} \mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)] \tag{4.1.23}$$

and

$$\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)] \otimes_{\mathfrak{S}_A} \mathfrak{S}_A[1/u] \xrightarrow{\sim} \mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)]. \tag{4.1.24}$$

Thus (since $\mathfrak{S}_A \to \mathfrak{S}_A[1/u]$ is a flat morphism of commutative rings) the morphism of rings $\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)] \to \mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)]$ is both left and right flat.

If $\mathfrak{M}$ is an object of $\mathcal{K}(A)$, then we see from (4.1.23) that $\mathfrak{M}[1/u] := \mathfrak{S}_A[1/u] \otimes_{\mathfrak{S}_A} \mathfrak{M} \xrightarrow{\sim} \mathfrak{S}_A[1/u][F, \mathrm{Gal}(K'/K)] \otimes_{\mathfrak{S}_A[F, \mathrm{Gal}(K'/K)]} \mathfrak{M}$ is naturally an object of $\mathcal{K}(A)[1/u]$. Our preceding remarks about flatness show that $\mathfrak{M} \mapsto \mathfrak{M}[1/u]$ is an exact functor $\mathcal{K}(A) \to \mathcal{K}(A)[1/u]$.

Lemma 4.1.25. *(1) If M and N are objects of $\mathcal{K}(A)[1/u]$, then there is a natural isomorphism*

$$\mathrm{Ext}^i_{\mathcal{K}(A)[1/u]}(M, N) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{K}(A)}(M, N).$$

(2) If $\mathfrak{M}$ is an object of $\mathcal{K}(A)$ and N is an object of $\mathcal{K}(A)[1/u]$, then there is a natural isomorphism

$$\mathrm{Ext}^i_{\mathcal{K}(A)}(\mathfrak{M}, N) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{K}(A)}(\mathfrak{M}[1/u], N),$$

for all $i \geq 0$.

Proof. The morphism of (1) can be understood in various ways; for example, by thinking in terms of Yoneda Exts, and recalling that $\mathcal{K}(A)[1/u]$ is a full subcategory of $\mathcal{K}(A)$. If instead we think in terms of projective resolutions, we can begin with a projective resolution $\mathfrak{P}^\bullet \to M$ in $\mathcal{K}(A)$, and then consider the induced projective resolution $\mathfrak{P}^\bullet[1/u]$ of $M[1/u]$. Noting that $M[1/u] \xrightarrow{\sim} M$ for any object M of $\mathcal{K}(A)[1/u]$, we then find (via tensor adjunction) that $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{P}^\bullet, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(A)[1/u]}(\mathfrak{P}^\bullet[1/u], N)$, which induces the desired isomorphism of Ext's by passing to cohomology.

Taking into account the isomorphism of (1), the claim of (2) is a general fact about tensoring over a flat ring map (as can again be seen by considering projective resolutions). ☐

Remark 4.1.26. The preceding lemma is fact an automatic consequence of the abstract categorical properties of our situation: the functor $\mathfrak{M} \mapsto \mathfrak{M}[1/u]$ is left adjoint to the inclusion $\mathcal{K}(A)[1/u] \subset \mathcal{K}(A)$, and restricts to (a functor naturally equivalent to) the identity functor on $\mathcal{K}(A)[1/u]$.

The following lemma expresses the Hom between étale φ-modules arising from Breuil–Kisin modules in terms of a certain direct limit.

Lemma 4.1.27. *Suppose that $\mathfrak{M}$ is a Breuil–Kisin module with descent data in a Noetherian $\mathcal{O}/\varpi^a$-algebra A, and that $\mathfrak{N}$ is an object of $\mathcal{K}(A)$ which is finitely generated and u-torsion free as an $\mathfrak{S}_A$-module. Then there is a natural isomorphism*

$$\varinjlim_i \mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}(A)[1/u]}(\mathfrak{M}[1/u], \mathfrak{N}[1/u]),$$

where the transition maps are induced by the inclusions $u^{i+1}\mathfrak{M} \subset u^i\mathfrak{M}$.

Remark 4.1.28. Note that since $\mathfrak{N}$ is u-torsion free, the transition maps in the colimit are injections, so the colimit is just an increasing union.

Proof. There are compatible injections $\mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}) \to \mathrm{Hom}_{\mathcal{K}(A)[1/u]}(\mathfrak{M}[1/u], \mathfrak{N}[1/u])$, taking $f' \in \mathrm{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N})$ to $f \in \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u])$ where $f(m) = u^{-i}f'(u^i m)$. Conversely, given $f \in \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u])$, there is some i such that $f(\mathfrak{M}) \subset u^{-i}\mathfrak{N}$, as required. ☐

We have the following analogue of Proposition 4.1.17.

Corollary 4.1.29. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data in a Noetherian $\mathcal{O}/\varpi^a$-algebra A. Consider the following conditions:*

(1) $\mathrm{Hom}_{\mathcal{K}(B)[1/u]}\big((\mathfrak{M} \widehat{\otimes}_A B)[1/u], (\mathfrak{N} \widehat{\otimes}_A B)[1/u]\big) = 0$ *for any finite type A-algebra B.*

(2) $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))[1/u]}\big((\mathfrak{M} \otimes_A \kappa(\mathfrak{m}))[1/u], (\mathfrak{N} \otimes_A \kappa(\mathfrak{m}))[1/u]\big) = 0$ *for each maximal ideal $\mathfrak{m}$ of A.*

(3) $\mathrm{Hom}_{\mathcal{K}(A)[1/u]}\big(\mathfrak{M}[1/u], (\mathfrak{N} \otimes_A Q)[1/u]\big) = 0$ *for any finitely generated A-module Q.*

Then we have (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3). If A is furthermore Jacobson, then all three conditions are equivalent.

Proof. By Lemma 4.1.27, the three conditions are respectively equivalent to the following conditions.

(1′) $\mathrm{Hom}_{\mathcal{K}(B)}\big(u^i(\mathfrak{M} \widehat{\otimes}_A B), \mathfrak{N} \widehat{\otimes}_A B\big) = 0$ for any finite type A-algebra B and all $i \geq 0$.

(2′) $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(u^i(\mathfrak{M} \otimes_A \kappa(\mathfrak{m})), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big) = 0$ for each maximal ideal $\mathfrak{m}$ of A and all $i \geq 0$.

(3′) $\mathrm{Hom}_{\mathcal{K}(A)}\big(u^i\mathfrak{M}, \mathfrak{N} \otimes_A Q\big) = 0$ for any finitely generated A-module Q and all $i \geq 0$.

Since $\mathfrak{M}$ is projective, the first two conditions are in turn equivalent to

(1″) $\mathrm{Hom}_{\mathcal{K}(B)}\big((u^i\mathfrak{M})\widehat{\otimes}_A B, \mathfrak{N}\widehat{\otimes}_A B\big) = 0$ for any finite type A-algebra B and all $i \geq 0$.

(2″) $\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big((u^i\mathfrak{M}) \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big) = 0$ for each maximal ideal $\mathfrak{m}$ of A and all $i \geq 0$.

The result then follows from Proposition 4.1.17. □

Definition 4.1.30. If $\mathfrak{M}$ and $\mathfrak{N}$ are objects of $\mathcal{K}(A)$, then we define

$$\text{ker-Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) := \ker\big(\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}[1/u], \mathfrak{N}[1/u])\big).$$

The point of this definition is to capture, in the setting of Lemma 2.3.3, the non-split extensions of Breuil–Kisin modules whose underlying extension of Galois representations is split.

Suppose now that $\mathfrak{M}$ is a Breuil–Kisin module. The exact sequence in $\mathcal{K}(A)$

$$0 \to \mathfrak{N} \to \mathfrak{N}[1/u] \to \mathfrak{N}[1/u]/\mathfrak{N} \to 0$$

gives an exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathfrak{N}) \arrow[r] \arrow[d] & C^0(\mathfrak{N}[1/u]) \arrow[r] \arrow[d] & C^0(\mathfrak{N}[1/u]/\mathfrak{N}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1(\mathfrak{N}) \arrow[r] & C^1(\mathfrak{N}[1/u]) \arrow[r] & C^1(\mathfrak{N}[1/u]/\mathfrak{N}) \arrow[r] & 0.
\end{tikzcd}$$

It follows from Corollary 4.1.8, Lemma 4.1.25(2), and the snake lemma that we have an exact sequence

$$\begin{aligned} &0 \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]) \\ &\to \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]/\mathfrak{N}) \to \text{ker-Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to 0. \end{aligned} \tag{4.1.31}$$

Lemma 4.1.32. *If $\mathfrak{M}$, $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A, and $\mathfrak{N}$ has height at most h, then $f(\mathfrak{M})$ is killed by u^i for any $f \in \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]/\mathfrak{N})$ and any $i \geq \lfloor e'ah/(p-1)\rfloor$.*

Proof. Suppose that f is an element of $\mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]/\mathfrak{N})$. Then $f(\mathfrak{M})$ is a finitely generated submodule of $\mathfrak{N}[1/u]/\mathfrak{N}$, and it therefore killed by u^i for some $i \geq 0$. Choosing i to be the exponent of $f(\mathfrak{M})$ (that is, choosing i to be minimal), it follows that $(\varphi^* f)(\varphi^*\mathfrak{M})$ has exponent precisely ip. (From the choice of i, we see that $u^{i-1}f(\mathfrak{M})$ is nonzero but killed by u, i.e., it is just a $W(k') \otimes A$-module, and so its pullback by $\varphi : \mathfrak{S}_A \to \mathfrak{S}_A$ has exponent precisely p. Then by the flatness of $\varphi : \mathfrak{S}_A \to \mathfrak{S}_A$ we have $u^{ip-1}(\varphi^* f)(\varphi^*\mathfrak{M}) = u^{p-1}\varphi^*(u^{i-1}f(\mathfrak{M})) \neq 0$.)

We claim that $u^{i+e'ah}(\varphi^* f)(\varphi^*\mathfrak{M}) = 0$; admitting this, we deduce that $i + e'ah \geq ip$, as required. To see the claim, take $x \in \varphi^*\mathfrak{M}$, so that $\Phi_{\mathfrak{N}}((u^i\varphi^* f)(x)) = u^i f(\Phi_{\mathfrak{M}}(x)) = 0$. It is therefore enough to show that the kernel of

$$\Phi_{\mathfrak{N}} : \varphi^*\mathfrak{N}[1/u]/\varphi^*\mathfrak{N} \to \mathfrak{N}[1/u]/\mathfrak{N}$$

is killed by $u^{e'ah}$; but this follows immediately from an application of the snake lemma to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \varphi^*\mathfrak{N} \arrow[r] \arrow[d, "\Phi_{\mathfrak{N}}"'] & \varphi^*\mathfrak{N}[1/u] \arrow[r] \arrow[d, "\Phi_{\mathfrak{N}}"'] & \varphi^*\mathfrak{N}[1/u]/\varphi^*\mathfrak{N} \arrow[r] \arrow[d, "\Phi_{\mathfrak{N}}"'] & 0 \\ 0 \arrow[r] & \mathfrak{N} \arrow[r] & \mathfrak{N}[1/u] \arrow[r] & \mathfrak{N}[1/u]/\mathfrak{N} \arrow[r] & 0 \end{tikzcd}$$

together with the assumption that $\mathfrak{N}$ has height at most h and an argument as in the first line of the proof of Lemma 4.1.10. □

Lemma 4.1.33. *If $\mathfrak{M}$, $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A, and $\mathfrak{N}$ has height at most h, then for any $i \geq \lfloor e'ah/(p-1)\rfloor$ we have an exact sequence*

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, u^i\mathfrak{N}) &\to \operatorname{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}) \\ &\to \operatorname{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}/u^i\mathfrak{N}) \to \text{ker-}\operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \to 0. \end{aligned}$$

Proof. Comparing Lemma 4.1.32 with the proof of Lemma 4.1.27, we see that the direct limit in that proof has stabilised at i, and we obtain an isomorphism $\operatorname{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N})$ sending a map f to $f' : u^i m \mapsto u^i f(m)$. The same formula evidently identifies $\operatorname{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ with $\operatorname{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, u^i\mathfrak{N})$ and $\operatorname{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]/\mathfrak{N})$ with $\operatorname{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}[1/u]/u^i\mathfrak{N})$. But any map in the latter group has image contained in $\mathfrak{N}/u^i\mathfrak{N}$ (by Lemma 4.1.32 applied to $\operatorname{Hom}_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}[1/u]/\mathfrak{N})$, together with the identification in the previous sentence), so that $\operatorname{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}[1/u]/u^i\mathfrak{N}) = \operatorname{Hom}_{\mathcal{K}(A)}(u^i\mathfrak{M}, \mathfrak{N}/u^i\mathfrak{N})$. □

Proposition 4.1.34. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-domain A. Then there is some nonzero $f \in A$ with the following property: if we write $\mathfrak{M}_{A_f} = \mathfrak{M}\widehat{\otimes}_A A_f$ and $\mathfrak{N}_{A_f} = \mathfrak{N}\widehat{\otimes}_A A_f$, then if B is any finitely generated A_f-algebra, and if Q is any finitely generated B-module, we have natural isomorphisms*

$$\begin{aligned} \text{ker-}\operatorname{Ext}^1_{\mathcal{K}(A_f)}(\mathfrak{M}, \mathfrak{N}) \otimes_{A_f} Q &\xrightarrow{\sim} \text{ker-}\operatorname{Ext}^1_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}\widehat{\otimes}_{A_f} B) \otimes_B Q \\ &\xrightarrow{\sim} \text{ker-}\operatorname{Ext}^1_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}\widehat{\otimes}_{A_f} Q). \end{aligned}$$

Proof. In view of Lemma 4.1.33, this follows from Lemma 4.1.21, with $\mathfrak{M}$ there being our $u^i\mathfrak{M}$, and $\mathfrak{N}$ being each of $\mathfrak{N}$, $\mathfrak{N}/u^i\mathfrak{N}$ in turn. □

The following result will be crucial in our investigation of the decomposition of $\mathcal{C}^{\mathrm{dd},1}$ and $\mathcal{R}^{\mathrm{dd},1}$ into irreducible components.

Proposition 4.1.35. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules with descent data and coefficients in a Noetherian $\mathcal{O}/\varpi^a$-algebra A which is furthermore a domain, and suppose that $\operatorname{Hom}_{\mathcal{K}(A)}\big(\mathfrak{M}\otimes_A \kappa(\mathfrak{m}), \mathfrak{N}\otimes_A \kappa(\mathfrak{m})\big)$ vanishes for each maximal ideal $\mathfrak{m}$ of A. Then there is some nonzero $f \in A$ with the following property: if we write $\mathfrak{M}_{A_f} = \mathfrak{M}\widehat{\otimes}_A A_f$ and $\mathfrak{N}_{A_f} = \mathfrak{N}\widehat{\otimes}_A A_f$, then for any finitely generated A_f-algebra B, each of* $\text{ker-}\operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} B)$, $\operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} B)$, *and*

$$\operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} B)/\,\text{ker-}\operatorname{Ext}^1_{\mathcal{K}(A_f)}(\mathfrak{M}_{A_f}\widehat{\otimes}_{A_f} B, \mathfrak{N}_{A_f}\widehat{\otimes}_{A_f} B)$$

is a finitely generated projective B-module.

Proof. Choose f as in Proposition 4.1.34, let B be a finitely generated A_f-algebra, and let Q be a finitely generated B-module. By Propositions 4.1.15 and 4.1.34, the morphism

$$\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)\otimes_B Q \to \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)\otimes_B Q$$

is naturally identified with the morphism

$$\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}Q) \to \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}Q);$$

in particular, it is injective. By Proposition 4.1.15 and Corollary 4.1.19 we see that $\mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)$ is a finitely generated projective B-module; hence it is also flat. Combining this with the injectivity just proved, we find that

$$\mathrm{Tor}^1_B\big(Q, \mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)/\,\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)\big) = 0$$

for every finitely generated B-module Q, and thus that

$$\mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)/\,\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)$$

is a finitely generated flat, and therefore finitely generated projective, B-module. Thus $\text{ker-Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)$ is a direct summand of the finitely generated projective B-module $\mathrm{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{A_f}\,\widehat{\otimes}_{A_f}B, \mathfrak{N}_{A_f}\,\widehat{\otimes}_{A_f}B)$, and so is itself a finitely generated projective B-module. ☐

4.2. **Families of extensions.** Let $\mathfrak{M}$ and $\mathfrak{N}$ be Breuil–Kisin modules with descent data and A-coefficients, so that $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is an A-module. Suppose that $\psi : V \to \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is a homomorphism of A-modules whose source is a projective A-module of finite rank. Then we may regard ψ as an element of

$$\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \otimes_A V^\vee = \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N} \otimes_A V^\vee),$$

and in this way ψ corresponds to an extension

$$0 \to \mathfrak{N} \otimes_A V^\vee \to \mathfrak{E} \to \mathfrak{M} \to 0, \tag{4.2.1}$$

which we refer to as the *family of extensions* of $\mathfrak{M}$ by $\mathfrak{N}$ parametrised by V (or by ψ, if we want to emphasise our choice of homomorphism). We let $\mathfrak{E}_v$ denote the pushforward of $\mathfrak{E}$ under the morphism $\mathfrak{N} \otimes_A V^\vee \to \mathfrak{N}$ given by evaluation on $v \in V$. In the special case that $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ itself is a projective A-module of finite rank, we can let V be $\mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ and take ψ be the identity map; in this case we refer to (4.2.1) as the *universal extension* of $\mathfrak{M}$ by $\mathfrak{N}$. The reason for this terminology is as follows: if $v \in \mathrm{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$, then $\mathfrak{E}_v$ is the extension of $\mathfrak{M}$ by $\mathfrak{N}$ corresponding to the element v.

Let $B := A[V^\vee]$ denote the symmetric algebra over A generated by $V^\vee$. The short exact sequence (4.2.1) is a short exact sequence of Breuil–Kisin modules with descent data, and so forming its u-adically completed tensor product with B over A, we obtain a short exact sequence

$$0 \to \mathfrak{N} \otimes_A V^\vee \,\widehat{\otimes}_A B \to \mathfrak{E}\,\widehat{\otimes}_A B \to \mathfrak{M}\,\widehat{\otimes}_A B \to 0$$

of Breuil–Kisin modules with descent data over B (see Lemma 2.1.6). Pushing this short exact sequence forward under the natural map

$$V^\vee\,\widehat{\otimes}_A B = V^\vee \otimes_A B \to B$$

induced by the inclusion of $V^\vee$ in B and the multiplication map $B \otimes_A B \to B$, we obtain a short exact sequence

$$0 \to \mathfrak{N} \widehat{\otimes}_A B \to \widetilde{\mathfrak{E}} \to \mathfrak{M} \widehat{\otimes}_A B \to 0 \tag{4.2.2}$$

of Breuil–Kisin modules with descent data over B, which we call the *family of extensions* of $\mathfrak{M}$ by $\mathfrak{N}$ parametrised by $\operatorname{Spec} B$ (which we note is (the total space of) the vector bundle over $\operatorname{Spec} A$ corresponding to the projective A-module V).

If $\alpha_v : B \to A$ is the morphism induced by the evaluation map $V^\vee \to A$ given by some element $v \in V$, then base-changing (4.2.2) by α_v, we recover the short exact sequence

$$0 \to \mathfrak{N} \to \mathfrak{E}_v \to \mathfrak{M} \to 0.$$

More generally, suppose that A is a $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let C be any A-algebra. Suppose that $\alpha_{\tilde{v}} : B \to C$ is the morphism induced by the evaluation map $V^\vee \to C$ corresponding to some element $\tilde{v} \in C \otimes_A V$. Then base-changing (4.2.2) by $\alpha_{\tilde{v}}$ yields a short exact sequence

$$0 \to \mathfrak{N} \widehat{\otimes}_A C \to \widetilde{\mathfrak{E}} \widehat{\otimes}_B C \to \mathfrak{M} \widehat{\otimes}_A C \to 0,$$

whose associated extension class corresponds to the image of $\tilde{v}$ under the natural morphism $C \otimes_A V \to C \otimes_A \operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N}) \cong \operatorname{Ext}^1_{\mathcal{K}(C)}(\mathfrak{M} \widehat{\otimes}_A C, \mathfrak{N} \otimes_A C)$, the first arrow being induced by ψ and the second arrow being the isomorphism of Proposition 4.1.15.

4.2.3. *The functor represented by a universal family.* We now suppose that the ring A and the Breuil–Kisin modules $\mathfrak{M}$ and $\mathfrak{N}$ have the following properties:

Assumption 4.2.4. Let A be a Noetherian and Jacobson $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and assume that for each maximal ideal $\mathfrak{m}$ of A, we have that

$$\operatorname{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M} \otimes_A \kappa(\mathfrak{m}), \mathfrak{N} \otimes_A \kappa(\mathfrak{m})\big) = \operatorname{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{N} \otimes_A \kappa(\mathfrak{m}), \mathfrak{M} \otimes_A \kappa(\mathfrak{m})\big) = 0.$$

By Corollary 4.1.19, this assumption implies in particular that $V := \operatorname{Ext}^1_{\mathcal{K}(A)}(\mathfrak{M}, \mathfrak{N})$ is projective of finite rank, and so we may form $\operatorname{Spec} B := \operatorname{Spec} A[V^\vee]$, which parametrised the universal family of extensions. We are then able to give the following precise description of the functor represented by $\operatorname{Spec} B$.

Proposition 4.2.5. *The scheme* $\operatorname{Spec} B$ *represents the functor which, to any $\mathcal{O}/\varpi^a$-algebra C, associates the set of isomorphism classes of tuples $(\alpha, \mathfrak{E}, \iota, \pi)$, where α is a morphism $\alpha : \operatorname{Spec} C \to \operatorname{Spec} A$, $\mathfrak{E}$ is a Breuil–Kisin module with descent data and coefficients in C, and ι and π are morphisms $\alpha^*\mathfrak{N} \to \mathfrak{E}$ and $\mathfrak{E} \to \alpha^*\mathfrak{M}$ respectively, with the property that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ is short exact.*

Proof. We have already seen that giving a morphism $\operatorname{Spec} C \to \operatorname{Spec} B$ is equivalent to giving the composite morphism $\alpha : \operatorname{Spec} C \to \operatorname{Spec} B \to \operatorname{Spec} A$, together with an extension class $[\mathfrak{E}] \in \operatorname{Ext}^1_{\mathcal{K}(C)}(\alpha^*\mathfrak{M}, \alpha^*\mathfrak{N})$. Thus to prove the proposition, we just have to show that any automorphism of $\mathfrak{E}$ which restricts to the identity on $\alpha^*\mathfrak{N}$ and induces the identity on $\alpha^*\mathfrak{M}$ is itself the identity on $\mathfrak{E}$. This follows from Corollary 4.1.18, together with Assumption 4.2.4. ☐

Fix an integer $h \geq 0$ so that $E(u)^h \in \operatorname{Ann}_{\mathfrak{S}_A}(\operatorname{coker} \Phi_{\mathfrak{M}}) \operatorname{Ann}_{\mathfrak{S}_A}(\operatorname{coker} \Phi_{\mathfrak{N}})$, so that by Lemma 4.1.1, every Breuil–Kisin module parametrised by $\operatorname{Spec} B$ has height at most h. There is a natural action of $\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m$ on $\operatorname{Spec} B$, given by rescaling each of ι and π. There is also an evident forgetful morphism $\operatorname{Spec} B \to \operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}$, given by forgetting ι and π, which is evidently invariant under the $\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m$-action. (Here and below, $\mathcal{C}^{\mathrm{dd},a}$ denotes the moduli stack defined in Section 3.9 for our fixed choice of h and for d equal to the sum of the ranks of $\mathfrak{M}$ and $\mathfrak{N}$.) We thus obtain a morphism

$$\operatorname{Spec} B \times_{\mathcal{O}} \mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m \to \operatorname{Spec} B \times_{\operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}} \operatorname{Spec} B. \tag{4.2.6}$$

Corollary 4.2.7. *Suppose that* $\operatorname{Aut}_{\mathcal{K}(C)}(\alpha^*\mathfrak{M}) = \operatorname{Aut}_{\mathcal{K}(C)}(\alpha^*\mathfrak{N}) = C^\times$ *for any morphism* $\alpha : \operatorname{Spec} C \to \operatorname{Spec} A$*. Then the morphism* (4.2.6) *is an isomorphism, and consequently the induced morphism*

$$[\operatorname{Spec} B/\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m] \to \operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}$$

is a finite type monomorphism.

Proof. By Proposition 4.2.5, a morphism

$$\operatorname{Spec} C \to \operatorname{Spec} B \times_{\operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}} \operatorname{Spec} B$$

corresponds to an isomorphism class of tuples $(\alpha, \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, where

- α is a morphism $\alpha : \operatorname{Spec} C \to \operatorname{Spec} A$,
- $\beta : \mathfrak{E} \to \mathfrak{E}'$ is an isomorphism of Breuil–Kisin modules with descent data and coefficients in C,
- $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$ and $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ are morphisms with the property that
$$0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$$
is short exact,
- $\iota' : \alpha^*\mathfrak{N} \to \mathfrak{E}'$ and $\pi' : \mathfrak{E}' \to \alpha^*\mathfrak{M}$ are morphisms with the property that
$$0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} \alpha^*\mathfrak{M} \to 0$$
is short exact.

Assumption 4.2.4 and Corollary 4.1.18 together show that $\operatorname{Hom}_{\mathcal{K}(C)}(\alpha^*\mathfrak{N}, \alpha^*\mathfrak{M}) = 0$. It follows that the composite $\alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\beta} \mathfrak{E}'$ factors through ι', and the induced endomorphism of $\alpha^*\mathfrak{N}$ is injective. Reversing the roles of $\mathfrak{E}$ and $\mathfrak{E}'$, we see that it is in fact an automorphism of $\alpha^*\mathfrak{N}$, and it follows easily that β also induces an automorphism of $\alpha^*\mathfrak{M}$. Again, Assumption 4.2.4 and Proposition 4.1.18 together show that $\operatorname{Hom}_{\mathcal{K}(C)}(\alpha^*\mathfrak{M}, \alpha^*\mathfrak{N}) = 0$, from which it follows easily that β is determined by the automorphisms of $\alpha^*\mathfrak{M}$ and $\alpha^*\mathfrak{N}$ that it induces.

Since $\operatorname{Aut}_{\mathcal{K}(C)}(\alpha^*\mathfrak{M}) = \operatorname{Aut}_{\mathcal{K}(C)}(\alpha^*\mathfrak{N}) = C^\times$ by assumption, we see that $\beta \circ \iota, \iota'$ and $\pi, \pi' \circ \beta$ differ only by the action of $\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m$, so the first claim of the corollary follows. The claim regarding the monomorphism is immediate from Lemma 4.2.8 below. Finally, note that $[\operatorname{Spec} B/\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m]$ is of finite type over $\operatorname{Spec} A$, while $\mathcal{C}^{\mathrm{dd},a}$ has finite type diagonal. It follows that the morphism $[\operatorname{Spec} B/\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m] \to \operatorname{Spec} A \times_{\mathcal{O}} \mathcal{C}^{\mathrm{dd},a}$ is of finite type, as required. $\square$

Lemma 4.2.8. *Let X be a scheme over a base scheme S, let G be a smooth affine group scheme over S, and let $\rho : X \times_S G \to X$ be a* (*right*) *action of G on X. Let $X \to \mathcal{Y}$ be a G-equivariant morphism, whose target is an algebraic stack over S on which G acts trivially. Then the induced morphism*

$$[X/G] \to \mathcal{Y}$$

is a monomorphism if and only if the natural morphism

$$X \times_S G \to X \times_{\mathcal{Y}} X$$

(*induced by the morphisms* $\mathrm{pr}_1, \rho : X \times_S G \to X$) *is an isomorphism.*

Proof. We have a Cartesian diagram as follows.

$$\begin{tikzcd} X \times_S G \arrow[r] \arrow[d] & X \times_{\mathcal{Y}} X \arrow[d] \\ {[X/G]} \arrow[r] & {[X/G] \times_{\mathcal{Y}} [X/G]} \end{tikzcd}$$

The morphism $[X/G] \to \mathcal{Y}$ is a monomorphism if and only if the bottom horizontal morphism of this square is an isomorphism; since the right hand vertical arrow is a smooth surjection, this is the case if and only if the top horizontal morphism is an isomorphism, as required. ☐

4.3. **Families of extensions of rank one Breuil–Kisin modules.** In this section we construct universal families of extensions of rank one Breuil–Kisin modules. We will use these rank two families to study our moduli spaces of Breuil–Kisin modules, and the corresponding spaces of étale φ-modules. We show how to compute the dimensions of these universal families; in the subsequent sections, we will combine these results with explicit calculations to determine the irreducible components of our moduli spaces. In particular, we will show that each irreducible component has a dense open substack given by a family of extensions.

4.3.1. *Universal unramified twists.* Fix a free Breuil–Kisin module with descent data $\mathfrak{M}$ over $\mathbf{F}$, and write Φ_i for $\Phi_{\mathfrak{M},i} : \varphi^*(\mathfrak{M}_{i-1}) \to \mathfrak{M}_i$. (Here we are using the notation of Section 2.1, so that $\mathfrak{M}_i = e_i\mathfrak{M}$ is cut out by the idempotent e_i of Section 1.7.) We will construct the "universal unramified twist" of $\mathfrak{M}$.

Definition 4.3.2. If Λ is an $\mathbf{F}$-algebra, and if $\lambda \in \Lambda^\times$, then we define $\mathfrak{M}_{\Lambda,\lambda}$ to be the free Breuil–Kisin module with descent data and Λ-coefficients whose underlying $\mathfrak{S}_\Lambda[\mathrm{Gal}(K'/K)]$-module is equal to $\mathfrak{M}\widehat{\otimes}_{\mathbf{F}}\Lambda$ (so the usual base change of $\mathfrak{M}$ to Λ), and for which $\Phi_{\mathfrak{M}_{\Lambda,\lambda}} : \varphi^*\mathfrak{M}_{\Lambda,\lambda} \to \mathfrak{M}_{\Lambda,\lambda}$ is defined via the f'-tuple $(\lambda\Phi_0, \Phi_1, \dots, \Phi_{f'-1})$. We refer to $\mathfrak{M}_{\Lambda,\lambda}$ as the *unramified twist* of $\mathfrak{M}$ by λ over Λ.

If M is a free étale φ-module with descent data, then we define $M_{\Lambda,\lambda}$ in the analogous fashion. If we write $X = \operatorname{Spec}\Lambda$, then we will sometimes write $\mathfrak{M}_{X,\lambda}$ (resp. $M_{X,\lambda}$) for $\mathfrak{M}_{\Lambda,\lambda}$ (resp. $M_{\Lambda,\lambda}$).

As usual, we write $\mathbf{G}_m := \operatorname{Spec}\mathbf{F}[x, x^{-1}]$. We may then form the rank one Breuil–Kisin module with descent data $\mathfrak{M}_{\mathbf{G}_m,x}$, which is the universal instance of an unramified twist: given $\lambda \in \Lambda^\times$, there is a corresponding morphism $\operatorname{Spec}\Lambda \to \mathbf{G}_m$ determined by the requirement that $x \in \Gamma(\mathbf{G}_m, \mathcal{O}_{\mathbf{G}_m}^\times)$ pulls-back to λ, and $\mathfrak{M}_{X,\lambda}$ is obtained by pulling back $\mathfrak{M}_{\mathbf{G}_m,x}$ under this morphism (that is, by base changing under the corresponding ring homomorphism $\mathbf{F}[x, x^{-1}] \to \Lambda$).

Lemma 4.3.3. *If $\mathfrak{M}_\Lambda$ is a Breuil–Kisin module of rank one with Λ-coefficients, then* $\operatorname{End}_{\mathcal{K}(\Lambda)}(\mathfrak{M}) = \Lambda$. *Similarly, if M_Λ is a étale φ-module of rank one with Λ-coefficients, then* $\operatorname{End}_{\mathcal{K}(\Lambda)}(M_\Lambda) = \Lambda$.

Proof. We give the proof for M_Λ, the argument for $\mathfrak{M}_\Lambda$ being essentially identical. One reduces easily to the case where M_Λ is free. Since an endomorphism ψ of M_Λ is in particular an endomorphism of the underlying $\mathfrak{S}_\Lambda[1/u]$-module, we see that there is some $\lambda \in \mathfrak{S}_\Lambda[1/u]$ such that ψ is given by multiplication by λ. The commutation relation with Φ_{M_Λ} means that we must have $\varphi(\lambda) = \lambda$, so that certainly (considering the powers of u in λ of lowest negative and positive degrees) $\lambda \in W(k') \otimes_{\mathbf{Z}_p} \Lambda$, and in fact $\lambda \in \Lambda$. Conversely, multiplication by any element of Λ is evidently an endomorphism of M_Λ, as required. $\square$

Lemma 4.3.4. *Let κ be a field of characteristic p, and let M_κ, N_κ be étale φ-modules of rank one with κ-coefficients and descent data. Then any nonzero element of $\mathrm{Hom}_{\mathcal{K}(\kappa)}(M_\kappa, N_\kappa)$ is an isomorphism.*

Proof. Since $\kappa((u))$ is a field, it is enough to show that if one of the induced maps $M_{\kappa,i} \to N_{\kappa,i}$ is nonzero, then they all are; but this follows from the commutation relation with φ. $\square$

Lemma 4.3.5. *If $\lambda, \lambda' \in \Lambda^\times$ and $\mathfrak{M}_{\Lambda,\lambda} \cong \mathfrak{M}_{\Lambda,\lambda'}$ (as Breuil–Kisin modules with descent data over Λ), then $\lambda = \lambda'$. Similarly, if $M_{\Lambda,\lambda} \cong M_{\Lambda,\lambda'}$, then $\lambda = \lambda'$.*

Proof. Again, we give the proof for M, the argument for $\mathfrak{M}$ being essentially identical. Write $M_i = \mathbf{F}((u))m_i$, and write $\Phi_i(1 \otimes m_{i-1}) = \theta_i m_i$, where $\theta_i \neq 0$. There are $\mu_i \in \Lambda[[u]][1/u]$ such that the given isomorphism $M_{\Lambda,\lambda} \cong M_{\Lambda,\lambda'}$ takes m_i to $\mu_i m_i$. The commutation relation between the given isomorphism and Φ_M imposes the condition

$$\lambda_i \mu_i \theta_i m_i = \lambda'_i \varphi(\mu_{i-1})\theta_i m_i$$

where λ_i (resp. λ'_i) equals 1 unless $i = 0$, when it equals λ (resp. λ').

Thus we have $\mu_i = (\lambda'_i/\lambda_i)\varphi(\mu_{i-1})$, so that in particular $\mu_0 = (\lambda'/\lambda)\varphi^{f'}(\mu_0)$. Considering the powers of u in μ_0 of lowest negative and positive degrees we conclude that $\mu_0 \in W(k') \otimes \Lambda$; but then $\mu_0 = \varphi^{f'}(\mu_0)$, so that $\lambda' = \lambda$, as required. $\square$

Remark 4.3.6. If $\mathfrak{M}$ has height at most h, and we let $\mathcal{C}$ (temporarily) denote the moduli stack of rank one Breuil–Kisin modules of height at most h with $\mathbf{F}$-coefficients and descent data then Lemma 4.3.5 can be interpreted as saying that the morphism $\mathbf{G}_m \to \mathcal{C}$ that classifies $\mathfrak{M}_{\mathbf{G}_m,x}$ is a monomorphism, i.e. the diagonal morphism $\mathbf{G}_m \to \mathbf{G}_m \times_{\mathcal{C}} \mathbf{G}_m$ is an isomorphism. Similarly, the morphism $\mathbf{G}_m \to \mathcal{R}$ (where we temporarily let $\mathcal{R}$ denote the moduli stack of rank one étale φ-modules with $\mathbf{F}$-coefficients and descent data) that classifies $M_{\mathbf{G}_m,x}$ is a monomorphism.

Now choose another rank one Breuil–Kisin module with descent data $\mathfrak{N}$ over $\mathbf{F}$. Let (x, y) denote the standard coordinates on $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$, and consider the rank one Breuil–Kisin modules with descent data $\mathfrak{M}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, x}$ and $\mathfrak{N}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, y}$ over $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$.

Lemma 4.3.7. *There is a non-empty irreducible affine open subset $\mathrm{Spec}\, A^{\mathrm{dist}}$ of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$ whose finite type points are exactly the maximal ideals $\mathfrak{m}$ of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$ such that*

$$\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\big) = 0$$

(where we have written $\bar{x}$ and $\bar{y}$ to denote the images of x and y in $\kappa(\mathfrak{m})^\times$).

Furthermore, if R is any finite-type A^{dist}-algebra, and if $\mathfrak{m}$ is any maximal ideal of R, then

$$\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = \mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\big) = 0,$$

and also

$$\mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}, \mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}\big) = \mathrm{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u], \mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u]\big) = 0.$$

In particular, Assumption 4.2.4 is satisfied by $\mathfrak{M}_{A^{\mathrm{dist}},x}$ and $\mathfrak{N}_{A^{\mathrm{dist}},y}$.

Proof. If $\operatorname{Hom}\bigl(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\bigr) = 0$ for all maximal ideals $\mathfrak{m}$ of $\mathbf{F}[x,y,x^{-1},y^{-1}]$, then we are done: $\operatorname{Spec} A^{\mathrm{dist}} = \mathbf{G}_m \times \mathbf{G}_m$. Otherwise, we see that for some finite extension $\mathbf{F}'/\mathbf{F}$ and some $a, a' \in \mathbf{F}'$, we have a non-zero morphism $\mathfrak{M}_{\mathbf{F}',a}[1/u] \to \mathfrak{N}_{\mathbf{F}',a'}[1/u]$. By Lemma 4.3.4, this morphism must in fact be an isomorphism. Since $\mathfrak{M}$ and $\mathfrak{N}$ are both defined over $\mathbf{F}$, we furthermore see that the ratio a'/a lies in $\mathbf{F}$. We then let $\operatorname{Spec} A^{\mathrm{dist}}$ be the affine open subset of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$ where $a'x \neq ay$; the claimed property of $\operatorname{Spec} A^{\mathrm{dist}}$ then follows easily from Lemma 4.3.5.

For the remaining statements of the lemma, note that if $\mathfrak{m}$ is a maximal ideal in a finite type A^{dist}-algebra, then its pull-back to A^{dist} is again a maximal ideal $\mathfrak{m}'$ of A^{dist} (since A^{dist} is Jacobson), and the vanishing of

$$\operatorname{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\bigl(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\bigr)$$

follows from the corresponding statement for $\kappa(\mathfrak{m}')$, together with Lemma 4.1.20.

Inverting u induces an embedding

$$\operatorname{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\bigl(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\bigr) \hookrightarrow \operatorname{Hom}_{\mathcal{K}(\kappa(\mathfrak{m}))}\bigl(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}[1/u], \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}[1/u]\bigr),$$

and so certainly the vanishing of the target implies the vanishing of the source.

The statements in which the roles of $\mathfrak{M}$ and $\mathfrak{N}$ are reversed follow from Lemma 4.3.4. □

Define $T := \operatorname{Ext}^1_{\mathcal{K}(\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m)}\bigl(\mathfrak{M}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, x}, \mathfrak{M}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, y}\bigr)$; it follows from Proposition 4.1.13 that T is finitely generated over $\mathbf{F}[x,x^{-1},y,y^{-1}]$, while Proposition 4.1.15 shows that $T_{A^{\mathrm{dist}}} := T \otimes_{\mathbf{F}[x^{\pm 1},y^{\pm 1}]} A^{\mathrm{dist}}$ is naturally isomorphic to $\operatorname{Ext}^1_{\mathcal{K}(A^{\mathrm{dist}})}\bigl(\mathfrak{M}_{A^{\mathrm{dist}},x}, \mathfrak{N}_{A^{\mathrm{dist}},y}\bigr)$. (Here and elsewhere we abuse notation by writing x, y for $x|_{A^{\mathrm{dist}}}$, $y|_{A^{\mathrm{dist}}}$.) Corollary 4.1.19 and Lemma 4.3.7 show that $T_{A^{\mathrm{dist}}}$ is in fact a finitely generated projective A^{dist}-module. If, for any A^{dist}-algebra B, we write $T_B := T_{A^{\mathrm{dist}}} \otimes_{A^{\mathrm{dist}}} B \xrightarrow{\sim} T \otimes_{\mathbf{F}[x^{\pm 1},y^{\pm 1}]} B$, then Proposition 4.1.15 again shows that $T_B \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{K}(B)}\bigl(\mathfrak{M}_{B,x}, \mathfrak{N}_{B,y}\bigr)$.

By Propositions 4.1.34 and 4.1.35, together with Lemma 4.3.7, there is a nonempty (so dense) affine open subset $\operatorname{Spec} A^{\text{k-free}}$ of $\operatorname{Spec} A^{\mathrm{dist}}$ with the properties that

$$U_{A^{\text{k-free}}} := \text{ker-}\operatorname{Ext}^1_{\mathcal{K}(A^{\text{k-free}})}(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y})$$

and

$$\begin{aligned} &T_{A^{\text{k-free}}}/U_{A^{\text{k-free}}} \\ &\quad \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{K}(A^{\text{k-free}})}(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y}) / \text{ker-}\operatorname{Ext}^1_{\mathcal{K}(A^{\text{k-free}})}(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y}) \end{aligned}$$

are finitely generated and projective over $A^{\text{k-free}}$, and furthermore so that for all finitely generated $A^{\text{k-free}}$-algebras B, the formation of $\text{ker-}\operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{B,x}, \mathfrak{N}_{B,y})$ and $\operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{B,x}, \mathfrak{N}_{B,y}) / \text{ker-}\operatorname{Ext}^1_{\mathcal{K}(B)}(\mathfrak{M}_{B,x}, \mathfrak{N}_{B,y})$ is compatible with base change from $U_{A^{\text{k-free}}}$ and $T_{A^{\text{k-free}}}/U_{A^{\text{k-free}}}$ respectively.

We choose a finite rank projective module V over $\mathbf{F}[x,x^{-1},y,y^{-1}]$ admitting a surjection $V \to T$. Thus, if we write $V_{A^{\mathrm{dist}}} := V \otimes_{\mathbf{F}[x^{\pm 1},y^{\pm 1}]} A^{\mathrm{dist}}$, then the induced morphism $V_{A^{\mathrm{dist}}} \to T_{A^{\mathrm{dist}}}$ is a (split) surjection of A^{dist}-modules.

Following the prescription of Subsection 4.2, we form the symmetric algebra $B^{\mathrm{twist}} := \mathbf{F}[x^{\pm 1}, y^{\pm 1}][V^\vee]$, and construct the family of extensions $\widetilde{\mathfrak{E}}$ over $\operatorname{Spec} B^{\mathrm{twist}}$. We may similarly form the symmetric algebras $B^{\mathrm{dist}} := A^{\mathrm{dist}}[T^\vee_{A^{\mathrm{dist}}}]$ and $B^{\text{k-free}} := A^{\text{k-free}}[T^\vee_{A^{\text{k-free}}}]$, and construct the families of extensions $\widetilde{\mathfrak{E}}^{\mathrm{dist}}$ and $\widetilde{\mathfrak{E}}^{\text{k-free}}$ over $\operatorname{Spec} B^{\mathrm{dist}}$ and $\operatorname{Spec} B^{\text{k-free}}$ respectively. Since $T_{A^{\text{k-free}}}/U_{A^{\text{k-free}}}$ is projective, the natural morphism $T^\vee_{A^{\text{k-free}}} \to U^\vee_{A^{\text{k-free}}}$ is surjective, and hence $C^{\text{k-free}} := A[U^\vee_{A^{\text{k-free}}}]$ is a quotient of $B^{\text{k-free}}$; geometrically, $\operatorname{Spec} C^{\text{k-free}}$ is a subbundle of the vector bundle $\operatorname{Spec} B^{\text{k-free}}$ over $\operatorname{Spec} A$.

We write $X := \operatorname{Spec} B^{\text{k-free}} \setminus \operatorname{Spec} C^{\text{k-free}}$; it is an open subscheme of the vector bundle $\operatorname{Spec} B^{\text{k-free}}$. The restriction of $\widetilde{\mathfrak{E}}'$ to X is the universal family of extensions over A which do not split after inverting u.

Remark 4.3.8. Since $\operatorname{Spec} A^{\text{dist}}$ and $\operatorname{Spec} A^{\text{k-free}}$ are irreducible, each of the vector bundles $\operatorname{Spec} B^{\text{dist}}$ and $\operatorname{Spec} B^{\text{k-free}}$ is also irreducible. In particular, $\operatorname{Spec} B^{\text{k-free}}$ is Zariski dense in $\operatorname{Spec} B^{\text{dist}}$, and if X is non-empty, then it is Zariski dense in each of $\operatorname{Spec} B^{\text{k-free}}$ and $\operatorname{Spec} B^{\text{dist}}$. Similarly, $\operatorname{Spec} B^{\text{twist}} \times_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m} \operatorname{Spec} A^{\text{dist}}$ is Zariski dense in $\operatorname{Spec} B^{\text{twist}}$.

The surjection $V_{A^{\text{dist}}} \to T_{A^{\text{dist}}}$ induces a surjection of vector bundles $\pi : \operatorname{Spec} B^{\text{twist}} \times_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m} \operatorname{Spec} A^{\text{dist}} \to \operatorname{Spec} B^{\text{dist}}$ over $\operatorname{Spec} A^{\text{dist}}$, and there is a natural isomorphism

$$\pi^* \widetilde{\mathfrak{E}}^{\text{dist}} \xrightarrow{\sim} \widetilde{\mathfrak{E}} \widehat{\otimes}_{\mathbf{F}[x^{\pm 1}, y^{\pm 1}]} A^{\text{dist}}. \tag{4.3.9}$$

The rank two Breuil–Kisin module with descent data $\widetilde{\mathfrak{E}}$ is classified by a morphism $\xi : \operatorname{Spec} B^{\text{twist}} \to \mathcal{C}^{\text{dd},1}$; similarly, the rank two Breuil–Kisin module with descent data $\widetilde{\mathfrak{E}}^{\text{dist}}$ is classified by a morphism $\xi^{\text{dist}} : \operatorname{Spec} B^{\text{dist}} \to \mathcal{C}^{\text{dd},1}$. If we write $\xi_{A^{\text{dist}}}$ for the restriction of ξ to the open subset $\operatorname{Spec} B^{\text{twist}} \times_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m} \operatorname{Spec} A^{\text{dist}}$ of $\operatorname{Spec} B^{\text{twist}}$, then the isomorphism (4.3.9) shows that $\xi^{\text{dist}} \circ \pi = \xi_{A^{\text{dist}}}$. We also write $\xi^{\text{k-free}}$ for the restriction of ξ^{dist} to $\operatorname{Spec} B^{\text{k-free}}$, and ξ_X for the restriction of $\xi^{\text{k-free}}$ to X.

Lemma 4.3.10. *The scheme-theoretic images* (*in the sense of* [EG19b, Def. 3.1.4]) *of* $\xi : \operatorname{Spec} B^{\text{twist}} \to \mathcal{C}^{\text{dd},1}$*,* $\xi^{\text{dist}} : \operatorname{Spec} B^{\text{dist}} \to \mathcal{C}^{\text{dd},1}$*, and* $\xi^{\text{k-free}} : \operatorname{Spec} B^{\text{k-free}} \to \mathcal{C}^{\text{dd},1}$ *all coincide; in particular, the scheme-theoretic image of* ξ *is independent of the choice of surjection* $V \to T$*, and the scheme-theoretic image of* $\xi^{\text{k-free}}$ *is independent of the choice of* $A^{\text{k-free}}$*. If* X *is non-empty, then the scheme-theoretic image of* $\xi_X : X \to \mathcal{C}^{\text{dd},1}$ *also coincides with these other scheme-theoretic images, and is independent of the choice of* $A^{\text{k-free}}$*.*

Proof. This follows from the various observations about Zariski density made in Remark 4.3.8. ∎

Definition 4.3.11. We let $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ denote the scheme-theoretic image of $\xi^{\text{dist}} : \operatorname{Spec} B^{\text{dist}} \to \mathcal{C}^{\text{dd},1}$, and we let $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$ denote the scheme-theoretic image of the composite $\xi^{\text{dist}} : \operatorname{Spec} B^{\text{dist}} \to \mathcal{C}^{\text{dd},1} \to \mathcal{Z}^{\text{dd},1}$. Equivalently, $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$ is the scheme-theoretic image of the composite $\operatorname{Spec} B^{\text{dist}} \to \mathcal{C}^{\text{dd},1} \to \mathcal{R}^{\text{dd},1}$ (*cf.* [EG19b, Prop. 3.2.31]), and the scheme-theoretic image of $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ under the morphism $\mathcal{C}^{\text{dd},1} \to \mathcal{Z}^{\text{dd},1}$. (Note that Lemma 4.3.10 provides various other alternative descriptions of $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ (and therefore also $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$) as a scheme-theoretic image.)

Remark 4.3.12. Note that $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ and $\overline{\mathcal{Z}}(\mathfrak{M}, \mathfrak{N})$ are both reduced (because they are each defined as a scheme-theoretic image of $\operatorname{Spec} B^{\text{dist}}$, which is reduced by definition).

As well as scheme-theoretic images, as in the preceding Lemma and Definition, we will need to consider images of underlying topological spaces.

Lemma 4.3.13. *The image of the morphism on underlying topological spaces* $|\operatorname{Spec} B^{\text{twist}}| \to |\mathcal{C}^{\text{dd},1}|$ *induced by* ξ *is a constructible subset of* $|\mathcal{C}^{\text{dd},1}|$*, and is independent of the choice of* V*.*

Proof. The fact that the image of $|\operatorname{Spec} B^{\text{twist}}|$ is a constructible subset of $|\mathcal{C}^{\text{dd},1}|$ follows from the fact that ξ is a morphism of finite presentation between Noetherian stacks; see [Ryd11, App. D]. Suppose now that V' is another choice of finite rank projective $\mathbf{F}[x, x^{-1}, y, y^{-1}]$-module surjecting onto T. Choose a finite rank projective module surjecting onto each of V and V', compatible with the given surjections of each these sheaves onto T. (E.g. one could take $W = V \oplus V'$.) Thus it suffices to prove the independence claim of the lemma in the case when V' admits a surjection onto V (compatible with the maps of each of V and V' onto T). If we write $B' := \mathbf{F}[x^{\pm 1}, y^{\pm 1}][(V')^\vee]$, then the natural morphism $\operatorname{Spec} B' \to \operatorname{Spec} B^{\text{twist}}$ is a surjection, and the morphism $\xi' : \operatorname{Spec} B' \to \mathcal{C}^{\text{dd},1}$ is the composite of this surjection with the morphism ξ. Thus indeed the images of $|\operatorname{Spec} B'|$ and of $|\operatorname{Spec} B^{\text{twist}}|$ coincide as subsets of $|\mathcal{C}^{\text{dd},1}|$. $\square$

Definition 4.3.14. We write $|\mathcal{C}(\mathfrak{M}, \mathfrak{N})|$ to denote the constructible subset of $|\mathcal{C}^{\text{dd},1}|$ described in Lemma 4.3.13.

Remark 4.3.15. We caution the reader that we don't define a substack $\mathcal{C}(\mathfrak{M}, \mathfrak{N})$ of $\mathcal{C}^{\text{dd},1}$. Rather, we have defined a closed substack $\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})$ of $\mathcal{C}^{\text{dd},1}$, and a constructible subset $|\mathcal{C}(\mathfrak{M}, \mathfrak{N})|$ of $|\mathcal{C}^{\text{dd},1}|$. It follows from the constructions that $|\overline{\mathcal{C}}(\mathfrak{M}, \mathfrak{N})|$ is the closure in $|\mathcal{C}^{\text{dd},1}|$ of $|\mathcal{C}(\mathfrak{M}, \mathfrak{N})|$.

As in Subsection 4.2, there is a natural action of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$ on T, and hence on each of $\operatorname{Spec} B^{\text{dist}}$, $\operatorname{Spec} B^{\text{k-free}}$ and X, given by the action of $\mathbf{G}_m$ as automorphisms on each of $\mathfrak{M}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, x}$ and $\mathfrak{N}_{\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m, y}$ (which induces a corresponding action on T, hence on $T_{A^{\text{dist}}}$ and $T_{A^{\text{k-free}}}$, and hence on $\operatorname{Spec} B^{\text{dist}}$ and $\operatorname{Spec} B^{\text{k-free}}$). Thus we may form the corresponding quotient stacks $[\operatorname{Spec} B^{\text{dist}} / \mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$ and $[X / \mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$, each of which admits a natural morphism to $\mathcal{C}^{\text{dd},1}$.

Remark 4.3.16. Note that we are making use of two independent copies of $\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$; one parameterises the different unramified twists of $\mathfrak{M}$ and $\mathfrak{N}$, and the other the automorphisms of (the pullbacks of) $\mathfrak{M}$ and $\mathfrak{N}$.

Definition 4.3.17. We say that the pair $(\mathfrak{M}, \mathfrak{N})$ is *strict* if $\operatorname{Spec} A^{\text{dist}} = \mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m$.

Before stating and proving the main result of this subsection, we prove some lemmas (the first two of which amount to recollections of standard — and simple — facts).

Lemma 4.3.18. *If $\mathcal{X} \to \mathcal{Y}$ is a morphism of stacks over S, with $\mathcal{X}$ algebraic and of finite type over S, and $\mathcal{Y}$ having diagonal which is representable by algebraic spaces and of finite type, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic stack of finite type over S.*

Proof. The fact that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic stack follows from [Sta13, Tag 04TF]. Since composites of morphisms of finite type are of finite type, in order to show that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is of finite type over S, it suffices to show that the natural morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is of finite type. Since this morphism is the base-change of the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times_S \mathcal{Y}$, this follows by assumption. $\square$

Lemma 4.3.19. *The following conditions are equivalent:*
(1) $\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = 0$ *for all maximal ideals* $\mathfrak{m}$ *of* $A^{\text{k-free}}$.
(2) $U_{A^{\text{k-free}}} = 0$.
(3) $\operatorname{Spec} C^{\text{k-free}}$ *is the trivial vector bundle over* $\operatorname{Spec} A^{\text{k-free}}$.

Proof. Conditions (2) and (3) are equivalent by definition. Since the formation of $\text{ker-Ext}^1_{\mathcal{K}(A^{\text{k-free}})}(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y})$ is compatible with base change, and since $A^{\text{k-free}}$ is Jacobson, (1) is equivalent to the assumption that

$$\text{ker-Ext}^1_{\mathcal{K}(A^{\text{k-free}})}(\mathfrak{M}_{A^{\text{k-free}},x}, \mathfrak{N}_{A^{\text{k-free}},y}) = 0,$$

i.e. that $U_{A^{\text{k-free}}} = 0$, as required. ∎

Lemma 4.3.20. *If the equivalent conditions of Lemma 4.3.19 hold, then the natural morphism*

$$\operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{C}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}} \\ \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}}$$

is an isomorphism.

Proof. Since $\mathcal{C}^{\text{dd},1} \to \mathcal{R}^{\text{dd},1}$ is separated (being proper) and representable, the diagonal morphism $\mathcal{C}^{\text{dd},1} \to \mathcal{C}^{\text{dd},1} \times_{\mathcal{R}^{\text{dd},1}} \mathcal{C}^{\text{dd},1}$ is a closed immersion, and hence the morphism in the statement of the lemma is a closed immersion. Thus, in order to show that it is an isomorphism, it suffices to show that it induces a surjection on R-valued points, for any $\mathbf{F}$-algebra R. Since the source and target are of finite type over $\mathbf{F}$, by Lemma 4.3.18, we may in fact restrict attention to finite type R-algebras.

A morphism $\operatorname{Spec} R \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{C}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}}$ corresponds to an isomorphism class of tuples $(\alpha, \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, where

- α is a morphism $\alpha : \operatorname{Spec} R \to \operatorname{Spec} A^{\text{k-free}}$,
- $\beta : \mathfrak{E} \to \mathfrak{E}'$ is an isomorphism of Breuil–Kisin modules with descent data and coefficients in R,
- $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$, $\iota' : \alpha^*\mathfrak{N} \to \mathfrak{E}'$, $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ and $\pi' : \mathfrak{E}' \to \alpha^*\mathfrak{M}$ are morphisms with the properties that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ and $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} \alpha^*\mathfrak{M} \to 0$ are both short exact.

Similarly, a morphism $\operatorname{Spec} R \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}}$ corresponds to an isomorphism class of tuples $(\alpha, \mathfrak{E}, \mathfrak{E}', \beta, \iota, \iota', \pi, \pi')$, where

- α is a morphism $\alpha : \operatorname{Spec} R \to \operatorname{Spec} A^{\text{k-free}}$,
- $\mathfrak{E}$ and $\mathfrak{E}'$ are Breuil–Kisin modules with descent data and coefficients in R, and β is an isomorphism $\beta : \mathfrak{E}[1/u] \to \mathfrak{E}'[1/u]$ of etale φ-modules with descent data and coefficients in R,
- $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$, $\iota' : \alpha^*\mathfrak{N} \to \mathfrak{E}'$, $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ and $\pi' : \mathfrak{E}' \to \alpha^*\mathfrak{M}$ are morphisms with the properties that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ and $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} \alpha^*\mathfrak{M} \to 0$ are both short exact.

Thus to prove the claimed surjectivity, we have to show that, given a tuple $(\alpha, \mathfrak{E}, \mathfrak{E}', \beta, \iota, \iota', \pi, \pi')$ associated to a morphism $\operatorname{Spec} R \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\text{dd},1}} \operatorname{Spec} B^{\text{k-free}}$, the isomorphism β restricts to an isomorphism $\mathfrak{E} \to \mathfrak{E}'$.

By Lemma 4.3.19, the natural map $\operatorname{Ext}^1(\alpha^*\mathfrak{M}, \alpha^*\mathfrak{N}) \to \operatorname{Ext}^1_{\mathcal{K}(R)}(\alpha^*\mathfrak{M}[1/u], \alpha^*\mathfrak{N}[1/u])$ is injective; so the Breuil–Kisin modules $\mathfrak{E}$ and $\mathfrak{E}'$ are isomorphic. Arguing as in the proof of Corollary 4.2.7, we see that β is equivalent to the data of an R-point of $\mathbf{G}_m \times_{\mathcal{O}} \mathbf{G}_m$, corresponding to the automorphisms of $\alpha^*\mathfrak{M}[1/u]$ and $\alpha^*\mathfrak{N}[1/u]$ that it induces. These restrict to automorphisms of $\alpha^*\mathfrak{M}$ and $\alpha^*\mathfrak{N}$, so that (again by the proof of Corollary 4.2.7) β indeed restricts to an isomorphism $\mathfrak{E} \to \mathfrak{E}'$, as required. ∎

We now present the main result of this subsection.

Proposition 4.3.21. (1) *The morphism ξ^{dist} induces a morphism*

$$[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \mathcal{C}^{\mathrm{dd},1}, \tag{4.3.22}$$

which is representable by algebraic spaces, of finite type, and unramified, whose fibres over finite type points are of degree ≤ 2. In the strict case, this induced morphism is in fact a monomorphism, while in general, the restriction ξ_X of ξ^{dist} induces a finite type monomorphism

$$[X/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \hookrightarrow \mathcal{C}^{\mathrm{dd},1}. \tag{4.3.23}$$

(2) *If* $\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = 0$ *for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$, then the composite morphism*

$$[\operatorname{Spec} B^{\text{k-free}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \mathcal{C}^{\mathrm{dd},1} \to \mathcal{R}^{\mathrm{dd},1} \tag{4.3.24}$$

is a representable by algebraic spaces, of finite type, and unramified, with fibres of degree ≤ 2. In the strict case, this induced morphism is in fact a monomorphism, while in general, the composite morphism

$$[X/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \hookrightarrow \mathcal{C}^{\mathrm{dd},1} \to \mathcal{R}^{\mathrm{dd},1} \tag{4.3.25}$$

is a finite type monomorphism.

Remark 4.3.26. The failure of (4.3.22) to be a monomorphism in general is due, effectively, to the possibility that an extension $\mathfrak{E}$ of some $\mathfrak{M}_{R,x}$ by $\mathfrak{N}_{R,y}$ and an extension $\mathfrak{E}'$ of some $\mathfrak{M}_{R,x'}$ by $\mathfrak{N}_{R,y'}$ might be isomorphic as Breuil–Kisin modules while nevertheless $(x, y) \neq (x', y')$. As we will see in the proof, whenever this happens the map $\mathfrak{N}_{\Lambda,y} \to \mathfrak{E} \to \mathfrak{E}' \to \mathfrak{M}_{\Lambda,x'}$ is nonzero, and then $\mathfrak{E}' \otimes_R \kappa(\mathfrak{m})[1/u]$ is split for some maximal ideal $\mathfrak{m}$ of R. This explains why, to obtain a monomorphism, we can restrict either to the strict case or to the substack of extensions that are non-split after inverting u.

Remark 4.3.27. We have stated this proposition in the strongest form that we are able to prove, but in fact its full strength is not required in the subsequent applications. In particular, we don't need the precise bounds on the degrees of the fibres.

Proof of Proposition 4.3.21. By Corollary 4.2.7 (which we can apply because Assumption 4.2.4 is satisfied, by Lemma 4.3.7) the natural morphism $[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \operatorname{Spec} A^{\mathrm{dist}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}$ is a finite type monomorphism, and hence so is its restriction to the open substack $[X/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$ of its source.

Let us momentarily write $\mathcal{X}$ to denote either $[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$ or $[X/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$. To show that the finite type morphism $\mathcal{X} \to \mathcal{C}^{\mathrm{dd},1}$ is representable by algebraic spaces, resp. unramified, resp. a monomorphism, it suffices to show that the corresponding diagonal morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{C}^{\mathrm{dd},1}} \mathcal{X}$ is a monomorphism, resp. étale, resp. an isomorphism.

Now since $\mathcal{X} \to \operatorname{Spec} A^{\mathrm{dist}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}$ is a monomorphism, the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_{\operatorname{Spec} A^{\mathrm{dist}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}} \mathcal{X}$ *is* an isomorphism, and so it is equivalent to show that the morphism of products

$$\mathcal{X} \times_{\operatorname{Spec} A^{\mathrm{dist}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{C}^{\mathrm{dd},1}} \mathcal{X}$$

is a monomorphism, resp. étale, resp. an isomorphism. This is in turn equivalent to showing the corresponding properties for the morphisms

$$\operatorname{Spec} B^{\mathrm{dist}} \times_{\operatorname{Spec} A^{\mathrm{dist}} \times \mathcal{C}^{\mathrm{dd},1}} \operatorname{Spec} B^{\mathrm{dist}} \to \operatorname{Spec} B^{\mathrm{dist}} \times_{\mathcal{C}^{\mathrm{dd},1}} \operatorname{Spec} B^{\mathrm{dist}} \tag{4.3.28}$$

or

$$X \times_{\operatorname{Spec} A^{\mathrm{dist}} \times \mathcal{C}^{\mathrm{dd},1}} X \to X \times_{\mathcal{C}^{\mathrm{dd},1}} X. \tag{4.3.29}$$

Now each of these morphisms is a base-change of the diagonal $\operatorname{Spec} A^{\mathrm{dist}} \to \operatorname{Spec} A^{\mathrm{dist}} \times_{\mathbf{F}} \operatorname{Spec} A^{\mathrm{dist}}$, which is a closed immersion (affine schemes being separated), and so is itself a closed immersion. In particular, it is a monomorphism, and so we have proved the representability by algebraic spaces of each of (4.3.22) and (4.3.23). Since the source and target of each of these monomorphisms is of finite type over $\mathbf{F}$, by Lemma 4.3.18, in order to show that either of these monomorphisms is an isomorphism, it suffices to show that it induces a surjection on R-valued points, for arbitrary finite type $\mathbf{F}$-algebras R. Similarly, to check that the closed immersion (4.3.28) is étale, it suffices to verify that it is formally smooth, and for this it suffices to verify that it satisfies the infinitesimal lifting property with respect to square zero thickenings of finite type $\mathbf{F}$-algebras.

A morphism $\operatorname{Spec} R \to \operatorname{Spec} B^{\mathrm{dist}} \times_{\mathcal{C}^{\mathrm{dd},1}} \operatorname{Spec} B^{\mathrm{dist}}$ corresponds to an isomorphism class of tuples $(\alpha, \alpha', \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, where

- α, α' are morphisms $\alpha, \alpha' : \operatorname{Spec} R \to \operatorname{Spec} A^{\mathrm{dist}}$,
- $\beta : \mathfrak{E} \to \mathfrak{E}'$ is an isomorphism of Breuil–Kisin modules with descent data and coefficients in R,
- $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$, $\iota' : (\alpha')^*\mathfrak{N} \to \mathfrak{E}'$, $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ and $\pi' : \mathfrak{E}' \to (\alpha')^*\mathfrak{M}$ are morphisms with the properties that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ and $0 \to (\alpha')^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} (\alpha')^*\mathfrak{M} \to 0$ are both short exact.

We begin by proving that (4.3.28) satisfies the infinitesimal lifting criterion (when R is a finite type $\mathbf{F}$-algebra). Thus we assume given a square-zero ideal $I \subset R$, such that the induced morphism

$$\operatorname{Spec} R/I \to \operatorname{Spec} B^{\mathrm{dist}} \times_{\mathcal{C}^{\mathrm{dd},1}} \operatorname{Spec} B^{\mathrm{dist}}$$

factors through $\operatorname{Spec} B^{\mathrm{dist}} \times_{\operatorname{Spec} A^{\mathrm{dist}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}} \operatorname{Spec} B^{\mathrm{dist}}$. In terms of the data $(\alpha, \alpha', \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, we are assuming that α and α' coincide when restricted to $\operatorname{Spec} R/I$, and we must show that α and α' themselves coincide.

To this end, we consider the composite

$$\alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\beta} \mathfrak{E}' \xrightarrow{\pi'} (\alpha')^*\mathfrak{M}. \tag{4.3.30}$$

If we can show the vanishing of this morphism, then by reversing the roles of $\mathfrak{E}$ and $\mathfrak{E}'$, we will similarly deduce the vanishing of $\pi \circ \beta^{-1} \circ \iota'$, from which we can conclude that β induces an isomorphism between $\alpha^*\mathfrak{N}$ and $(\alpha')^*\mathfrak{N}$. Consequently, it also induces an isomorphism between $\alpha^*\mathfrak{M}$ and $(\alpha')^*\mathfrak{M}$, so it follows from Lemma 4.3.5 that $\alpha = \alpha'$, as required.

We show the vanishing of (4.3.30). Suppose to the contrary that it doesn't vanish, so that we have a non-zero morphism $\alpha^*\mathfrak{N} \to (\alpha')^*\mathfrak{M}$. It follows from Proposition 4.1.17 that, for some maximal ideal $\mathfrak{m}$ of R, there exists a non-zero morphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m}) \to (\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}).$$

By assumption α and α' coincide modulo I. Since $I^2 = 0$, there is an inclusion $I \subset \mathfrak{m}$, and so in particular we find that

$$(\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}) \xrightarrow{\sim} \alpha^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}).$$

Thus there exists a non-zero morphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m}) \to \alpha^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}).$$

Then, by Lemma 4.3.4, after inverting u we obtain an isomorphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u] \xrightarrow{\sim} \alpha^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m})[1/u],$$

contradicting the assumption that α maps $\operatorname{Spec} R$ into $\operatorname{Spec} A^{\mathrm{dist}}$. This completes the proof that (4.3.28) is formally smooth, and hence that (4.3.22) is unramified.

We next show that, in the strict case, the closed immersion (4.3.28) is an isomorphism, and thus that (4.3.22) is actually a monomorphism. As noted above, it suffices to show that (4.3.28) induces a surjection on R-valued points for finite type $\mathbf{F}$-algebras R, which in terms of the data $(\alpha, \alpha', \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, amounts to showing that necessarily $\alpha = \alpha'$. Arguing just as we did above, it suffices show the vanishing of (4.3.30).

Again, we suppose for the sake of contradiction that (4.3.30) does not vanish. It then follows from Proposition 4.1.17 that for some maximal ideal $\mathfrak{m}$ of R there exists a non-zero morphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m}) \to (\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m}).$$

Then, by Lemma 4.3.4, after inverting u we obtain an isomorphism

$$\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u] \xrightarrow{\sim} (\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m})[1/u]. \tag{4.3.31}$$

In the strict case, such an isomorphism cannot exist by assumption, and thus (4.3.30) must vanish.

We now turn to proving that (4.3.29) is an isomorphism. Just as in the preceding arguments, it suffices to show that (4.3.30) vanishes, and if not then we obtain an isomorphism (4.3.31). Since we are considering points of $X \times X$, we are given that the induced extension $\mathfrak{E}' \otimes_R \kappa(\mathfrak{m})[1/u]$ is non-split, so that the base change of the morphism (4.3.30) from $R[[u]]$ to $\kappa(\mathfrak{m})((u))$ must vanish. Consequently the composite $\beta \circ \iota$ induces a non-zero morphism $\alpha^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u] \to (\alpha')^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u]$, which, by Lemma 4.3.4, must in fact be an isomorphism. Comparing this isomorphism with the isomorphism (4.3.31), we find that $(\alpha')^*(\mathfrak{N}) \otimes_R \kappa(\mathfrak{m})[1/u]$ and $(\alpha')^*(\mathfrak{M}) \otimes_R \kappa(\mathfrak{m})[1/u]$ are isomorphic, contradicting the fact that α' maps $\operatorname{Spec} R$ to $\operatorname{Spec} A^{\mathrm{dist}}$. Thus in fact the composite (4.3.30) must vanish, and we have completed the proof that (4.3.23) is a monomorphism.

To complete the proof of part (1) of the proposition, we have to show that the fibres of (4.3.22) are of degree at most 2. We have already observed that $[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \operatorname{Spec} A^{\mathrm{dist}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}$ is a monomorphism, so it is enough to check that given a finite extension $\mathbf{F}'/\mathbf{F}$ and an isomorphism class of tuples $(\alpha, \alpha', \beta : \mathfrak{E} \to \mathfrak{E}', \iota, \iota', \pi, \pi')$, where

- α, α' are distinct morphisms $\alpha, \alpha' : \operatorname{Spec} \mathbf{F}' \to \operatorname{Spec} A^{\mathrm{dist}}$,
- $\beta : \mathfrak{E} \to \mathfrak{E}'$ is an isomorphism of Breuil–Kisin modules with descent data and coefficients in $\mathbf{F}'$,

- $\iota : \alpha^*\mathfrak{N} \to \mathfrak{E}$, $\iota' : (\alpha')^*\mathfrak{N} \to \mathfrak{E}'$, $\pi : \mathfrak{E} \to \alpha^*\mathfrak{M}$ and $\pi' : \mathfrak{E}' \to (\alpha')^*\mathfrak{M}$ are morphisms with the properties that $0 \to \alpha^*\mathfrak{N} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \alpha^*\mathfrak{M} \to 0$ and $0 \to (\alpha')^*\mathfrak{N} \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} (\alpha')^*\mathfrak{M} \to 0$ are both short exact.

then α' is determined by the data of α and $\mathfrak{E}$. To see this, note that since we are assuming that $\alpha' \neq \alpha$, the arguments above show that (4.3.30) does not vanish, so that (since $\mathbf{F}'$ is a field), we have an isomorphism $\alpha^*\mathfrak{N}[1/u] \xrightarrow{\sim} (\alpha')^*\mathfrak{M}[1/u]$. Since we are over A^{dist}, it follows that $\mathfrak{E}[1/u] \cong \mathfrak{E}'[1/u]$ is split, and that we also have an isomorphism $\alpha^*\mathfrak{M}[1/u] \xrightarrow{\sim} (\alpha')^*\mathfrak{N}[1/u]$. Thus if α'' is another possible choice for α', we have $(\alpha'')^*\mathfrak{M}[1/u] \xrightarrow{\sim} (\alpha')^*\mathfrak{M}[1/u]$ and $(\alpha'')^*\mathfrak{N}[1/u] \xrightarrow{\sim} (\alpha')^*\mathfrak{N}[1/u]$, whence $\alpha'' = \alpha'$ by Lemma 4.3.5, as required.

We turn to proving (2), and thus assume that

$$\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = 0$$

for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$.

Lemma 4.3.20 shows that

$$\operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{C}^{\mathrm{dd},1}} \operatorname{Spec} B^{\text{k-free}} \to \operatorname{Spec} B^{\text{k-free}} \times_{\operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\mathrm{dd},1}} \operatorname{Spec} B^{\text{k-free}}$$

is an isomorphism, from which we deduce that

$$[\operatorname{Spec} B^{\text{k-free}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \operatorname{Spec} A^{\text{k-free}} \times_{\mathbf{F}} \mathcal{R}^{\mathrm{dd},1}$$

is a monomorphism. Using this as input, the claims of (2) may be proved in an essentially identical fashion to those of (1). □

Corollary 4.3.32. *The dimension of $\overline{\mathcal{C}}(\mathfrak{M},\mathfrak{N})$ is equal to the rank of $T_{A^{\mathrm{dist}}}$ as a projective A^{dist}-module. If*

$$\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = 0$$

for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$, then the dimension of $\overline{\mathcal{Z}}(\mathfrak{M},\mathfrak{N})$ is also equal to this rank, while if

$$\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) \neq 0$$

for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$, then the dimension of $\overline{\mathcal{Z}}(\mathfrak{M},\mathfrak{N})$ is strictly less than this rank.

Proof. The dimension of $[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m]$ is equal to the rank of $T_{A^{\mathrm{dist}}}$ (it is the quotient by a two-dimensional group of a vector bundle over a two-dimensional base of rank equal to the rank of $T_{A^{\mathrm{dist}}}$). By Lemma 4.3.10, $\overline{\mathcal{C}}(\mathfrak{M},\mathfrak{N})$ is the scheme-theoretic image of the morphism $[\operatorname{Spec} B^{\mathrm{dist}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \mathcal{C}^{\mathrm{dd},1}$ provided by Proposition 4.3.21(1), which (by that proposition) is representable by algebraic spaces and unramified. Since such a morphism is locally quasi-finite (in fact, in this particular case, we have shown that the fibres of this morphism have degree at most 2), [Sta13, Tag 0DS6] ensures that $\overline{\mathcal{C}}(\mathfrak{M},\mathfrak{N})$ has the claimed dimension.

If $\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) = 0$ for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$, then an identical argument using Proposition 4.3.21(2) implies the claim regarding the dimension of $\overline{\mathcal{Z}}(\mathfrak{M},\mathfrak{N})$.

Finally, suppose that

$$\text{ker-Ext}^1_{\mathcal{K}(\kappa(\mathfrak{m}))}\big(\mathfrak{M}_{\kappa(\mathfrak{m}),\bar{x}}, \mathfrak{N}_{\kappa(\mathfrak{m}),\bar{y}}\big) \neq 0$$

for all maximal ideals $\mathfrak{m}$ of $A^{\text{k-free}}$. Then the composite $[\operatorname{Spec} B^{\text{k-free}}/\mathbf{G}_m \times_{\mathbf{F}} \mathbf{G}_m] \to \mathcal{C}^{\mathrm{dd},1} \to \mathcal{R}^{\mathrm{dd},1}$ has the property that for every point t in the source, the fibre over the image of t has a positive dimensional fibre. [Sta13, Tag 0DS6] then implies the remaining claim of the present lemma. ☐

4.4. **Rank one modules over finite fields, and their extensions.** We now wish to apply the results of the previous subsections to study the geometry of our various moduli stacks. In order to do this, it will be convenient for us to have an explicit description of the rank one Breuil–Kisin modules of height at most one with descent data over a finite field of characteristic p, and of their possible extensions. Many of the results in this section are proved (for $p > 2$) in [DS15, §1] in the context of Breuil modules, and in those cases it is possible simply to translate the relevant statements to the Breuil–Kisin module context.

Assume from now on that $e(K'/K)$ is divisible by $p^f - 1$, so that we are in the setting of [DS15, Remark 1.7]. (Note that the parallel in [DS15] of our field extension K'/K, with ramification and inertial indices e', f' and e, f respectively, is the extension K/L with indices e, f and e', f' respectively.)

Let $\mathbf{F}$ be a finite subfield of $\overline{\mathbf{F}}_p$ containing the image of some (so all) embedding(s) $k' \hookrightarrow \overline{\mathbf{F}}_p$. Recall that for each $g \in \mathrm{Gal}(K'/K)$ we write $g(\pi')/\pi' = h(g)$ with $h(g) \in \mu_{e(K'/K)}(K') \subset W(k')$. We abuse notation and denote the image of $h(g)$ in k' again by $h(g)$, so that we obtain a map $h\colon \mathrm{Gal}(K'/K) \to (k')^\times$. Note that h restricts to a character on the inertia subgroup $I(K'/K)$, and is itself a character when $e(K'/K) = p^f - 1$.

Lemma 4.4.1. *Every rank one Breuil–Kisin module of height at most one with descent data and $\mathbf{F}$-coefficients is isomorphic to one of the modules $\mathfrak{M}(r, a, c)$ defined by:*

- $\mathfrak{M}(r, a, c)_i = \mathbf{F}[[u]] \cdot m_i$,
- $\Phi_{\mathfrak{M}(r,a,c),i}(1 \otimes m_{i-1}) = a_i u^{r_i} m_i$,
- $\hat{g}(\sum_i m_i) = \sum_i h(g)^{c_i} m_i$ *for all* $g \in \mathrm{Gal}(K'/K)$,

where $a_i \in \mathbf{F}^\times$, $r_i \in \{0, \dots, e'\}$ and $c_i \in \mathbf{Z}/e(K'/K)$ are sequences satisfying $pc_{i-1} \equiv c_i + r_i \pmod{e(K'/K)}$, the sums in the third bullet point run from 0 to $f' - 1$, and the r_i, c_i, a_i are periodic with period dividing f.

Furthermore, two such modules $\mathfrak{M}(r, a, c)$ and $\mathfrak{M}(s, b, d)$ are isomorphic if and only if $r_i = s_i$ and $c_i = d_i$ for all i, and $\prod_{i=0}^{f-1} a_i = \prod_{i=0}^{f-1} b_i$.

Proof. The proof is elementary; see e.g. [Sav08, Thm. 2.1, Thm. 3.5] for proofs of analogous results. ☐

We will sometimes refer to the element $m = \sum_i m_i \in \mathfrak{M}(r, a, c)$ as the standard generator of $\mathfrak{M}(r, a, c)$.

Remark 4.4.2. When $p > 2$ many of the results in this section (such as the above) can be obtained by translating [DS15, Lem. 1.3, Cor. 1.8] from the Breuil module context to the Breuil–Kisin module context. We briefly recall the dictionary between these two categories (*cf.* [Kis09, §1.1.10]). If A is a finite local $\mathbf{Z}_p$-algebra, write $S_A = S \otimes_{\mathbf{Z}_p} A$, where S is Breuil's ring. We regard S_A as a $\mathfrak{S}_A$-algebra via $u \mapsto u$, and we let $\varphi : \mathfrak{S}_A \to S_A$ be the composite of this map with φ on $\mathfrak{S}_A$. Then given a Breuil–Kisin module of height at most 1 with descent data $\mathfrak{M}$, we set $\mathcal{M} := S_A \otimes_{\varphi, \mathfrak{S}_A} \mathfrak{M}$. We have a map $1 \otimes \varphi_{\mathfrak{M}} : S_A \otimes_{\varphi, \mathfrak{S}_A} \mathfrak{M} \to S_A \otimes_{\mathfrak{S}_A} \mathfrak{M}$, and we set

$$\mathrm{Fil}^1 \mathcal{M} := \{x \in \mathcal{M} \ : \ (1 \otimes \varphi_{\mathfrak{M}})(x) \in \mathrm{Fil}^1 S_A \otimes_{\mathfrak{S}_A} \mathfrak{M} \subset S_A \otimes_{\mathfrak{S}_A} \mathfrak{M}\}$$

and define $\varphi_1 : \operatorname{Fil}^1 \mathcal{M} \to \mathcal{M}$ as the composite

$$\operatorname{Fil}^1 \mathcal{M} \xrightarrow{1\otimes\varphi_{\mathfrak{M}}} \operatorname{Fil}^1 S_A \otimes_{\mathfrak{S}_A} \mathfrak{M} \xrightarrow{\varphi_1\otimes 1} S_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M} = \mathcal{M}.$$

Finally, we define $\hat{g}$ on $\mathcal{M}$ via $\hat{g}(s\otimes m) = g(s)\otimes\hat{g}(m)$. One checks without difficulty that this makes $\mathcal{M}$ a strongly divisible module with descent data (*cf.* the proofs of [Kis09, Proposition 1.1.11, Lemma 1.2.4]).

In the correspondence described above, the Breuil–Kisin module $\mathfrak{M}((r_i),(a_i),(c_i))$ corresponds to the Breuil module $\mathcal{M}((e'-r_i),(a_i),(pc_{i-1}))$ of [DS15, Lem. 1.3].

Definition 4.4.3. If $\mathfrak{M} = \mathfrak{M}(r,a,c)$ is a rank one Breuil–Kisin module as described in the preceding lemma, we set $\alpha_i(\mathfrak{M}) := (p^{f'-1}r_{i-f'+1} + \cdots + r_i)/(p^{f'}-1)$ (equivalently, $(p^{f-1}r_{i-f+1} + \cdots + r_i)/(p^f-1)$). We may abbreviate $\alpha_i(\mathfrak{M})$ simply as α_i when $\mathfrak{M}$ is clear from the context.

It follows easily from the congruence $r_i \equiv pc_{i-1} - c_i \pmod{e(K'/K)}$ together with the hypothesis that $p^f-1 \mid e(K'/K)$ that $\alpha_i \in \mathbf{Z}$ for all i. Note that the α_i's are the unique solution to the system of equations $p\alpha_{i-1} - \alpha_i = r_i$ for all i. Note also that $(p^f-1)(c_i-\alpha_i) \equiv 0 \pmod{e(K'/K)}$, so that $h^{c_i-\alpha_i}$ is a character with image in $k^\times$.

Lemma 4.4.4. *We have $T(\mathfrak{M}(r,a,c)) = \left(\sigma_i \circ h^{c_i-\alpha_i}\cdot \mathrm{ur}_{\prod_{i=0}^{f-1} a_i}\right)|_{G_{K_\infty}}$, where ur_λ is the unramified character of G_K sending geometric Frobenius to λ.*

Proof. Set $\mathfrak{N} = \mathfrak{M}(0,(a_i),0)$, so that $\mathfrak{N}$ is effectively a Breuil–Kisin module without descent data. Then for $\mathfrak{N}$ this result follows from the second paragraph of the proof [GLS14, Lem. 6.3]. (Note that the functor $T_{\mathfrak{S}}$ of *loc. cit.* is dual to our functor T; *cf.* [Fon90, A 1.2.7]. Note also that the fact that the base field is unramified in *loc. cit.* does not change the calculation.) If $n = \sum n_i$ is the standard generator of $\mathfrak{N}$ as in Lemma 4.4.1, let $\gamma \in \mathbf{Z}_p^{\mathrm{un}} \otimes_{\mathbf{Z}_p} (k' \otimes_{\mathbf{F}_p} \mathbf{F})$ be an element so that $\gamma m \in (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}[1/u]} \mathfrak{N}[1/u])^{\varphi=1}$.

Now for $\mathfrak{M}$ as in the statement of the lemma it is straightforward to verify that

$$\gamma \sum_{i=0}^{f'-1} [\underline{\pi}']^{-\alpha_i} \otimes m_i \in (\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}[1/u]} \mathfrak{M}[1/u])^{\varphi=1},$$

and the result follows. $\square$

One immediately deduces the following.

Corollary 4.4.5. *Let $\mathfrak{M} = \mathfrak{M}(r,a,c)$ and $\mathfrak{N} = \mathfrak{M}(s,b,d)$ be rank one Breuil–Kisin modules with descent data as above. We have $T(\mathfrak{M}) = T(\mathfrak{N})$ if and only if $c_i - \alpha_i(\mathfrak{M}) \equiv d_i - \alpha_i(\mathfrak{N}) \pmod{e(K'/K)}$ for some i (hence for all i) and $\prod_{i=0}^{f-1} a_i = \prod_{i=0}^{f-1} b_i$.*

Lemma 4.4.6. *In the notation of the previous Corollary, there is a nonzero map $\mathfrak{M} \to \mathfrak{N}$ (equivalently, $\dim_{\mathbf{F}} \operatorname{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) = 1$) if and only if $T(\mathfrak{M}) = T(\mathfrak{N})$ and $\alpha_i(\mathfrak{M}) \geq \alpha_i(\mathfrak{N})$ for each i.*

Proof. The proof is essentially the same as that of [DS15, Lem. 1.6]. (Indeed, when $p > 2$ this lemma can once again be proved by translating directly from [DS15] to the Breuil–Kisin module context.) $\square$

Using the material of Section 4.1, one can compute $\mathrm{Ext}^1(\mathfrak{M},\mathfrak{N})$ for any pair of rank one Breuil–Kisin modules $\mathfrak{M},\mathfrak{N}$ of height at most one. We begin with the following explicit description of the complex $C^\bullet(\mathfrak{N})$ of Section 4.1.

Definition 4.4.7. We write $\mathscr{C}^0_u = \mathscr{C}^0_u(\mathfrak{M},\mathfrak{N}) \subset \mathbf{F}((u))^{\mathbf{Z}/f\mathbf{Z}}$ for the space of f-tuples (μ_i) such that each nonzero term of μ_i has degree congruent to $c_i - d_i \pmod{e(K'/K)}$, and set $\mathscr{C}^0 = \mathscr{C}^0_u \cap \mathbf{F}[[u]]^{\mathbf{Z}/f\mathbf{Z}}$.

We further define $\mathscr{C}^1_u = \mathscr{C}^1_u(\mathfrak{M},\mathfrak{N}) \subset \mathbf{F}((u))^{\mathbf{Z}/f\mathbf{Z}}$ to be the space of f-tuples (h_i) such that each nonzero term of h_i has degree congruent to $r_i+c_i-d_i \pmod{e(K'/K)}$, and set $\mathscr{C}^1 = \mathscr{C}^1_u \cap \mathbf{F}[[u]]^{\mathbf{Z}/f\mathbf{Z}}$. There is a map $\partial\colon \mathscr{C}^0_u \to \mathscr{C}^1_u$ defined by

$$\partial(\mu_i) = (-a_i u^{r_i}\mu_i + b_i\varphi(\mu_{i-1})u^{s_i})$$

Evidently this restricts to a map $\partial\colon \mathscr{C}^0 \to \mathscr{C}^1$.

Lemma 4.4.8. *There is an isomorphism of complexes*

$$[\mathscr{C}^0 \xrightarrow{\partial} \mathscr{C}^1] \xrightarrow{\sim} C^\bullet(\mathfrak{N})$$

in which $(\mu_i) \in \mathscr{C}^0$ is sent to the map $m_i \mapsto \mu_i n_i$ in $C^0(\mathfrak{N})$, and $(h_i) \in \mathscr{C}^1$ is sent to the map $(1\otimes m_{i-1}) \mapsto h_i n_i$ in $C^1(\mathfrak{N})$.

Proof. Each element of $\mathrm{Hom}_{\mathfrak{S}_\mathbf{F}}(\mathfrak{M},\mathfrak{N})$ has the form $m_i \mapsto \mu_i n_i$ for some f'-tuple $(\mu_i)_{i\in\mathbf{Z}/f'\mathbf{Z}}$ of elements of $\mathbf{F}[[u]]$. The condition that this map is $\mathrm{Gal}(K'/K)$-equivariant is easily seen to be equivalent to the conditions that (μ_i) is periodic with period dividing f, and that each nonzero term of μ_i has degree congruent to $c_i - d_i \pmod{e(K'/K)}$. (For the former consider the action of a lift to $g \in \mathrm{Gal}(K'/K)$ satisfying $h(g)=1$ of a generator of $\mathrm{Gal}(k'/k)$, and for the latter consider the action of $I(K'/K)$; *cf.* the proof of [DS15, Lem. 1.5].) It follows that the map $\mathscr{C}^0 \to C^0(\mathfrak{N})$ in the statement of the Lemma is an isomorphism. An essentially identical argument shows that the given map $\mathscr{C}^1 \to C^1(\mathfrak{N})$ is an isomorphism.

To conclude, it suffices to observe that if $\alpha \in C^0(\mathfrak{N})$ is given by $m_i \mapsto \mu_i n_i$ with $(\mu_i)_i \in \mathscr{C}^0$ then $\delta(\alpha) \in C^1(\mathfrak{N})$ is the map given by

$$(1\otimes m_{i-1}) \mapsto (-a_i u^{r_i}\mu_i + b_i\varphi(\mu_{i-1})u^{s_i})n_i,$$

which follows by a direct calculation. ☐

It follows from Corollary 4.1.8 that $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) \cong \operatorname{coker}\partial$. If $h \in \mathscr{C}^1$, we write $\mathfrak{P}(h)$ for the element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ represented by h under this isomorphism.

Remark 4.4.9. Let $\mathfrak{M} = \mathfrak{M}(r,a,c)$ and $\mathfrak{N} = \mathfrak{M}(s,b,d)$ be rank one Breuil–Kisin modules with descent data as in Lemma *4.4.1*. It follows from the proof of Lemma 4.1.5, and in particular the description of the map (4.1.6) found there, that the extension $\mathfrak{P}(h)$ is given by the formulas

- $\mathfrak{P}_i = \mathbf{F}[[u]]\cdot m_i + \mathbf{F}[[u]]\cdot n_i$,
- $\Phi_{\mathfrak{P},i}(1\otimes n_{i-1}) = b_i u^{s_i} n_i$, $\Phi_{\mathfrak{P},i}(1\otimes m_{i-1}) = a_i u^{r_i} m_i + h_i n_i$.
- $\hat{g}(\sum_i m_i) = \sum_i h(g)^{c_i} m_i$, $\hat{g}(\sum_i n_i) = h(g)^{d_i}\sum_i n_i$ for all $g \in \mathrm{Gal}(K'/K)$.

From this description it is easy to see that the extension $\mathfrak{P}(h)$ has height at most 1 if and only if each h_i is divisible by $u^{r_i+s_i-e'}$.

Theorem 4.4.10. *The dimension of* $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ *is given by the formula*

$$\Delta + \sum_{i=0}^{f-1} \#\Big\{ j \in [0,r_i) : j \equiv r_i + c_i - d_i \pmod{e(K'/K)} \Big\}$$

where $\Delta = \dim_{\mathbf{F}} \operatorname{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ *is* 1 *if there is a nonzero map* $\mathfrak{M} \to \mathfrak{N}$ *and* 0 *otherwise, while the subspace consisting of extensions of height at most* 1 *has dimension*

$$\Delta + \sum_{i=0}^{f-1} \#\Big\{ j \in [\max(0, r_i + s_i - e'), r_i) : j \equiv r_i + c_i - d_i \pmod{e(K'/K)} \Big\}.$$

Proof. When $p > 2$, this result (for extensions of height at most 1) can be obtained by translating [DS15, Thm. 1.11] from Breuil modules to Breuil–Kisin modules. We argue in the same spirit as [DS15] using the generalities of Section 4.1.

Choose N as in Lemma 4.1.10(2). For brevity we write $C^\bullet$ in lieu of $C^\bullet(\mathfrak{N})$. We now use the description of $C^\bullet$ provided by Lemma 4.4.8. As we have noted, C^0 consists of the maps $m_i \mapsto \mu_i n_i$ with $(\mu_i) \in \mathfrak{C}^0$. Since $(\varphi^*_{\mathfrak{M}})^{-1}(v^N C^1)$ contains precisely the maps $m_i \mapsto \mu_i n_i$ in C^0 such that $v^N \mid u^{r_i}\mu_i$, we compute that $\dim_{\mathbf{F}} C^0/\big((\varphi^*_{\mathfrak{M}})^{-1}(v^N C^1)\big)$ is the quantity

$$Nf - \sum_{i=0}^{f-1} \#\Big\{ j \in [e(K'/K)N - r_i, e(K'/K)N) : j \equiv c_i - d_i \pmod{e(K'/K)} \Big\}.$$

We have $\dim_{\mathbf{F}} C^1/v^N C^1 = Nf$, so our formula for the dimension of $\operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ now follows from Lemma 4.1.10. □

Remark 4.4.11. One can show exactly as in [DS15] that each element of $\operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ can be written uniquely in the form $\mathfrak{P}(h)$ for $h \in \mathfrak{C}^1$ with $\deg(h_i) < r_i$, except that when there exists a nonzero morphism $\mathfrak{M} \to \mathfrak{N}$, the polynomials h_i for $f \mid i$ may also have a term of degree $\alpha_0(\mathfrak{M}) - \alpha_0(\mathfrak{N}) + r_0$ in common. Since we will not need this fact we omit the proof.

4.5. **Extensions of shape J.** We now begin the work of showing, for each non-scalar tame type τ, that $\mathcal{C}^{\tau,\mathrm{BT},1}$ has 2^f irreducible components, indexed by the subsets J of $\{0,1,\dots,f-1\}$. We will also describe the irreducible components of $\mathcal{Z}^{\tau,1}$. The proof of this hinges on examining the extensions considered in Theorem 4.4.10, and then applying the results of Subsection 4.3. We will show that most of these families of extensions have positive codimension in $\mathcal{C}^{\tau,\mathrm{BT},1}$, and are thus negligible from the point of view of determining irreducible components. By a base change argument, we will also be able to show that we can neglect the irreducible Breuil–Kisin modules. The rest of Section 4 is devoted to establishing the necessary bounds on the dimension of the various families of extensions, and to studying the map from $\mathcal{C}^{\tau,\mathrm{BT},1}$ to $\mathcal{R}^{\mathrm{dd},1}$.

We now introduce notation that we will use for the remainder of the paper. We fix a tame inertial type $\tau = \eta \oplus \eta'$ with coefficients in $\overline{\mathbf{Q}}_p$. We allow the case of scalar types (that is, the case $\eta = \eta'$). Assume that the subfield $\mathbf{F}$ of $\overline{\mathbf{F}}_p$ is large enough so that the reductions modulo $\mathfrak{m}_{\mathbf{Z}_p}$ of η and η' (which by abuse of notation we continue to denote η, η') have image in $\mathbf{F}$. We also fix a uniformiser π of K.

Remark 4.5.1. We stress that when we write $\tau = \eta \oplus \eta'$, we are implicitly ordering η, η'. Much of the notation in this section depends on distinguishing η, η', as do some of the constructions later in paper (in particular, the map to the Dieudonné stack of Section 4.9).

As in Subsection 3.11, we make the following "standard choice" for the extension K'/K: if τ is a tame principal series type, we take $K' = K(\pi^{1/(p^f-1)})$, while if τ is a tame cuspidal type, we let L be an unramified quadratic extension of K, and set $K' = L(\pi^{1/(p^{2f}-1)})$. In either case K'/K is a Galois extension and η, η' both factor through $I(K'/K)$. In the principal series case, we have $e' = (p^f - 1)e$, $f' = f$, and in the cuspidal case we have $e' = (p^{2f} - 1)e$, $f' = 2f$. Either way, we have $e(K'/K) = p^{f'} - 1$.

In either case, it follows from Lemma 4.4.1 that a Breuil–Kisin module of rank one with descent data from K' to K is described by the data of the quantities r_i, a_i, c_i for $0 \le i \le f-1$, and similarly from Lemma 4.4.8 that extensions between two such Breuil–Kisin modules are described by the h_i for $0 \le i \le f-1$. This common description will enable us to treat the principal series and cuspidal cases largely in parallel.

The character $h|_{I_K}$ of Section 4.4 is identified via the Artin map $\mathcal{O}_L^\times \to I_L^{\mathrm{ab}} = I_K^{\mathrm{ab}}$ with the reduction map $\mathcal{O}_L^\times \to (k')^\times$. Thus for each $\sigma \in \mathrm{Hom}(k', \overline{\mathbf{F}}_p)$ the map $\sigma \circ h|_{I_L}$ is the fundamental character ω_σ defined in Section 1.7. Define $k_i, k_i' \in \mathbf{Z}/(p^{f'} - 1)\mathbf{Z}$ for all i by the formulas $\eta = \sigma_i \circ h^{k_i}|_{I(K'/K)}$ and $\eta' = \sigma_i \circ h^{k_i'}|_{I(K'/K)}$. In particular we have $k_i = p^i k_0$, $k_i' = p^i k_0'$ for all i.

Definition 4.5.2. Let $\mathfrak{M} = \mathfrak{M}(r, a, c)$ and $\mathfrak{N} = \mathfrak{M}(s, b, d)$ be Breuil–Kisin modules of rank one with descent data. We say that the pair $(\mathfrak{M}, \mathfrak{N})$ has *type* τ provided that for all i:

- the multisets $\{c_i, d_i\}$ and $\{k_i, k_i'\}$ are equal, and
- $r_i + s_i = e'$.

Lemma 4.5.3. *The following are equivalent.*

(1) The pair $(\mathfrak{M}, \mathfrak{N})$ has type τ.
(2) Some element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ of height at most one satisfies the strong determinant condition and is of type τ.
(3) Every element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ has height at most one, satisfies the strong determinant condition, and is of type τ.

(Accordingly, we will sometimes refer to the condition that $r_i + s_i = e'$ for all i as the determinant condition.)

Proof. Suppose first that the pair $(\mathfrak{M}, \mathfrak{N})$ has type τ. The last sentence of Remark 4.4.9 shows that every element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ has height at most one. Let $\mathfrak{P}$ be such an element. The condition on the multisets $\{c_i, d_i\}$ guarantees that $\mathfrak{P}$ has unmixed type τ. By Proposition 3.5.12 we see that $\dim_{\mathbf{F}}(\mathrm{im}_{\mathfrak{P},i}/E(u)\mathfrak{P}_i)_{\tilde{\eta}}$ is independent of $\tilde{\eta}$. From the condition that $r_i + s_i = e'$ we know that the sum over all $\tilde{\eta}$ of these dimensions is equal to e'; since they are all equal, each is equal to e, and Lemma 3.5.11 tells us that $\mathfrak{P}$ satisfies the strong determinant condition. This proves that (1) implies (3).

Certainly (3) implies (2), so it remains to check that (2) implies (1). Suppose that $\mathfrak{P} \in \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ has height at most one, satisfies the strong determinant condition, and has type τ. The condition that $\{c_i, d_i\} = \{k_i, k_i'\}$ follows from $\mathfrak{P}$ having type τ, and the condition that $r_i + s_i = e'$ follows from the last part of Lemma 3.5.11. ∎

Definition 4.5.4. If $(\mathfrak{M}, \mathfrak{N})$ is a pair of type τ (resp. $\mathfrak{P}$ is an extension of type τ), we define the *shape* of $(\mathfrak{M}, \mathfrak{N})$ (resp. of $\mathfrak{P}$) to be the subset $J := \{i \mid c_i = k_i\} \subseteq \mathbf{Z}/f'\mathbf{Z}$, unless τ is scalar, in which case we define the shape to be the subset $\varnothing$. (Equivalently, J is in all cases the complement in $\mathbf{Z}/f'\mathbf{Z}$ of the set $\{i \mid c_i = k_i'\}$.)

Observe that in the cuspidal case the equality $c_i = c_{i+f}$ means that $i \in J$ if and only if $i + f \notin J$, so that the set J is determined by its intersection with any f consecutive integers modulo $f' = 2f$.

In the cuspidal case we will say that a subset $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ is a shape if it satisfies $i \in J$ if and only if $i + f \notin J$; in the principal series case, we may refer to any subset $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ as a shape.

We define the *refined shape* of the pair $(\mathfrak{M}, \mathfrak{N})$ (resp. of $\mathfrak{P}$) to consist of its shape J, together with the f-tuple of invariants $r := (r_i)_{i=0}^{f-1}$. If (J, r) is a refined shape that arises from some pair (or extension) of type τ, then we refer to (J, r) as a refined shape for τ.

We say the pair $(i-1, i)$ is a *transition* for J if $i - 1 \in J$, $i \notin J$ or vice-versa. (In the first case we sometimes say that the pair $(i-1, i)$ is a transition out of J, and in the latter case a transition into J.) Implicit in many of our arguments below is the observation that in the cuspidal case $(i-1, i)$ is a transition if and only if $(i + f - 1, i + f)$ is a transition.

4.5.5. *An explicit description of refined shapes.* The refined shapes for τ admit an explicit description. If $\mathfrak{P}$ is of shape J, for some fixed $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ then, since c_i, d_i are fixed, we see that the r_i and s_i appearing in $\mathfrak{P}$ are determined modulo $e(K'/K) = p^{f'} - 1$. Furthermore, we see that $r_i + s_i \equiv 0 \pmod{p^{f'} - 1}$, so that these values are consistent with the determinant condition; conversely, if we make any choice of the r_i in the given residue class modulo $(p^{f'} - 1)$, then the s_i are determined by the determinant condition, and the imposed values are consistent with the descent data. There are of course only finitely many choices for the r_i, and so there are only finitely many possible refined shapes for τ.

To make this precise, recall that we have the congruence

$$r_i \equiv pc_{i-1} - c_i \pmod{p^{f'} - 1}.$$

We will write $[n]$ for the least non-negative residue class of n modulo $e(K'/K) = p^{f'} - 1$.

If both $i - 1$ and i lie in J, then we have $c_{i-1} = k_{i-1}$ and $c_i = k_i$. The first of these implies that $pc_{i-1} = k_i$, and therefore $r_i \equiv 0 \pmod{p^{f'} - 1}$. The same conclusion holds if neither $i-1$ and i lie in J. Therefore if $(i-1, i)$ is not a transition we may write

$$r_i = (p^{f'} - 1)y_i \quad \text{and} \quad s_i = (p^{f'} - 1)(e - y_i).$$

with $0 \le y_i \le e$.

Now suppose instead that $(i-1, i)$ is a transition. (In particular the type τ is not scalar.) This time $pc_{i-1} = d_i$ (instead of $pc_{i-1} = c_i$), so that $r_i \equiv d_i - c_i \pmod{p^{f'} - 1}$. In this case we write

$$r_i = (p^{f'} - 1)y_i - [c_i - d_i] \quad \text{and} \quad s_i = (p^{f'} - 1)(e + 1 - y_i) - [d_i - c_i]$$

with $1 \le y_i \le e$.

Conversely, for fixed shape J one checks that each choice of integers y_i in the ranges described above gives rise to a refined shape for τ.

If $(i-1, i)$ is not a transition and $(h_i) \in \mathcal{C}^1_u(\mathfrak{M}, \mathfrak{N})$ then non-zero terms of h_i have degree congruent to $r_i + c_i - d_i \equiv c_i - d_i \pmod{p^{f'} - 1}$. If instead $(i-1, i)$ is a transition and $(h_i) \in \mathcal{C}^1_u(\mathfrak{M}, \mathfrak{N})$ then non-zero terms of h_i have degree congruent to $r_i + c_i - d_i \equiv 0 \pmod{p^{f'} - 1}$. In either case, comparing with the preceding paragraphs we see that $\#\{j \in [0, r_i) : j \equiv r_i + c_i - d_i \pmod{e(K'/K)}\}$ is exactly y_i.

By Theorem 4.4.10, we conclude that for a fixed choice of the r_i the dimension of the corresponding Ext^1 is $\Delta + \sum_{i=0}^{f-1} y_i$ (with Δ as in the statement of *loc. cit.*). We say that the refined shape $\bigl(J, (r_i)_{i=0}^{f-1}\bigr)$ is *maximal* if the r_i are chosen to be maximal subject to the above conditions, or equivalently if the y_i are all chosen to be e; for each shape J, there is a unique maximal refined shape (J, r).

4.5.6. *The sets $\mathcal{P}_\tau$.* To each tame type τ we now associate a set $\mathcal{P}_\tau$, which will be a subset of the set of shapes in $\mathbf{Z}/f'\mathbf{Z}$. (In Appendix B we will recall, following [Dia07], that the set $\mathcal{P}_\tau$ parameterises the Jordan–Hölder factors of the reduction mod p of $\sigma(\tau)$.)

We write $\eta(\eta')^{-1} = \prod_{j=0}^{f'-1} (\sigma_j \circ h)^{\gamma_j}$ for uniquely defined integers $0 \le \gamma_j \le p - 1$ not all equal to $p - 1$, so that

$$[k_i - k_i'] = \sum_{j=0}^{f'-1} p^j \gamma_{i-j} \tag{4.5.6}$$

with subscripts taken modulo f'.

If τ is scalar then we set $\mathcal{P}_\tau = \{\varnothing\}$. Otherwise we let $\mathcal{P}_\tau$ be the collection of shapes $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ satisfying the conditions:

- if $i - 1 \in J$ and $i \notin J$ then $\gamma_i \neq p - 1$, and
- if $i - 1 \notin J$ and $i \in J$ then $\gamma_i \neq 0$.

When τ is a cuspidal type, so that $\eta' = \eta^q$, the integers γ_j satisfy $\gamma_{i+f} = p - 1 - \gamma_i$ for all i; thus the condition that if $(i - 1, i)$ is a transition out of J then $\gamma_i \neq p - 1$ translates exactly into the condition that if $(i + f - 1, i + f)$ is a transition into J then $\gamma_{i+f} \neq 0$.

4.5.7. *Moduli stacks of extensions.* We now apply the constructions of stacks and topological spaces of Definitions 4.3.11 and 4.3.14 to the families of extensions considered in Section 4.5.

Definition 4.5.8. If (J, r) is a refined shape for τ, then we let $\mathfrak{M}(J, r) := \mathfrak{M}(r, 1, c)$ and let $\mathfrak{N}(J, r) := \mathfrak{M}(s, 1, d)$, where c, d, and s are determined from J, r, and τ according to the discussion of (4.5.5); for instance we take $c_i = k_i$ when $i \in J$ and $c_i = k_i'$ when $i \notin J$. For the unique maximal shape (J, r) refining J, we write simply $\mathfrak{M}(J)$ and $\mathfrak{N}(J)$.

Definition 4.5.9. If (J, r) is a refined shape for τ, then following Definition 4.3.11, we may construct the reduced closed substack $\overline{\mathcal{C}}\bigl(\mathfrak{M}(J, r), \mathfrak{N}(J, r)\bigr)$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$, as well as the reduced closed substack $\overline{\mathcal{Z}}\bigl(\mathfrak{M}(J, r), \mathfrak{N}(J, r)\bigr)$ of $\mathcal{Z}^{\tau,1}$. We introduce the notation $\overline{\mathcal{C}}(J, r)$ and $\overline{\mathcal{Z}}(J, r)$ for these two stacks, and note that (by definition) $\overline{\mathcal{Z}}(J, r)$ is the scheme-theoretic image of $\overline{\mathcal{C}}(J, r)$ under the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$.

Theorem 4.5.10. *If (J, r) is any refined shape for τ, then* $\dim \overline{\mathcal{C}}(J, r) \le [K : \mathbf{Q}_p]$, *with equality if and only if (J, r) is maximal.*

Proof. This follows from Corollary 4.3.32, Theorem 4.4.10, and Proposition 4.1.15. (See also the discussion following Definition 4.5.4, and note that over $\operatorname{Spec} A^{\mathrm{dist}}$, we have $\Delta = 0$ by definition.) $\square$

Definition 4.5.11. If $J \subseteq \mathbf{Z}/f'\mathbf{Z}$ is a shape, and if r is chosen so that (J,r) is a maximal refined shape for τ, then we write $\overline{\mathcal{C}}(J)$ to denote the closed substack $\overline{\mathcal{C}}(J,r)$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$, and $\overline{\mathcal{Z}}(J)$ to denote the closed substack $\overline{\mathcal{Z}}(J,r)$ of $\mathcal{Z}^{\tau,1}$. Again, we note that by definition $\overline{\mathcal{Z}}(J)$ is the scheme-theoretic image of $\overline{\mathcal{C}}(J)$ in $\mathcal{Z}^{\tau,1}$.

We will see later that the $\overline{\mathcal{C}}(J)$ are precisely the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$; in particular, their finite type points can correspond to irreducible Galois representations. While we do not need it in the sequel, we note the following definition and result, describing the underlying topological spaces of the loci of reducible Breuil–Kisin modules of fixed refined shape.

Definition 4.5.12. For each refined type (J,r), we write $|\mathcal{C}(J,r)^\tau|$ for the constructible subset $|\mathcal{C}(\mathfrak{M}(J,r),\mathfrak{N}(J,r))|$ of $|\mathcal{C}^{\tau,\mathrm{BT},1}|$ of Definition 4.3.14 (where $\mathfrak{M}(J,r)$, $\mathfrak{N}(J,r)$ are the Breuil–Kisin modules of Definition 4.5.8). We write $|\mathcal{Z}(J,r)^\tau|$ for the image of $|\mathcal{C}(J,r)^\tau|$ in $|\mathcal{Z}^{\tau,1}|$ (which is again a constructible subset).

Lemma 4.5.13. *The $\overline{\mathbf{F}}_p$-points of $|\mathcal{C}(J,r)^\tau|$ are precisely the reducible Breuil–Kisin modules with $\overline{\mathbf{F}}_p$-coefficients of type τ and refined shape (J,r).*

Proof. This is immediate from the definition. □

4.6. ker-Ext^1 **and vertical components.** In this section we will establish some basic facts about $\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$, and use these results to study the images of our irreducible components in $\mathcal{Z}^{\tau,1}$. Let $\mathfrak{M}=\mathfrak{M}(r,a,c)$ and $\mathfrak{N}=\mathfrak{M}(s,b,c)$ be Breuil–Kisin modules as in Section 4.4.

Recall from (4.1.31) that the dimension of $\text{ker-Ext}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ is bounded above by the dimension of $\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$; more precisely, by Lemma 2.3.3 we find in this setting that

$$\begin{aligned}\dim_{\mathbf{F}} \text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) &= \dim_{\mathbf{F}} \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N}) \\ &-(\dim_{\mathbf{F}} \mathrm{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}),T(\mathfrak{N})) - \dim_{\mathbf{F}} \mathrm{Hom}_{\mathcal{K}(A)}(\mathfrak{M},\mathfrak{N})).\end{aligned} \tag{4.6.1}$$

A map $f:\mathfrak{M}\to\mathfrak{N}[1/u]/\mathfrak{N}$ has the form $f(m_i)=\mu_i n_i$ for some f'-tuple of elements $\mu_i\in\mathbf{F}((u))/\mathbf{F}[[u]]$. By the same argument as in the first paragraph of the proof of Lemma 4.4.8, such a map belongs to $C^0(\mathfrak{N}[1/u]/\mathfrak{N})$ (i.e., it is $\mathrm{Gal}(K'/K)$-equivariant) if and only if the μ_i are periodic with period dividing f, and each nonzero term of μ_i has degree congruent to $c_i-d_i \pmod{e(K'/K)}$. One computes that $\delta(f)(1\otimes m_{i-1}) = (u^{s_i}\varphi(\mu_{i-1}) - u^{r_i}\mu_i)n_i$ and so $f\in C^0(\mathfrak{N}[1/u]/\mathfrak{N})$ lies in $\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$ precisely when

$$a_i u^{r_i}\mu_i = b_i\varphi(\mu_{i-1})u^{s_i} \tag{4.6.2}$$

for all i.

Remark 4.6.3. Let $f\in\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$ be given as above. Choose any lifting $\tilde{\mu}_i$ of μ_i to $\mathbf{F}((u))$. Then (with notation as in Definition 4.4.7) the tuple $(\tilde{\mu}_i)$ is an element of $\mathfrak{C}^0_u$, and we define $h_i=\partial(\tilde{\mu}_i)$. Then h_i lies in $\mathbf{F}[[u]]$ for all i, so that $(h_i)\in\mathfrak{C}^1$, and a comparison with Lemma 4.4.8 shows that f maps to the extension class in $\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ represented by $\mathfrak{P}(h)$.

Recall that Lemma 4.1.32 implies that nonzero terms appearing in μ_i have degree at least $-\lfloor e'/(p-1)\rfloor$. From this we obtain the following trivial bound on ker-Ext.

Lemma 4.6.4. *We have* $\dim_{\mathbf{F}} \text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) \le \lceil e/(p-1)\rceil f$.

Proof. The degrees of nonzero terms of μ_i all lie in a single congruence class modulo $e(K'/K)$, and are bounded below by $-e'/(p-1)$. Therefore there are at most $\lceil e/(p-1)\rceil$ nonzero terms, and since the μ_i are periodic with period dividing f the lemma follows. $\square$

Remark 4.6.5. It follows directly from Corollary 4.6.4 that if $p > 3$ and $e \neq 1$ then we have $\dim_{\mathbf{F}} \text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}) \le [K:\mathbf{Q}_p]/2$, for then $\lceil e/(p-1)\rceil \le e/2$. Moreover these inequalities are strict if $e > 2$.

We will require a more precise computation of $\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ in the setting of Section 4.5 where the pair $(\mathfrak{M},\mathfrak{N})$ has maximal refined shape (J,r). We now return to that setting and its notation.

Let τ be a tame type. We will find the following notation to be helpful. We let $\gamma_i^* = \gamma_i$ if $i-1 \notin J$, and $\gamma_i^* = p-1-\gamma_i$ if $i-1 \in J$. (Here the integers γ_i are as in Section 4.5.6. In the case of scalar types this means that we have $\gamma_i^* = 0$ for all i.) Since $p[k_{i-1}-k'_{i-1}] - [k_i - k'_i] = (p^{f'}-1)\gamma_i$, an elementary but useful calculation shows that

$$p[d_{i-1}-c_{i-1}] - [c_i - d_i] = \gamma_i^*(p^{f'}-1), \tag{4.6.6}$$

when $(i-1,i)$ is a transition, and that in this case $\gamma_i^* = 0$ if and only if $[d_{i-1}-c_{i-1}] < p^{f'-1}$. Similarly, if τ is not a scalar type and $(i-1,i)$ is not a transition then

$$p[d_{i-1}-c_{i-1}] + [c_i - d_i] = (\gamma_i^*+1)(p^{f'}-1). \tag{4.6.7}$$

The main computational result of this section is the following.

Proposition 4.6.8. *Let (J,r) be any maximal refined shape for τ, and suppose that the pair $(\mathfrak{M},\mathfrak{N})$ has refined shape (J,r). Then $\dim_{\mathbf{F}} \text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ is equal to*

$$\#\{0 \le i < f \,:\, \text{the pair } (i-1,i) \text{ is a transition and } \gamma_i^* = 0\},$$

except that when $e=1$, $\prod_i a_i = \prod_i b_i$, and the quantity displayed above is f, then the dimension of $\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ is equal to $f-1$.

Proof. The argument has two parts. First we show that $\dim_{\mathbf{F}} \operatorname{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N}[1/u]/\mathfrak{N})$ is precisely the displayed quantity in the statement of the Proposition; then we check that $\dim_{\mathbf{F}} \operatorname{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}),T(\mathfrak{N})) - \dim_{\mathbf{F}} \operatorname{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ is equal to 1 in the exceptional case of the statement, and 0 otherwise. The result then follows from (4.6.1).

Let $f : m_i \mapsto \mu_i n_i$ be an element of $C^0(\mathfrak{N}[1/u]/\mathfrak{N})$. Since $u^{e'}$ kills μ_i, and all nonzero terms of μ_i have degree congruent to $c_i - d_i \pmod{p^{f'}-1}$, certainly all nonzero terms of μ_i have degree at least $-e' + [c_i - d_i]$. On the other hand since the shape (J,r) is maximal we have $r_i = e' - [c_i - d_i]$ when $(i-1,i)$ is a transition and $r_i = e'$ otherwise. In either case u^{r_i} kills μ_i, so that (4.6.2) becomes simply the condition that u^{s_i} kills $\varphi(\mu_{i-1})$.

If $(i-1,i)$ is not a transition then $s_i = 0$, and we conclude that $\mu_{i-1} = 0$. Suppose instead that $(i-1,i)$ is a transition, so that $s_i = [c_i - d_i]$. Then all nonzero terms of μ_{i-1} have degree at least $-s_i/p > -p^{f'-1} > -e(K'/K)$. Since those terms must have degree congruent to $c_{i-1} - d_{i-1} \pmod{p^{f'}-1}$, it follows that μ_{i-1} has at most one nonzero term (of degree $-[d_{i-1}-c_{i-1}]$), and this only if $[d_{i-1}-c_{i-1}] < p^{f'-1}$, or equivalently $\gamma_i^* = 0$ (as noted above). Conversely if $\gamma_i^* = 0$ then

$$u^{s_i}\varphi(u^{-[d_{i-1}-c_{i-1}]}) = u^{[c_i-d_i]-p[d_{i-1}-c_{i-1}]} = u^{-\gamma_i^*(p^{f'}-1)}$$

vanishes in $\mathbf{F}((u))/\mathbf{F}[[u]]$. We conclude that μ_{i-1} may have a single nonzero term if and only if $(i-1,i)$ is a transition and $\gamma_i^*=0$, and this completes the first part of the argument.

Turn now to the second part. Looking at Corollary 4.4.5 and Lemma 4.4.6, to compare $\mathrm{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}),T(\mathfrak{N}))$ and $\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ we need to compute the quantities $\alpha_i(\mathfrak{M})-\alpha_i(\mathfrak{N})$. By definition this quantity is equal to

$$\frac{1}{p^{f'}-1}\sum_{j=1}^{f'} p^{f'-j}\left(r_{i+j}-s_{i+j}\right). \tag{4.6.9}$$

Suppose first that τ is non-scalar. When $(i+j-1,i+j)$ is a transition, we have $r_{i+j}-s_{i+j}=(e-1)(p^{f'}-1)+[d_{i+j}-c_{i+j}]-[c_{i+j}-d_{i+j}]$, and otherwise we have $r_{i+j}-s_{i+j}=e(p^{f'}-1)=(e-1)(p^{f'}-1)+[d_{i+j}-c_{i+j}]+[c_{i+j}-d_{i+j}]$. Substituting these expressions into (4.6.9), adding and subtracting $\frac{1}{p^{f'}-1}p^{f'}[d_i-c_i]$, and regrouping gives

$$-[d_i-c_i]+(e-1)\cdot\frac{p^{f'}-1}{p-1}+\frac{1}{p^{f'}-1}\sum_{j=1}^{f'}p^{f'-j}\left(p[d_{i+j-1}-c_{i+j-1}]\mp[c_{i+j}-d_{i+j}]\right),$$

where the sign is $-$ if $(i+j-1,i+j)$ is a transition and $+$ if not. Applying the formulas (4.6.6) and (4.6.7) we conclude that

$$\alpha_i(\mathfrak{M})-\alpha_i(\mathfrak{N})=-[d_i-c_i]+(e-1)\cdot\frac{p^{f'}-1}{p-1}+\sum_{j=1}^{f'}p^{f'-j}\gamma_{i+j}^*+\sum_{j\in S_i}p^{f'-j} \tag{4.6.10}$$

where the set S_i consists of $1\le j\le f$ such that $(i+j-1,i+j)$ is not a transition. Finally, a moment's inspection shows that the same formula still holds if τ is scalar (recalling that $J=\varnothing$ in that case).

Suppose that we are in the exceptional case of the proposition, so that $e=1$, $\gamma_i^*=0$ for all i, and every pair $(i-1,i)$ is a transition. The formula (4.6.10) gives $\alpha_i(\mathfrak{M})-\alpha_i(\mathfrak{N})=-[d_i-c_i]$. Since also $\prod_i a_i=\prod_i b_i$ the conditions of Corollary 4.4.5 are satisfied, so that $T(\mathfrak{M})=T(\mathfrak{N})$; but on the other hand $\alpha_i(\mathfrak{M})<\alpha_i(\mathfrak{N})$, so that by Lemma 4.4.6 there are no nonzero maps $\mathfrak{M}\to\mathfrak{N}$, and $\dim_{\mathbf{F}}\mathrm{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}),T(\mathfrak{N}))-\dim_{\mathbf{F}}\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})=1$.

If instead we are not in the exceptional case of the proposition, then either $\prod_i a_i\neq\prod_i b_i$, or else (4.6.10) gives $\alpha_i(\mathfrak{M})-\alpha_i(\mathfrak{N})>-[d_i-c_i]$ for all i. Suppose that $T(\mathfrak{M})\cong T(\mathfrak{N})$. It follows from Corollary 4.4.5 that $\alpha_i(\mathfrak{M})-\alpha_i(\mathfrak{N})\equiv-[d_i-c_i]$ (mod $e(K'/K)$). Combined with the previous inequality we deduce that $\alpha_i(\mathfrak{M})-\alpha_i(\mathfrak{N})>0$, and Lemma 4.4.6 guarantees the existence of a nonzero map $\mathfrak{M}\to\mathfrak{N}$. We deduce that in any event $\dim_{\mathbf{F}}\mathrm{Hom}_{\mathbf{F}[G_K]}(T(\mathfrak{M}),T(\mathfrak{N}))=\dim_{\mathbf{F}}\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$, completing the proof. □

Corollary 4.6.11. *Let (J,r) be any maximal refined shape for τ, and suppose that the pair $(\mathfrak{M},\mathfrak{N})$ has refined shape (J,r). Then $\dim_{\mathbf{F}}\text{ker-}\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})=0$ if and only if $J\in\mathcal{P}_\tau$.*

Proof. This is immediate from Proposition 4.6.8, comparing the definition of γ_i^* with the defining condition on elements of $\mathcal{P}_\tau$, and noting that the exceptional case in Proposition 4.6.8 can occur only if f is even (so in particular $f-1\neq0$ in these exceptional cases). □

Recall that $\overline{\mathcal{Z}}(J)$ is by definition the scheme-theoretic image of $\overline{\mathcal{C}}(J)$ in $\mathcal{Z}^{\tau,1}$. In the remainder of this section, we show that the $\overline{\mathcal{Z}}(J)$ with $J \in \mathcal{P}_\tau$ are pairwise distinct irreducible components of $\mathcal{Z}^{\tau,1}$. In Section 4.8 below we will show that they in fact exhaust the irreducible components of $\mathcal{Z}^{\tau,1}$.

Theorem 4.6.12. *$\overline{\mathcal{Z}}(J)$ has dimension at most $[K : \mathbf{Q}_p]$, with equality occurring if and only if $J \in \mathcal{P}_\tau$. Consequently, the $\overline{\mathcal{Z}}(J)$ with $J \in \mathcal{P}_\tau$ are irreducible components of $\mathcal{Z}^{\tau,1}$.*

Proof. The first part is immediate from Corollary 4.3.32, Proposition 4.1.15, Corollary 4.6.11 and Theorem 4.5.10. Since $\mathcal{Z}^{\tau,1}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$ by Proposition 3.10.19, and the $\overline{\mathcal{Z}}(J)$ are closed and irreducible by construction, the second part follows from the first together with [Sta13, Tag 0DS2]. ☐

We also note the following result.

Proposition 4.6.13. *If $J \in \mathcal{P}_\tau$, then there is a dense open substack $\mathcal{U}$ of $\overline{\mathcal{C}}(J)$ such that the canonical morphism $\overline{\mathcal{C}}(J) \to \overline{\mathcal{Z}}(J)$ restricts to an open immersion on $\mathcal{U}$.*

Proof. This follows from Proposition 4.3.21 and Corollary 4.6.11. ☐

For later use, we note the following computation. Recall that we write $\mathfrak{N}(J) = \mathfrak{N}(J, r)$ for the maximal shape (J, r) refining J, and that $\tau = \eta \oplus \eta'$.

Proposition 4.6.14. *For each shape J we have*

$$T(\mathfrak{N}(J)) \cong \eta \cdot \left(\prod_{i=0}^{f'-1} (\sigma_i \circ h)^{t_i} \right)^{-1} |_{G_{K_\infty}}$$

where

$$t_i = \begin{cases} \gamma_i + \delta_{J^c}(i) & \text{if } i-1 \in J \\ 0 & \text{if } i-1 \notin J. \end{cases}$$

Here δ_{J^c} is the characteristic function of the complement of J in $\mathbf{Z}/f'\mathbf{Z}$, and we are abusing notation by writing η for the function $\sigma_i \circ h^{k_i}$, which agrees with η on I_K.

In particular the map $J \mapsto T(\mathfrak{N}(J))$ is injective on $\mathcal{P}_\tau$.

Remark 4.6.15. In the cuspidal case it is not *a priori* clear that the formula in Proposition 4.6.14 gives a character of G_{K_∞} (rather than a character only when restricted to G_{L_∞}), but this is an elementary (if somewhat painful) calculation using the definition of the γ_i's and the relation $\gamma_i + \gamma_{i+f} = p - 1$.

Proof. We begin by explaining how the final statement follows from the rest of the Proposition. First observe that if $J \in \mathcal{P}_\tau$ then $0 \le t_i \le p-1$ for all i. Indeed the only possibility for a contradiction would be if $\gamma_i = p-1$ and $i \notin J$, but then the definition of $\mathcal{P}_\tau$ requires that we cannot have $i - 1 \in J$. Next, note that we never have $t_i = p - 1$ for all i. Indeed, this would require $J = \mathbf{Z}/f'\mathbf{Z}$ and $\gamma_i = p - 1$ for all i, but by definition the γ_i are not all equal to $p - 1$.

The observations in the previous paragraph imply that (for $J \in \mathcal{P}_\tau$) the character $T(\mathfrak{N}(J))$ uniquely determines the integers t_i, and so it remains to show that the integers t_i determine the set J. If $t_i = 0$ for all i, then either $J = \varnothing$ or $J = \mathbf{Z}/f'\mathbf{Z}$ (for otherwise there is a transition out of J, and $\delta_{J^c}(i) \neq 0$ for some $i - 1 \in J$). But if $J = \mathbf{Z}/f'\mathbf{Z}$ then $\gamma_i = 0$ for all i and τ is scalar; but for scalar types we have $\mathbf{Z}/f'\mathbf{Z} \notin \mathcal{P}_\tau$, a contradiction. Thus $t_i = 0$ for all i implies $J = \varnothing$.

For the rest of this part of the argument, we may therefore suppose $t_i \neq 0$ for some i, which forces $i-1 \in J$. The entire set J will then be determined by recursion if we can show that knowledge of t_i along with whether or not $i \in J$, determines whether or not $i-1 \in J$. Given the defining formula for t_i, the only possible ambiguity is if $t_i = 0$ and $\gamma_i + \delta_{J^c}(i) = 0$, so that $\gamma_i = 0$ and $i \in J$. But the definition of $\mathcal{P}_\tau$ requires $i-1 \in J$ in this case. This completes the proof.

We now turn to proving the formula for $T(\mathfrak{N}(J))$. We will use Lemma 4.4.4 applied at $i = 0$, for which we have to compute $\alpha_0 - d_0$ writing $\alpha_0 = \alpha_0(\mathfrak{N})$. Recall that we have already computed $\alpha_0(\mathfrak{M}(J)) - \alpha_0(\mathfrak{N}(J))$ in the proof of Proposition 4.6.8. Since $\alpha_0(\mathfrak{M}(J)) + \alpha_0(\mathfrak{N}(J)) = e(p^{f'}-1)/(p-1)$, taking the difference between these formulas gives

$$2\alpha_0 = [d_0 - c_0] - \sum_{j=1}^{f'} p^{f'-j}\gamma_j^* + \sum_{j \in S_0^c} p^{f'-j}$$

where S_0^c consists of those $1 \leq j \leq f$ such that $(j-1, j)$ is a transition. Subtract $2[d_0]$ from both sides, and add the expression $-[k_0 - k_0'] + \sum_{j=1}^{f'} p^{f'-j}\gamma_j$ (which vanishes by definition) to the right-hand side. Note that $[d_0 - c_0] - [k_0 - k_0'] - 2[d_0]$ is equal to $-2[k_0]$ if $0 \notin J$, and to $e(K'/K) - 2[k_0 - k_0'] - 2[k_0']$ if $0 \in J$. Since $\gamma_j - \gamma_j^* = 2\gamma_j - (p-1)$ if $j-1 \in J$ and is 0 otherwise, the preceding expression rearranges to give (after dividing by 2)

$$\alpha_0 - [d_0] = -\kappa_0 + \sum_{j-1 \in J} p^{f'-j}\gamma_j + \sum_{j-1 \in J, j \notin J} p^{f'-j} = -\kappa_0 + \sum_{j=1}^{f'} p^{f'-j} t_j$$

where $\kappa_0 = [k_0]$ if $0 \notin J$ and $\kappa_0 = [k_0 - k_0'] + [k_0']$ if $0 \in J$. Since in either case $\kappa_0 \equiv k_0 \pmod{e(K'/K)}$ the result now follows from Lemma 4.4.4. $\square$

Definition 4.6.16. Let $\overline{r} : G_K \to \mathrm{GL}_2(\mathbf{F}')$ be representation. Then we say that a Breuil–Kisin module $\mathfrak{M}$ with $\mathbf{F}'$-coefficients is a *Breuil–Kisin model of* $\overline{r}$ *of type* τ if $\mathfrak{M}$ is an $\mathbf{F}'$-point of $\mathcal{C}^{\tau,\mathrm{BT},1}$, and $T_{\mathbf{F}'}(\mathfrak{M}) \cong \overline{r}|_{G_{K_\infty}}$.

Theorem 4.6.17. *The $\overline{\mathcal{Z}}(J)$, with $J \in \mathcal{P}_\tau$, are pairwise distinct closed substacks of $\mathcal{Z}^{\tau,1}$. For each $J \in \mathcal{P}_\tau$, there is a dense set of finite type points of $\overline{\mathcal{Z}}(J)$ with the property that the corresponding Galois representations have $\overline{\sigma}_J$ as a Serre weight, and which furthermore admit a unique Breuil–Kisin model of type τ.*

Proof. Recall from Definition 4.3.11 that $\overline{\mathcal{Z}}(J)$ is defined to be the scheme-theoretic image of a morphism $\operatorname{Spec} B^{\mathrm{dist}} \to \mathcal{Z}^{\mathrm{dd},1}$. As in the proof of Lemma 4.3.13, since the source and target of this morphism are of finite presentation over $\mathbf{F}$, its image is a dense constructible subset of its scheme-theoretic image, and so contains a dense open subset, which we may interpret as a dense open substack $\mathcal{U}$ of $\overline{\mathcal{Z}}(J)$. From the definition of B^{dist}, the finite type points of $\mathcal{U}$ correspond to reducible Galois representations admitting a model of type τ and refined shape (J, r), for which (J, r) is maximal.

That the $\overline{\mathcal{Z}}(J)$ are pairwise distinct is immediate from the above and Proposition 4.6.14. Combining this observation with Theorem 4.5.10, we see that by deleting the intersections of $\overline{\mathcal{Z}}(J)$ with the $\overline{\mathcal{Z}}(J', r')$ for all refined shapes $(J', r') \neq (J, r)$, we obtain a dense open substack $\mathcal{U}'$ whose finite type points have the property that every Breuil–Kisin model of type τ of the corresponding Galois representation has shape (J, r). The unicity of such a Breuil–Kisin model then follows from Corollary 4.6.11.

It remains to show that every such Galois representation $\overline{r}$ has $\overline{\sigma}_J$ as a Serre weight. Suppose first that τ is a principal series type. We claim that (writing $\overline{\sigma}_J = \overline{\sigma}_{\vec{t},\vec{s}} \otimes (\eta' \circ \det)$ as in Appendix B) we have

$$T(\mathfrak{N}(J))|_{I_K} = \eta'|_{I_K} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i+t_i}.$$

To see this, note that by Proposition 4.6.14 it is enough to show that $\eta|_{I_K} = \eta'|_{I_K} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i+2t_i}$, which follows by comparing the central characters of $\overline{\sigma}_J$ and $\overline{\sigma}(\tau)$ (or from a direct computation with the quantities s_i, t_i).

Since $\det \overline{r}|_{I_K} = \eta\eta'\overline{\varepsilon}^{-1}$, we have

$$\overline{r}|_{I_K} \cong \eta'|_{I_K} \otimes \begin{pmatrix} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i+t_i} & * \\ 0 & \overline{\varepsilon}^{-1} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{t_i} \end{pmatrix}.$$

The result then follows from Lemma B.6, using Lemma B.5(2) and the fact that the fibre of the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{R}^{\mathrm{dd},1}$ above $\overline{r}$ is nonempty to see that $\overline{r}$ is not très ramifiée.

The argument in the cuspidal case proceeds analogously, noting that if the character θ (as in Appendix B) corresponds to $\widetilde{\theta}$ under local class field theory then $\widetilde{\theta}|_{I_K} = \eta' \prod_{i=0}^{f'-1} \omega_{\sigma_i'}^{t_i}$, and that from central characters we have $\eta\eta' = (\widetilde{\theta}|_{I_K})^2 \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i}$. ☐

Remark 4.6.18. With more work, we could use the results of [GLS15] and our results on dimensions of families of extensions to strengthen Theorem 4.6.17, showing that there is a dense set of finite type points of $\overline{\mathcal{Z}}(J)$ with the property that the corresponding Galois representations have $\overline{\sigma}_J$ as their *unique* Serre weight. In fact, we will prove this as part of our work on the geometric Breuil–Mézard conjecture, and it is an immediate consequence of Theorem 5.2.2 below (which uses Theorem 4.6.17 as an input).

4.7. Irreducible Galois representations.

We now show that the points of $\mathcal{C}^{\tau,\mathrm{BT},1}$ which are irreducible (that is, cannot be written as an extension of rank one Breuil–Kisin modules) lie in a closed substack of positive codimension. We begin with the following useful observation.

Lemma 4.7.1. *The rank two Breuil–Kisin modules with descent data and $\overline{\mathbf{F}}_p$-coefficients which are irreducible (that is, which cannot be written as an extension of rank* 1 *Breuil–Kisin modules with descent data) are precisely those whose corresponding étale φ-modules are irreducible, or equivalently whose corresponding G_K-representations are irreducible.*

Proof. Let $\mathfrak{M}$ be a Breuil–Kisin module with descent data corresponding to a finite type point of $\mathcal{C}_{\mathrm{dd}}^{\tau,\mathrm{BT},1}$, let $M = \mathfrak{M}[1/u]$, and let $\overline{\rho}$ be the G_K-representation corresponding to M. As noted in the proof of Lemma 2.3.5, $\overline{\rho}$ is reducible if and only if $\overline{\rho}|_{G_{K_\infty}}$ is reducible, and by Lemma 2.3.3, this is equivalent to M being reducible. That this is in turn equivalent to $\mathfrak{M}$ being reducible may be proved in the same way as [GLS14, Lem. 5.5]. ☐

Recall that L/K denotes the unramified quadratic extension; then the irreducible representations $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ are all induced from characters of G_L. Bearing in mind Lemma 4.7.1, this means that we can study irreducible Breuil–Kisin modules via a consideration of base-change of Breuil–Kisin modules from K to L, and our previous study of reducible Breuil–Kisin modules. Since this will require us to consider Breuil–Kisin modules (and moduli stacks thereof) over both K and L, we will have to introduce additional notation in order to indicate over which of the two fields we might be working. We do this simply by adding a subscript 'K' or 'L' to our current notation. We will also omit other decorations which are being held fixed throughout the present discussion. Thus we write $\mathcal{C}^{\tau}_K$ to denote the moduli stack that was previously denoted $\mathcal{C}^{\tau,\mathrm{BT},1}$, and $\mathcal{C}^{\tau_{|L}}_L$ to denote the corresponding moduli stack for Breuil–Kisin modules over L, with the type taken to be the restriction $\tau_{|L}$ of τ from K to L. (Note that whether τ is principal series or cuspidal, the restriction $\tau_{|L}$ is principal series.)

As usual we fix a uniformiser π of K, which we also take to be our fixed uniformiser of L. Also, throughout this section we take $K' = L(\pi^{1/(p^{2f}-1)})$, so that K'/L is the standard choice of extension for τ and π regarded as a type and uniformiser for L.

If $\mathfrak{P}$ is a Breuil–Kisin module with descent data from K' to L, then we let $\mathfrak{P}^{(f)}$ be the Breuil–Kisin module $W(k')\otimes_{\mathrm{Gal}(k'/k),W(k')}\mathfrak{P}$, where the pullback is given by the non-trivial automorphism of k'/k. In particular, we have $\mathfrak{M}(r,a,c)^{(f)} = \mathfrak{M}(r',a',c')$ where $r'_i = r_{i+f}$, $a'_i = a_{i+f}$, and $c'_i = c_{i+f}$.

We let σ denote the non-trivial automorphism of L over K, and write $G := \mathrm{Gal}(L/K) = \langle\sigma\rangle$, a cyclic group of order two. There is an action α of G on $\mathcal{C}_L$ defined via $\alpha_\sigma : \mathfrak{P} \mapsto \mathfrak{P}^{(f)}$. More precisely, this induces an action of $G := \langle\sigma\rangle$ on $\mathcal{C}^{\tau_{|L}}_L$ in the strict[4] sense that

$$\alpha_\sigma \circ \alpha_\sigma = \mathrm{id}_{\mathcal{C}^{\tau_{|L}}_L}.$$

We now define the fixed point stack for this action.

Definition 4.7.2. We let the fixed point stack $(\mathcal{C}^{\tau_{|L}}_L)^G$ denote the stack whose A-valued points consist of an A-valued point $\mathfrak{M}$ of $\mathcal{C}^{\tau_{|L}}_L$, together with an isomorphism $\imath : \mathfrak{M} \xrightarrow{\sim} \mathfrak{M}^{(f)}$ which satisfies the cocycle condition that the composite

$$\mathfrak{M} \xrightarrow{\imath} \mathfrak{M}^{(f)} \xrightarrow{\imath^{(f)}} (\mathfrak{M}^{(f)})^{(f)} = \mathfrak{M}$$

is equal to the identity morphism $\mathrm{id}_{\mathfrak{M}}$.

We now give another description of $(\mathcal{C}^{\tau_{|L}}_L)^G$, in terms of various fibre products, which is technically useful. This alternate description involves two steps. In the first step, we define fixed points of the automorphism α_σ, without imposing the additional condition that the fixed point data be compatible with the relation $\sigma^2 = 1$ in G. Namely, we define

$$(\mathcal{C}^{\tau_{|L}}_L)^{\alpha_\sigma} := \mathcal{C}^{\tau_{|L}}_L \underset{\mathcal{C}^{\tau_{|L}}_L \times \mathcal{C}^{\tau_{|L}}_L}{\times} \mathcal{C}^{\tau_{|L}}_L$$

[4]From a 2-categorical perspective, it is natural to relax the notion of group action on a stack so as to allow natural transformations, rather than literal equalities, when relating multiplication in the group to the compositions of the corresponding equivalences of categories arising in the definition of an action. An action in which actual equalities hold is then called *strict*. Since our action is strict, we are spared from having to consider the various 2-categorical aspects of the situation that would otherwise arise.

where the first morphism $\mathcal{C}_L^{\tau|_L} \to \mathcal{C}_L^{\tau|_L} \times \mathcal{C}_L^{\tau|_L}$ is the diagonal, and the second is $\mathrm{id} \times \alpha_\sigma$. Working through the definitions, one finds that an A-valued point of $(\mathcal{C}_L^{\tau|_L})^{\alpha_\sigma}$ consists of a pair $(\mathfrak{M}, \mathfrak{M}')$ of objects of $\mathcal{C}_L^{\tau|_L}$ over A, equipped with isomorphisms $\alpha : \mathfrak{M} \xrightarrow{\sim} \mathfrak{M}'$ and $\beta : \mathfrak{M} \xrightarrow{\sim} (\mathfrak{M}')^{(f)}$. The morphism

$$(\mathfrak{M}, \mathfrak{M}', \alpha, \beta) \mapsto (\mathfrak{M}, \imath),$$

where $\imath := (\alpha^{-1})^{(f)} \circ \beta : \mathfrak{M} \to \mathfrak{M}^{(f)}$, induces an isomorphism between $(\mathcal{C}_L^{\tau|_L})^{\alpha_\sigma}$ and the stack classifying points $\mathfrak{M}$ of $\mathcal{C}_L^{\tau|_L}$ equipped with an isomorphism $\imath : \mathfrak{M} \to \mathfrak{M}^{(f)}$. (However, no cocycle condition has been imposed on $\imath$.)

Let $I_{\mathcal{C}_L^{\tau|_L}}$ denote the inertia stack of $\mathcal{C}_L^{\tau|_L}$. We define a morphism

$$(\mathcal{C}_L^{\tau|_L})^{\alpha_\sigma} \to I_{\mathcal{C}_L^{\tau|_L}}$$

via

$$(\mathfrak{M}, \imath) \mapsto (\mathfrak{M}, \imath^{(f)} \circ \imath),$$

where, as in Definition 4.7.2, we regard the composite $\imath^{(f)} \circ \imath$ as an automorphism of $\mathfrak{M}$ via the identity $(\mathfrak{M}^{(f)})^{(f)} = \mathfrak{M}$. Of course, we also have the identity section $e : \mathcal{C}_L^{\tau|_L} \to I_{\mathcal{C}_L^{\tau|_L}}$. We now define

$$(\mathcal{C}_L^{\tau|_L})^G := (\mathcal{C}_L^{\tau|_L})^{\alpha_\sigma} \underset{I_{\mathcal{C}_L^{\tau|_L}}}{\times} \mathcal{C}_L^{\tau|_L}.$$

If we use the description of $(\mathcal{C}_L^{\tau|_L})^{\alpha_\sigma}$ as classifying pairs $(\mathfrak{M}, \imath)$, then (just unwinding definitions) this fibre product classifies tuples $(\mathfrak{M}, \imath, \mathfrak{M}', \alpha)$, where α is an isomorphism $\mathfrak{M} \xrightarrow{\sim} \mathfrak{M}'$ which furthermore identifies $\imath^{(f)} \circ \imath$ with $\mathrm{id}_{\mathfrak{M}'}$. Forgetting $\mathfrak{M}'$ and α then induces an isomorphism between $(\mathcal{C}_L^{\tau|_L})^G$, as defined via the above fibre product, and the stack defined in Definition 4.7.2.

To compare this fixed point stack to $\mathcal{C}_K^\tau$, we make the following observations. Given a Breuil–Kisin module with descent data from K' to K, we obtain a Breuil–Kisin module with descent data from K' to L via the obvious forgetful map. Conversely, given a Breuil–Kisin module $\mathfrak{P}$ with descent data from K' to L, the additional data required to enrich this to a Breuil–Kisin module with descent data from K' to K can be described as follows as follows: let $\theta \in \mathrm{Gal}(K'/K)$ denote the unique element which fixes $\pi^{1/(p^{2f}-1)}$ and acts nontrivially on L. Then to enrich the descent data on $\mathfrak{P}$ to descent data from K' to K, it is necessary and sufficient to give an additive map $\hat{\theta} : \mathfrak{P} \to \mathfrak{P}$ satisfying $\hat{\theta}(sm) = \theta(s)\hat{\theta}(m)$ for all $s \in \mathfrak{S}_{\mathbf{F}}$ and $m \in \mathfrak{P}$, and such that $\hat{\theta}\hat{g}\hat{\theta} = \hat{g}^{p^f}$ for all $g \in \mathrm{Gal}(K'/L)$.

In turn, the data of the additive map $\hat{\theta} : \mathfrak{P} \to \mathfrak{P}$ is equivalent to giving the data of the map $\theta(\hat{\theta}) : \mathfrak{P} \to \mathfrak{P}^{(f)}$ obtained by composing $\hat{\theta}$ with the Frobenius on L/K. The defining properties of $\hat{\theta}$ are equivalent to asking that this map is an isomorphism of Breuil–Kisin modules with descent data satisfying the cocycle condition of Definition 4.7.2; accordingly, we have a natural morphism $\mathcal{C}_K^\tau \to (\mathcal{C}_L^{\tau|_L})^G$, and a restriction morphism

$$\mathcal{C}_K^\tau \to \mathcal{C}_L^{\tau|_L}. \tag{4.7.3}$$

The following simple lemma summarises the basic facts about base-change in the situation we are considering.

Lemma 4.7.4. *There is an isomorphism* $\mathcal{C}_K^\tau \xrightarrow{\sim} (\mathcal{C}_L^{\tau|_L})^G$.

Proof. This follows immediately from the preceding discussion. □

Remark 4.7.5. In the proof of Theorem 4.7.9 we will make use of the following analogue of Lemma 4.7.4 for étale φ-modules. Write $\mathcal{R}_K$, $\mathcal{R}_L$ for the moduli stacks of Definition 3.1.4, i.e. for the moduli stacks of rank 2 étale φ-modules with descent data respectively to K or to L. Then we have an action of G on $\mathcal{R}_L$ defined via $M \mapsto M^{(f)} := W(k') \otimes_{\mathrm{Gal}(k'/k),W(k')} M$, and we define the fixed point stack $(\mathcal{R}_L)^G$ exactly as in Definition 4.7.2: namely an A-valued point of $(\mathcal{R}_L)^G$ consists of an A-valued point M of $\mathcal{R}_L$, together with an isomorphism $\iota : M \xrightarrow{\sim} M^{(f)}$ satisfying the cocycle condition. The preceding discussion goes through in this setting, and shows that there is an isomorphism $\mathcal{R}_K \xrightarrow{\sim} (\mathcal{R}_L)^G$.

We also note that the morphisms $\mathcal{C}_K^\tau \to \mathcal{C}_L^{\tau_{|L}}$ and $\mathcal{C}_K^\tau \to \mathcal{R}_K$ induce a monomorphism

$$\mathcal{C}_K^\tau \hookrightarrow \mathcal{C}_L^{\tau_{|L}} \times_{\mathcal{R}_L} \mathcal{R}_K \tag{4.7.6}$$

One way to see this is to rewrite this morphism (using the previous discussion) as a morphism

$$(\mathcal{C}_L^{\tau_{|L}})^G \to \mathcal{C}_L^{\tau_{|L}} \times_{\mathcal{R}_L} (\mathcal{R}_L)^G,$$

and note that the descent data via G on an object classified by the source of this morphism is determined by the induced descent data on its image in $(\mathcal{R}_L)^G$.

We now use the Lemma 4.7.4 to study the locus of finite type points of $\mathcal{C}_K^\tau$ which correspond to irreducible Breuil–Kisin modules. Any irreducible Breuil–Kisin module over K becomes reducible when restricted to L, and so may be described as an extension

$$0 \to \mathfrak{N} \to \mathfrak{P} \to \mathfrak{M} \to 0,$$

where $\mathfrak{M}$ and $\mathfrak{N}$ are Breuil–Kisin modules of rank one with descent data from K' to L, and $\mathfrak{P}$ is additionally equipped with an isomorphism $\mathfrak{P} \cong \mathfrak{P}^{(f)}$, satisfying the cocycle condition of Definition 4.7.2.

Note that the characters $T(\mathfrak{M})$, $T(\mathfrak{N})$ of G_{L_∞} are distinct and cannot be extended to characters of G_K. Indeed, this condition is plainly necessary for an extension $\mathfrak{P}$ to arise as the base change of an irreducible Breuil–Kisin module (see the proof of Lemma 2.3.5). Conversely, if $T(\mathfrak{M})$, $T(\mathfrak{N})$ of G_{L_∞} are distinct and cannot be extended to characters of G_K, then for any $\mathfrak{P} \in \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ whose descent data can be enriched to give descent data from K' to K, this enrichment is necessarily irreducible. In particular, the existence of such a $\mathfrak{P}$ implies that the descent data on $\mathfrak{M}$ and $\mathfrak{N}$ cannot be enriched to give descent data from K' to K.

We additionally have the following observation.

Lemma 4.7.7. *If $\mathfrak{M}, \mathfrak{N}$ are such that there is an extension*

$$0 \to \mathfrak{N} \to \mathfrak{P} \to \mathfrak{M} \to 0$$

whose descent data can be enriched to give an irreducible Breuil–Kisin module over K, then there exists a nonzero map $\mathfrak{N} \to \mathfrak{M}^{(f)}$.

Proof. The composition $\mathfrak{N} \to \mathfrak{P} \xrightarrow{\hat{\theta}} \mathfrak{P} \to \mathfrak{M}$, in which first and last arrows are the natural inclusions and projections, must be nonzero (or else $\hat{\theta}$ would give descent data on $\mathfrak{N}$ from K' to K). It is not itself a map of Breuil–Kisin modules, because $\hat{\theta}$ is semilinear, but is a map of Breuil–Kisin modules when viewed as a map $\mathfrak{N} \to \mathfrak{M}^{(f)}$. □

We now consider (for our fixed $\mathfrak{M}$, $\mathfrak{N}$, and working over L rather than over K) the scheme $\operatorname{Spec} B^{\text{dist}}$ as in Subsection 4.3. Following Lemma 4.7.7, we assume that there exists a nonzero map $\mathfrak{N} \to \mathfrak{M}^{(f)}$. The observations made above show that we are in the strict case, and thus that $\operatorname{Spec} A^{\text{dist}} = \mathbf{G}_m \times \mathbf{G}_m$ and that furthermore we may (and do) set $V = T$. We consider the fibre product with the restriction morphism (4.7.3)

$$Y(\mathfrak{M}, \mathfrak{N}) := \operatorname{Spec} B^{\text{dist}} \times_{\mathcal{C}_L^{\tau|_L}} \mathcal{C}_K^{\tau}.$$

Let $\mathbf{G}_m \hookrightarrow \mathbf{G}_m \times \mathbf{G}_m$ be the diagonal closed immersion, and let $(\operatorname{Spec} B^{\text{dist}})_{|\mathbf{G}_m}$ denote the pull-back of $\operatorname{Spec} B^{\text{dist}}$ along this closed immersion. By Lemma 4.7.7, the projection $Y(\mathfrak{M}, \mathfrak{N}) \to \operatorname{Spec} B^{\text{dist}}$ factors through $(\operatorname{Spec} B^{\text{dist}})_{|\mathbf{G}_m}$, and combining this with Lemma 4.7.4 we see that $Y(\mathfrak{M}, \mathfrak{N})$ may also be described as the fibre product

$$(\operatorname{Spec} B^{\text{dist}})_{|\mathbf{G}_m} \times_{\mathcal{C}_L^{\tau|_L}} (\mathcal{C}_L^{\tau|_L})^G.$$

Recalling the warning of Remark 4.3.16, Proposition 4.3.21 now shows that there is a monomorphism

$$[(\operatorname{Spec} B^{\text{dist}})_{|\mathbf{G}_m} / \mathbf{G}_m \times \mathbf{G}_m] \hookrightarrow \mathcal{C}_L^{\tau|_L},$$

and thus, by Lemma 4.2.8, that there is an isomorphism

$$(\operatorname{Spec} B^{\text{dist}})_{|\mathbf{G}_m} \times_{\mathcal{C}_L^{\tau|_L}} (\operatorname{Spec} B^{\text{dist}})_{|\mathbf{G}_m} \xrightarrow{\sim} (\operatorname{Spec} B^{\text{dist}})_{|\mathbf{G}_m} \times \mathbf{G}_m \times \mathbf{G}_m.$$

(An inspection of the proof of Proposition 4.3.21 shows that in fact this result is more-or-less proved directly, as the key step in proving the proposition.) An elementary manipulation with fibre products then shows that there is an isomorphism

$$Y(\mathfrak{M}, \mathfrak{N}) \times_{(\mathcal{C}_L^{\tau|_L})^G} Y(\mathfrak{M}, \mathfrak{N}) \xrightarrow{\sim} Y(\mathfrak{M}, \mathfrak{N}) \times \mathbf{G}_m \times \mathbf{G}_m,$$

and thus, by another application of Lemma 4.2.8, we find that there is a monomorphism

$$[Y(\mathfrak{M}, \mathfrak{N}) / \mathbf{G}_m \times \mathbf{G}_m] \hookrightarrow (\mathcal{C}_L^{\tau|_L})^G. \tag{4.7.8}$$

We define $\mathcal{C}_{\text{irred}}$ to be the union over all such pairs $(\mathfrak{M}, \mathfrak{N})$ of the scheme-theoretic images of the various projections $Y(\mathfrak{M}, \mathfrak{N}) \to (\mathcal{C}_L^{\tau|_L})^G$. Note that this image depends on $(\mathfrak{M}, \mathfrak{N})$ up to simultaneous unramified twists of $\mathfrak{M}$ and $\mathfrak{N}$, and there are only finitely many such pairs $(\mathfrak{M}, \mathfrak{N})$ up to such unramified twist. By definition, $\mathcal{C}_{\text{irred}}$ is a closed substack of $\mathcal{C}_K^{\tau}$ which contains every finite type point of $\mathcal{C}_K^{\tau}$ corresponding to an irreducible Breuil–Kisin module.

The following is the main result of this section.

Theorem 4.7.9. *The closed substack $\mathcal{C}_{\text{irred}}$ of $\mathcal{C}_K^{\tau} = \mathcal{C}^{\tau, \mathrm{BT}, 1}$, which contains every finite type point of $\mathcal{C}_K^{\tau}$ corresponding to an irreducible Breuil–Kisin module, has dimension strictly less than $[K : \mathbf{Q}_p]$.*

Proof. As noted above, there are only finitely many pairs $(\mathfrak{M}, \mathfrak{N})$ up to unramified twist, so it is enough to show that for each of them, the scheme-theoretic image of the monomorphism (4.7.8) has dimension less than $[K : \mathbf{Q}_p]$.

By [Sta13, Tag 0DS6], to prove the present theorem, it then suffices to show that $\dim Y(\mathfrak{M}, \mathfrak{N}) \leq [K : \mathbf{Q}_p] + 1$ (since $\dim \mathbf{G}_m \times \mathbf{G}_m = 2$). To establish this, it suffices to show, for each point $x \in \mathbf{G}_m(\mathbf{F}')$, where $\mathbf{F}'$ is a finite extension of $\mathbf{F}$, that the dimension of the fibre $Y(\mathfrak{M}, \mathfrak{N})_x$ is bounded by $[K : \mathbf{Q}_p]$. After relabelling, as we may, the field $\mathbf{F}'$ as $\mathbf{F}$ and the Breuil–Kisin modules $\mathfrak{M}_x$ and $\mathfrak{N}_x$ as $\mathfrak{M}$ and $\mathfrak{N}$, we may suppose that in fact $\mathbf{F}' = \mathbf{F}$ and $x = 1$.

Manipulating the fibre product appearing in the definition of $Y(\mathfrak{M}, \mathfrak{N})$, we find that

$$Y(\mathfrak{M}, \mathfrak{N})_1 = \operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}) \times_{\mathcal{C}_L^{\tau|_N}} \mathcal{C}_K^{\tau}, \tag{4.7.10}$$

where the fibre product is taken with respect to the morphism $\operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}) \to \mathcal{C}_L^{\tau}$ that associates the corresponding rank two extension to an extension of rank one Breuil–Kisin modules, and the restriction morphism (4.7.3).

In order to bound the dimension of $Y(\mathfrak{M}, \mathfrak{N})_1$, it will be easier to first embed it into another, larger, fibre product, which we now introduce. Namely, the monomorphism (4.7.6) induces a monomorphism

$$Y(\mathfrak{M}, \mathfrak{N})_1 \hookrightarrow Y'(\mathfrak{M}, \mathfrak{N})_1 := \operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}) \times_{\mathcal{R}_L} \mathcal{R}_K.$$

Any finite type point of this fibre product lies over a fixed isomorphism class of finite type points of $\mathcal{R}_K$ (corresponding to some fixed irreducible Galois representation); we let P be a choice of such a point. The restriction of P then lies in a fixed isomorphism class of finite type points of $\mathcal{R}_L$ (namely, the isomorphism class of the direct sum $\mathfrak{M}[1/u] \oplus \mathfrak{N}[1/u] \cong \mathfrak{M}[1/u] \oplus \mathfrak{M}^{(f)}[1/u]$). Thus the projection $Y'(\mathfrak{M}, \mathfrak{N})_1 \to \mathcal{R}_K$ factors through the residual gerbe of P, while the morphism $Y'(\mathfrak{M}, \mathfrak{N})_1 \to \mathcal{R}_L$ factors through the residual gerbe of $\mathfrak{M}[1/u] \oplus \mathfrak{N}[1/u] \cong \mathfrak{M}[1/u] \oplus \mathfrak{M}^{(f)}[1/u]$. Since P corresponds via Lemma 2.3.3 to an irreducible Galois representation, we find that $\operatorname{Aut}(P) = \mathbf{G}_m$. Since $\mathfrak{M}[1/u] \oplus \mathfrak{N}[1/u]$ corresponds via Lemma 2.3.3 to the direct sum of two non-isomorphic Galois characters, we find that $\operatorname{Aut}(\mathfrak{M}[1/u] \oplus \mathfrak{N}[1/u]) = \mathbf{G}_m \times \mathbf{G}_m$.

Thus we obtain monomorphisms

$$\begin{aligned} &Y(\mathfrak{M}, \mathfrak{N})_1 \hookrightarrow Y'(\mathfrak{M}, \mathfrak{N})_1 \\ &\hookrightarrow \operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}) \times_{[\operatorname{Spec} F'//\mathbf{G}_m \times \mathbf{G}_m]} [\operatorname{Spec} F'//\mathbf{G}_m] \cong \operatorname{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}) \times \mathbf{G}_m. \end{aligned} \tag{4.7.11}$$

In Proposition 4.7.12 we obtain a description of the image of $Y(\mathfrak{M}, \mathfrak{N})_1$ under this monomorphism which allows us to bound its dimension by $[K : \mathbf{Q}_p]$, as required. $\square$

We now prove the bound on the dimension of $Y(\mathfrak{M}, \mathfrak{N})_1$ that we used in the proof of Theorem 4.7.9. Before establishing this bound, we make some further remarks. To begin with, we remind the reader that we are working with Breuil–Kisin modules, étale φ-modules, etc., over L rather than K, so that e.g. the structure parameters of $\mathfrak{M}, \mathfrak{N}$ are periodic modulo $f' = 2f$ (not modulo f), and the pair $(\mathfrak{M}, \mathfrak{N})$ has type $\tau|_L$. We will readily apply various pieces of notation that were introduced above in the context of the field K, adapted in the obvious manner to the context of the field L. (This applies in particular to the notation $\mathcal{C}_u^1$, $\mathcal{C}_u^0$, etc. introduced in Definition 4.4.7.)

We write m, n for the standard generators of $\mathfrak{M}$ and $\mathfrak{N}$. The existence of the nonzero map $\mathfrak{N} \to \mathfrak{M}^{(f)}$ implies that $\alpha_i(\mathfrak{N}) \geq \alpha_{i+f}(\mathfrak{M})$ for all i, and also that $\prod_i a_i = \prod_i b_i$. Thanks to the latter we will lose no generality by assuming that $a_i = b_i = 1$ for all i. Let $\tilde{m}$ be the standard generator for $\mathfrak{M}^{(f)}$. The map $\mathfrak{N} \to \mathfrak{M}^{(f)}$ will (up to a scalar) have the form $n_i \mapsto u^{x_i} \tilde{m}_i$ for integers x_i satisfying $px_{i-1} - x_i = s_i - r_{i+f}$ for all i; thus $x_i = \alpha_i(\mathfrak{N}) - \alpha_{i+f}(\mathfrak{M})$ for all i. Since the characters $T(\mathfrak{M})$ and $T(\mathfrak{N})$ are conjugate we must have $x_i \equiv d_i - c_{i+f} \pmod{p^{f'} - 1}$

for all i (*cf.* Lemma 4.4.4). Moreover, the strong determinant condition $s_i + r_i = e'$ for all i implies that $x_i = x_{i+f}$.

We stress that we make no claims about the optimality of the following result; we merely prove "just what we need" for our applications. Indeed the estimates of [Hel09, Car17] suggest that improvement should be possible.

Proposition 4.7.12. *We have* $\dim Y(\mathfrak{M},\mathfrak{N})_1 \le [K:\mathbf{Q}_p]$.

Remark 4.7.13. Since the image of $Y(\mathfrak{M},\mathfrak{N})_1$ in $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ lies in $\ker\text{-}\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ with fibres that can be seen to have dimension at most one, many cases of Proposition 4.7.12 will already follow from Remark 4.6.5 (applied with L in place of K).

Proof of Proposition 4.7.12. Let $\mathfrak{P} = \mathfrak{P}(h)$ be an element of $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$ whose descent data can be enriched to give descent data from K' to K, and let $\widetilde{\mathfrak{P}}$ be such an enrichment. By Lemma 4.7.7 (and the discussion preceding that lemma) the étale φ-module $\mathfrak{P}[\frac{1}{u}]$ is isomorphic to $\mathfrak{M}[\frac{1}{u}] \oplus \mathfrak{M}^{(f)}[\frac{1}{u}]$. All extensions of the G_{L_∞}-representation $T(\mathfrak{M}[\frac{1}{u}] \oplus \mathfrak{M}^{(f)}[\frac{1}{u}])$ to a representation of G_{K_∞} are isomorphic (and given by the induction of $T(\mathfrak{M}[\frac{1}{u}])$ to G_{K_∞}), so the same is true of the étale φ-modules with descent data from K' to K that enrich the descent data on $\mathfrak{M}[\frac{1}{u}] \oplus \mathfrak{M}^{(f)}[\frac{1}{u}]$. One such enrichment, which we denote P, has $\hat{\theta}$ that interchanges m and $\tilde{m}$. Thus $\widetilde{\mathfrak{P}}[\frac{1}{u}]$ is isomorphic to P.

As in the proof of Lemma 4.7.7, the hypothesis that $T(\mathfrak{M}) \not\cong T(\mathfrak{N})$ implies that any non-zero map (equivalently, isomorphism) of étale φ-modules with descent data $\lambda : \widetilde{\mathfrak{P}}[\frac{1}{u}] \to P$ takes the submodule $\mathfrak{N}[\frac{1}{u}]$ to $\mathfrak{M}^{(f)}[\frac{1}{u}]$. We may scale the map λ so that it restricts to the map $n_i \to u^{x_i}\tilde{m}_i$ on $\mathfrak{N}$. Then there is an element $\xi \in \mathbf{F}^\times$ so that λ induces multiplication by ξ on the common quotients $\mathfrak{M}[\frac{1}{u}]$. That is, the map λ may be assumed to have the form

$$\begin{pmatrix} n_i \\ m_i \end{pmatrix} \mapsto \begin{pmatrix} u^{x_i} & 0 \\ \nu_i & \xi \end{pmatrix} \begin{pmatrix} \tilde{m}_i \\ m_i \end{pmatrix} \tag{4.7.14}$$

for some $(\nu_i) \in \mathbf{F}((u))^{f'}$. The condition that the map λ commutes with the descent data from K' to L is seen to be equivalent to the condition that nonzero terms in ν_i have degree congruent to $c_i - d_i + x_i \pmod{p^{f'}-1}$; or equivalently, if we define $\mu_i := \nu_i u^{-x_i}$ for all i, that the tuple $\mu = (\mu_i)$ is an element of the set $\mathcal{C}^0_u = \mathcal{C}^0_u(\mathfrak{M},\mathfrak{N})$ of Definition 4.4.7.

The condition that λ commutes with φ can be checked to give

$$\varphi \begin{pmatrix} n_{i-1} \\ m_{i-1} \end{pmatrix} = \begin{pmatrix} u^{s_i} & 0 \\ \varphi(\nu_{i-1})u^{r_{i+f}-x_i} - \nu_i u^{r_i - x_i} & u^{r_i} \end{pmatrix} \begin{pmatrix} n_i \\ m_i \end{pmatrix}.$$

The extension $\mathfrak{P}$ is of the form $\mathfrak{P}(h)$, for some $h \in \mathcal{C}^1$ as in Definition 4.4.7. The lower-left entry of the first matrix on the right-hand side of the above equation must then be h_i. Since $r_{i+f} - x_i = s_i - px_{i-1}$, the resulting condition can be rewritten as

$$h_i = \varphi(\mu_{i-1})u^{s_i} - \mu_i u^{r_i},$$

or equivalently that $h = \partial(\mu)$. Comparing with Remark 4.6.3, we recover the fact that the extension class of $\mathfrak{P}$ is an element of $\ker\text{-}\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M},\mathfrak{N})$, and the tuple μ determines an element of the space $\mathcal{H}$ defined as follows.

Definition 4.7.15. The map $\partial\colon \mathcal{C}^0_u \to \mathcal{C}^1_u$ induces a map $\mathcal{C}^0_u/\mathcal{C}^0 \to \mathcal{C}^1_u/\partial(\mathcal{C}^0)$, which we also denote ∂. We let $\mathcal{H} \subset \mathcal{C}^0_u/\mathcal{C}^0$ denote the subspace consisting of elements μ such that $\partial(\mu) \in \mathcal{C}^1/\partial(\mathcal{C}^0)$.

By the discussion following Lemma 4.4.8, an element $\mu \in \mathcal{H}$ determines an extension $\mathfrak{P}(\partial(\mu))$. Indeed, Remark 4.6.3 and the proof of (4.1.31) taken together show that there is a natural isomorphism, in the style of Lemma 4.4.8, between the morphism $\partial : \mathcal{H} \to \mathcal{C}^1/\partial(\mathcal{C}^0)$ and the connection map $\mathrm{Hom}_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N}[1/u]/\mathfrak{N}) \to \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$, with $\mathrm{im}\,\partial$ corresponding to $\text{ker-Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$.

Conversely, let h be an element of $\partial(\mathcal{C}^0_u) \cap \mathcal{C}^1$, and set $\nu_i = u^{x_i}\mu_i$. The condition that there is a Breuil–Kisin module $\widetilde{\mathfrak{P}}$ with descent data from K' to K and $\xi \in \mathbf{F}^\times$ such that $\lambda : \widetilde{\mathfrak{P}}[\frac{1}{u}] \to P$ defined as above is an isomorphism is precisely the condition that the map $\hat{\theta}$ on P pulls back via λ to a map that preserves $\mathfrak{P}$. One computes that this pullback is

$$\hat{\theta}\begin{pmatrix} n_i \\ m_i \end{pmatrix} = \xi^{-1}\begin{pmatrix} -\nu_{i+f} & u^{x_i} \\ (\xi^2 - \nu_i\nu_{i+f})u^{-x_i} & \nu_i \end{pmatrix}\begin{pmatrix} n_{i+f} \\ m_{i+f} \end{pmatrix}$$

recalling that $x_i = x_{i+f}$.

We deduce that $\hat{\theta}$ preserves $\mathfrak{P}$ precisely when the ν_i are integral and $\nu_i\nu_{i+f} \equiv \xi^2 \pmod{u^{x_i}}$ for all i. For i with $x_i = 0$ the latter condition is automatic given the former, which is equivalent to the condition that μ_i and μ_{i+f} are both integral. If instead $x_i > 0$, then we have the nontrivial condition $\nu_{i+f} \equiv \xi^2\nu_i^{-1} \pmod{u^{x_i}}$; in other words that μ_i, μ_{i+f} have u-adic valuation exactly $-x_i$, and their principal parts determine one another via the equation $\mu_{i+f} \equiv \xi^2(u^{2x_i}\mu_i)^{-1} \pmod{1}$.

Let $\mathbf{G}_{m,\xi}$ be the multiplicative group with parameter ξ. We now (using the notation of Definition 4.7.15) define $\mathcal{H}' \subset \mathcal{C}^0_u/\mathcal{C}^0 \times \mathbf{G}_{m,\xi}$ to be the subvariety consisting of the pairs (μ, ξ) with exactly the preceding properties; that is, we regard $\mathcal{C}^0_u/\mathcal{C}^0$ as an Ind-affine space in the obvious way, and define $\mathcal{H}'$ to be the pairs (μ, ξ) satisfying

- if $x_i = 0$ then $\mathrm{val}_i\,\mu = \mathrm{val}_{i+f}\,\mu = \infty$, and
- if $x_i > 0$ then $\mathrm{val}_i\,\mu = \mathrm{val}_{i+f}\,\mu = -x_i$ and $\mu_{i+f} \equiv \xi^2(u^{2x_i}\mu_i)^{-1} \pmod{u^0}$

where we write $\mathrm{val}_i\,\mu$ for the u-adic valuation of μ_i, putting $\mathrm{val}_i\,\mu = \infty$ when μ_i is integral.

Putting all this together with (4.7.10), we find that the map

$$\mathcal{H}' \cap (\mathcal{H} \times \mathbf{G}_{m,\xi}) \to Y(\mathfrak{M}, \mathfrak{N})_1$$

sending (μ, ξ) to the pair $(\mathfrak{P}, \widetilde{\mathfrak{P}})$ is a well-defined surjection, where $\mathfrak{P} = \mathfrak{P}(\partial(\mu))$, $\widetilde{\mathfrak{P}}$ is the enrichment of $\mathfrak{P}$ to a Breuil–Kisin module with descent data from K' to K in which $\hat{\theta}$ is pulled back to $\mathfrak{P}$ from P via the map λ as in (4.7.14). (Note that $Y(\mathfrak{M}, \mathfrak{N})_1$ is reduced and of finite type, for example by (4.7.11), so the surjectivity can be checked on $\overline{\mathbf{F}}_p$-points.) In particular $\dim Y(\mathfrak{M}, \mathfrak{N})_1 \le \dim \mathcal{H}'$.

Note that $\mathcal{H}'$ will be empty if for some i we have $x_i > 0$ but $x_i + c_i - d_i \not\equiv 0 \pmod{p^{f'} - 1}$ (so that ν_i cannot be a u-adic unit). Otherwise, the dimension of $\mathcal{H}'$ is easily computed to be $D = 1 + \sum_{i=0}^{f-1}\lceil x_i/(p^{f'} - 1)\rceil$ (indeed if d is the number of nonzero x_i's, then $\mathcal{H}' \cong \mathbf{G}_m^{d+1} \times \mathbf{G}_a^{D-d}$), and since $x_i \le e'/(p-1)$ we find that $\mathcal{H}'$ has dimension at most $1 + \lceil e/(p-1)\rceil f$. This establishes the bound $\dim Y(\mathfrak{M}, \mathfrak{N})_1 \le 1 + \lceil e/(p-1)\rceil f$.

Since $p > 2$ this bound already establishes the theorem when $e > 1$. If instead $e = 1$ the above bound gives $\dim Y(\mathfrak{M}, \mathfrak{N}) \le [K : \mathbf{Q}_p] + 1$. Suppose for the sake of contradiction that equality holds. This is only possible if $\mathcal{H}' \cong \mathbf{G}_m^{f+1}$, $\mathcal{H}' \subset \mathcal{H} \times \mathbf{G}_{m,\xi}$, and $x_i = [d_i - c_i] > 0$ for all i. Define $\mu^{(i)} \in \mathcal{C}^0_u$ to be the element such that $\mu_i = u^{-[d_i - c_i]}$, and $\mu_j = 0$ for $j \neq i$. Let $\mathbf{F}''/\mathbf{F}$ be any finite extension such that $\#\mathbf{F}'' > 3$. For each nonzero $z \in \mathbf{F}''$ define $\mu_z = \sum_{j \neq i, i+f} \mu^{(i)} + z\mu^{(i)} + z^{-1}\mu^{(i+f)}$, so that $(\mu_z, 1)$ is an element of $\mathcal{H}'(\mathbf{F}'')$. Since $\mathcal{H}' \subset \mathcal{H} \times \mathbf{G}_{m,\xi}$ and $\mathcal{H}$ is linear, the differences between the μ_z for varying z lie in $\mathcal{H}(\mathbf{F}'')$, and (e.g. by considering $\mu_1 - \mu_{-1}$ and $\mu_1 - \mu_z$ for any $z \in \mathbf{F}''$ with $z \neq z^{-1}$) we deduce that each $\mu^{(i)}$ lies in $\mathcal{H}$. In particular each $\partial(\mu^{(i)})$ lies in $\mathcal{C}^1$.

If $(i-1, i)$ were not a transition then (since $e = 1$) we would have either $r_i = 0$ or $s_i = 0$. The former would contradict $\partial(\mu^{(i)}) \in \mathcal{C}^1$ (since the ith component of $\partial(\mu^{(i)})$ would be $u^{-[d_i - c_i]}$, of negative degree), and similarly the latter would contradict $\partial(\mu^{(i-1)}) \in \mathcal{C}^1$. Thus $(i-1, i)$ is a transition for all i. In fact the same observations show more precisely that $r_i \geq x_i = [d_i - c_i]$ and $s_i \geq px_{i-1} = p[d_{i-1} - c_{i-1}]$. Summing these inequalities and subtracting e' we obtain $0 \geq p[d_{i-1} - c_{i-1}] - [c_i - d_i]$, and comparing with (4.6.6) shows that we must also have $\gamma_i^* = 0$ for all i. Since $e = 1$ and $(i-1, i)$ is a transition for all i the refined shape of the pair $(\mathfrak{M}, \mathfrak{N})$ is automatically maximal; but then we are in the exceptional case of Proposition 4.6.8, which (recalling the proof of that Proposition) implies that $T(\mathfrak{M}) \cong T(\mathfrak{N})$. This is the desired contradiction. ◻

4.8. **Irreducible components.** We can now use our results on families of extensions of characters to classify the irreducible components of the stacks $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$. In Section 5 we will combine these results with results coming from Taylor–Wiles patching (in particular the results of [GK14, EG14], which we combine in Appendix C) to describe the closed points of each irreducible component of $\mathcal{Z}^{\tau,1}$ in terms of the weight part of Serre's conjecture.

Corollary 4.8.1. *Each irreducible component of $\mathcal{C}^{\tau,\mathrm{BT},1}$ is of the form $\overline{\mathcal{C}}(J)$ for some J; conversely, each $\overline{\mathcal{C}}(J)$ is an irreducible component of $\mathcal{C}^{\tau,\mathrm{BT},1}$.*

Remark 4.8.2. Note that at this point we have not established that different sets J give distinct irreducible components $\overline{\mathcal{C}}(J)$; we will prove this in Section 4.9 below by a consideration of Dieudonné modules.

Proof of Corollary 4.8.1. By Proposition 3.10.20, $\mathcal{C}^{\tau,\mathrm{BT},1}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$. By construction, the $\overline{\mathcal{C}}(J)$ are irreducible substacks of $\mathcal{C}^{\tau,\mathrm{BT},1}$, and by Theorem 4.5.10 they also have dimension $[K : \mathbf{Q}_p]$, so they are in fact irreducible components by [Sta13, Tag 0DS2].

By Theorem 4.7.9 and Theorem 4.5.10, we see that there is a closed substack $\mathcal{C}_{\mathrm{small}}$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$ of dimension strictly less than $[K : \mathbf{Q}_p]$, with the property that every finite type point of $\mathcal{C}^{\tau,\mathrm{BT},1}$ is a point of at least one of the $\overline{\mathcal{C}}(J)$ or of $\mathcal{C}_{\mathrm{small}}$ (or both). (Indeed, we can take $\mathcal{C}_{\mathrm{small}}$ to be the union of the stack $\mathcal{C}_{\mathrm{irred}}$ of Theorem 4.7.9 and the stacks $\overline{\mathcal{C}}(J, r)$ for non-maximal shapes (J, r).) Since $\mathcal{C}^{\tau,\mathrm{BT},1}$ is equidimensional of dimension $[K : \mathbf{Q}_p]$, it follows that the $\overline{\mathcal{C}}(J)$ exhaust the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$, as required. ◻

We now deduce a classification of the irreducible components of $\mathcal{Z}^{\tau,1}$; Theorem 5.2.2 below is a considerable refinement of this, giving a precise description of the finite type points of the irreducible components in terms of the weight part of Serre's conjecture.

Corollary 4.8.3. *The irreducible components of $\mathcal{Z}^{\tau,1}$ are precisely the $\overline{\mathcal{Z}}(J)$ for $J \in \mathcal{P}_\tau$, and if $J \neq J'$ then $\overline{\mathcal{Z}}(J) \neq \overline{\mathcal{Z}}(J')$.*

Proof. By Theorem 4.6.12, if $J \in \mathcal{P}_\tau$ then $\overline{\mathcal{Z}}(J)$ is an irreducible component of $\mathcal{Z}^{\tau,1}$. Furthermore, these $\overline{\mathcal{Z}}(J)$ are pairwise distinct by Theorem 4.6.17.

Since the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$ is scheme-theoretically dominant, it follows from Corollary 4.8.1 that each irreducible component of $\mathcal{Z}^{\tau,1}$ is dominated by some $\overline{\mathcal{C}}(J)$. Applying Theorem 4.6.12 again, we see that if $J \notin \mathcal{P}_\tau$ then $\overline{\mathcal{C}}(J)$ does not dominate an irreducible component, as required. $\square$

4.9. **Dieudonné modules and the morphism to the gauge stack.** We now study the images of the irreducible components $\overline{\mathcal{C}}(J)$ in the gauge stack $\mathcal{G}_\eta$; this amounts to computing the Dieudonné modules and Galois representations associated to the extensions of Breuil–Kisin modules that we considered in Section 4. Suppose throughout this subsection that τ is a non-scalar type, and that (J, r) is a maximal refined shape. Recall that in the cuspidal case this entails that $i \in J$ if and only if $i + f \notin J$.

Lemma 4.9.1. *Let $\mathfrak{P} \in \mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$ be an extension of type τ and refined shape (J, r). Then for $i \in \mathbf{Z}/f'\mathbf{Z}$ we have $F = 0$ on $D(\mathfrak{P})_{\eta,i-1}$ if $i \in J$, while $V = 0$ on $D(\mathfrak{P})_{\eta,i}$ if $i \notin J$.*

Proof. Recall that $D(\mathfrak{P}) = \mathfrak{P}/u\mathfrak{P}$. Let w_i be the image of m_i in $D(\mathfrak{P})$ if $i \in J$, and let w_i be the image of n_i in $D(\mathfrak{P})$ if $i \notin J$. It follows easily from the definitions that $D(\mathfrak{P})_{\eta,i}$ is generated over $\mathbf{F}$ by w_i.

Recall that the actions of F, V on $D(\mathfrak{P})$ are as specified in Definition 2.2.1. In particular F is induced by φ, while V is $c^{-1}\mathfrak{V}$ mod u where $\mathfrak{V}$ is the unique map on $\mathfrak{P}$ satisfying $\mathfrak{V} \circ \varphi = E(u)$, and $c = E(0)$. For the Breuil–Kisin module $\mathfrak{P}$, we have

$$\varphi(n_{i-1}) = b_i u^{s_i} n_i, \qquad \varphi(m_{i-1}) = a_i u^{r_i} m_i + h_i n_i,$$

and so one checks (using that $E(u) = u^{e'}$ in $\mathbf{F}$) that

$$\mathfrak{V}(m_i) = a_i^{-1} u^{s_i} m_{i-1} - a_i^{-1} b_i^{-1} h_i n_{i-1}, \qquad \mathfrak{V}(n_i) = b_i^{-1} u^{r_i} n_{i-1}.$$

From Definition 4.5.4 and the discussion immediately following it, we recall that if $(i-1, i)$ is not a transition then $r_i = e'$, $s_i = 0$, and h_i is divisible by u (the latter because nonzero terms of h_i have degrees congruent to $r_i + c_i - d_i \pmod{p^{f'} - 1}$, and $c_i \not\equiv d_i$ since τ is non-scalar). On the other hand if $(i-1, i)$ is a transition, then $r_i, s_i > 0$, and nonzero terms of h_i have degrees divisible by $p^{f'} - 1$; in that case we write h_i^0 for the constant coefficient of h_i, and we remark that h_i^0 does not vanish identically on $\mathrm{Ext}^1_{\mathcal{K}(\mathbf{F})}(\mathfrak{M}, \mathfrak{N})$.

Suppose, for instance, that $i-1 \in J$ and $i \in J$. Then w_{i-1} and w_i are the images in $D(\mathfrak{P})$ of m_{i-1} and m_i. From the above formulas we see that $u^{r_i} = u^{e'}$ and h_i are both divisible by u, while on the other hand $u^{s_i} = 1$. We deduce that $F(w_{i-1}) = 0$ and $V(w_i) = c^{-1} a_i^{-1} w_{i-1}$. Computing along similar lines, it is easy to check the following four cases.

(1) $i-1 \in J, i \in J$. Then $F(w_{i-1}) = 0$ and $V(w_i) = c^{-1} a_i^{-1} w_{i-1}$.
(2) $i-1 \notin J, i \notin J$. Then $F(w_{i-1}) = b_i w_i$, $V(w_i) = 0$.
(3) $i-1 \in J$, $i \notin J$. Then $F(w_{i-1}) = h_i^0 w_i$, $V(w_i) = 0$.
(4) $i-1 \notin J$, $i \in J$. Then $F(w_{i-1}) = 0$, $V(w_i) = -c^{-1} a_i^{-1} b_i^{-1} h_i^0 w_{i-1}$.

In particular, if $i \in J$ then $F(w_i) = 0$, while if $i \notin J$ then $V(w_{i+1}) = 0$. $\square$

Since $\mathcal{C}^{\tau,\mathrm{BT}}$ is flat over $\mathcal{O}$ by Corollary 3.8.3, it follows from Lemma 3.11.17 that the natural morphism $\mathcal{C}^{\tau,\mathrm{BT}} \to \mathcal{G}_\eta$ is determined by an f-tuple of effective Cartier divisors $\{\mathcal{D}_j\}_{0\le j<f}$ lying in the special fibre $\mathcal{C}^{\tau,\mathrm{BT},1}$. Concretely, $\mathcal{D}_j$ is the zero locus of X_j, which is the zero locus of $F : D_{\eta,j} \to D_{\eta,j+1}$. The zero locus of Y_j (which is the zero locus of $V : D_{\eta,j+1} \to D_{\eta,j}$) is another Cartier divisor $\mathcal{D}'_j$. Since $\mathcal{C}^{\tau,\mathrm{BT},1}$ is reduced, we conclude that each of $\mathcal{D}_j$ and $\mathcal{D}'_j$ is simply a union of irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$, each component appearing precisely once in precisely one of either $\mathcal{D}_j$ or $\mathcal{D}'_j$.

Proposition 4.9.2. *$\mathcal{D}_j$ is equal to the union of the irreducible components $\overline{\mathcal{C}}(J)$ of $\mathcal{C}^{\tau,\mathrm{BT},1}$ for those J that contain $j+1$.*

Proof. Lemma 4.9.1 shows that if $j+1 \in J$, then $X_j = 0$, while if $j+1 \notin J$, then $Y_j = 0$. In the latter case, by an inspection of case (3) of the proof of Lemma 4.9.1, we have $X_j = 0$ if and only if $j \in J$ and $h^0_{j+1} = 0$. Since h^0_{j+1} does not vanish identically on an irreducible component, we see that the irreducible components on which X_j vanishes identically are precisely those for which $j+1 \in J$, as claimed. $\square$

Theorem 4.9.3. *The algebraic stack $\mathcal{C}^{\tau,\mathrm{BT},1}$ has precisely 2^f irreducible components, namely the irreducible substacks $\overline{\mathcal{C}}(J)$.*

Proof. By Corollary 4.8.1, we need only show that if $J \neq J'$ then $\overline{\mathcal{C}}(J) \neq \mathcal{C}(J')$; but this is immediate from Proposition 4.9.2. $\square$

5. Moduli stacks of Galois representations and the geometric Breuil–Mézard conjecture

We now make a more detailed study of the stacks $\mathcal{Z}^{\mathrm{dd}}$ and $\mathcal{Z}^{\tau}$. In particular, we prove a "geometric Breuil–Mézard" result, showing in particular that the finite type points of each irreducible components are precisely described by the weight part of Serre's conjecture. We also prove a new result on the structure of potentially Barsotti–Tate deformation rings, Proposition 5.1.1, showing that their special fibres are generically reduced.

5.1. Generic reducedness of $\operatorname{Spec} R^{\tau,\mathrm{BT}}_{\overline{r}}/\varpi$.

We return to the setting of Subsection 3.10: that is, we fix a finite type point $\operatorname{Spec} \mathbf{F}' \to \mathcal{Z}^{\tau,a}$, where $\mathbf{F}'/\mathbf{F}$ is a finite extension, and let $\overline{r} : G_K \to \mathrm{GL}_2(\mathbf{F}')$ be the corresponding Galois representation. It follows from Corollary 3.10.18 that $\operatorname{Spec} R^{\tau,a}$ is a closed subscheme of $\operatorname{Spec} R^{\tau,\mathrm{BT}}_{\overline{r}}/\varpi^a$, but we have no reason to believe that equality holds. It follows from Lemma 3.9.8, together with Lemma 5.1.6 below, that $\operatorname{Spec} R^{\tau,1}$ is the underlying reduced subscheme of $\operatorname{Spec} R^{\tau,\mathrm{BT}}_{\overline{r}}/\varpi$, so that equality holds in the case $a = 1$ if and only if $\operatorname{Spec} R^{\tau,\mathrm{BT}}_{\overline{r}}/\varpi$ is reduced. Again, we have no reason to believe that this holds in general, but the main result of this section is Proposition 5.1.1 below, showing that $\operatorname{Spec} R^{\tau,\mathrm{BT}}_{\overline{r}}/\varpi$ is generically reduced. We will use this in the proof of our geometric Breuil–Mézard result below. (Recall that a scheme is generically reduced if it contains an open reduced subscheme whose underlying topological space is dense. In the case of a Noetherian affine scheme $\operatorname{Spec} A$, this is equivalent to requiring that the localisation of A at each of its minimal primes is reduced.)

Proposition 5.1.1. *For any tame type τ, the scheme $\operatorname{Spec} R^{\tau,\mathrm{BT}}_{\overline{r}}/\varpi$ is generically reduced, with underlying reduced subscheme $\operatorname{Spec} R^{\tau,1}$.*

We will deduce Proposition 5.1.1 from the following global statement.

Proposition 5.1.2. *Let τ be a tame type. There is a dense open substack $\mathcal{U}$ of $\mathcal{Z}^\tau$ such that $\mathcal{U}_{/\mathbf{F}}$ is reduced.*

Proof. The proposition will follow from an application of Proposition A.11, and the key to this application will be to find a candidate open substack $\mathcal{U}^1$ of $\mathcal{Z}^{\tau,1}$, which we will do using our study of the irreducible components of $\mathcal{C}^{\tau,\mathrm{BT},1}$ and $\mathcal{Z}^{\tau,1}$.

Recall that, for each $J \in \mathcal{P}_\tau$, we let $\overline{\mathcal{Z}}(J)$ denote the scheme-theoretic image of $\overline{\mathcal{C}}(J)$ under the proper morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$. Each $\overline{\mathcal{Z}}(J)$ is a closed substack of $\mathcal{Z}^{\tau,1}$, and so, if we let $\mathcal{V}(J)$ be the complement in $\mathcal{Z}^{\tau,1}$ of the union of the $\overline{\mathcal{Z}}(J')$ for all $J' \neq J$, $J' \in \mathcal{P}_\tau$, then $\mathcal{V}(J)$ is a dense open open substack of $\mathcal{Z}^{\tau,1}$, by Corollary 4.8.3. The preimage $\mathcal{W}(J)$ of $\mathcal{V}(J)$ in $\mathcal{C}^{\tau,\mathrm{BT},1}$ is therefore a dense open substack of $\overline{\mathcal{C}}(J)$. Possibly shrinking $\mathcal{W}(J)$ further, we may suppose by Proposition 4.6.13 that the morphism $\mathcal{W}(J) \to \mathcal{Z}^{\tau,1}$ is a monomorphism.

The complement $|\overline{\mathcal{C}}(J)| \setminus |\mathcal{W}(J)|$ is a closed subset of $|\overline{\mathcal{C}}(J)|$, and thus of $|\mathcal{C}^{\tau,\mathrm{BT},1}|$, and its image under the proper morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \to \mathcal{Z}^{\tau,1}$ is a closed subset of $|\mathcal{Z}^{\tau,\mathrm{BT},1}|$, which is (e.g. for dimension reasons) a proper closed subset of $|\overline{\mathcal{Z}}(J)|$; so if we let $\mathcal{U}(J)$ be the complement in $\mathcal{V}(J)$ of this image, then $\mathcal{U}(J)$ is open and dense in $\overline{\mathcal{Z}}(J)$, and the morphism $\mathcal{C}^{\tau,\mathrm{BT},1} \times_{\mathcal{Z}^{\tau,1}} \mathcal{U}(J) \to \mathcal{U}(J)$ is a monomorphism. Set $\mathcal{U}^1 = \cup_J \mathcal{U}(J)$. Since the $\mathcal{U}(J)$ are pairwise disjoint by construction, $\mathcal{C}^{\tau,\mathrm{BT},1} \times_{\mathcal{Z}^{\tau,1}} \mathcal{U}^1 \to \mathcal{U}^1$ is again a monomorphism. By construction (taking into account Corollary 4.8.3), $\mathcal{U}^1$ is dense in $\mathcal{Z}^{\tau,1}$.

Now let $\mathcal{U}$ denote the open substack of $\mathcal{Z}^\tau$ corresponding to $\mathcal{U}^1$. Since $|\mathcal{Z}^\tau| = |\mathcal{Z}^{\tau,1}|$, we see that $\mathcal{U}$ is dense in $\mathcal{Z}^\tau$. We have seen in the previous paragraph that the statement of Proposition A.11 (5) holds (taking $a = 1$, $\mathcal{X} = \mathcal{C}^{\tau,\mathrm{BT}}$, and $\mathcal{Y} = \mathcal{Z}$); so Proposition A.11 implies that, for each $a \geq 1$, the closed immersion $\mathcal{U} \times_{\mathcal{Z}^\tau} \mathcal{Z}^{\tau,a} \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ is an isomorphism.

In particular, since the closed immersion $\mathcal{U}^1 = \mathcal{U} \times_{\mathcal{Z}^\tau} \mathcal{Z}^{\tau,1} \to \mathcal{U}_{/\mathbf{F}}$ is an isomorphism, we may regard $\mathcal{U}_{/\mathbf{F}}$ as an open substack of $\mathcal{Z}^{\tau,1}$. Since $\mathcal{Z}^{\tau,1}$ is reduced, by Lemma 3.9.8, so is its open substack $\mathcal{U}_{/\mathbf{F}}$. This completes the proof of the proposition. ☐

Corollary 5.1.3. *Let τ be a tame type. There is a dense open substack $\mathcal{U}$ of $\mathcal{Z}^\tau$ such that we have an isomorphism $\mathcal{C}^{\tau,\mathrm{BT}} \times_{\mathcal{Z}^\tau} \mathcal{U} \xrightarrow{\sim} \mathcal{U}$, as well as isomorphisms*

$$\mathcal{U} \times_{\mathcal{Z}^\tau} \mathcal{C}^{\tau,\mathrm{BT},a} \xrightarrow{\sim} \mathcal{U} \times_{\mathcal{Z}^\tau} \mathcal{Z}^{\tau,a} \xrightarrow{\sim} \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a,$$

for each $a \geq 1$.

Proof. This follows from Proposition 5.1.2 and Proposition A.11. ☐

Remark 5.1.4. More colloquially, Corollary 5.1.3 shows that for each tame type τ, there is an open dense substack $\mathcal{U}$ of $\mathcal{Z}^\tau$ consisting of Galois representations which have a unique Breuil–Kisin model of type τ.

Lemma 5.1.5. *If $\mathcal{U}$ is an open substack of $\mathcal{Z}^\tau$ satisfying the conditions of Proposition 5.1.2, and if $T \to \mathcal{Z}^\tau_{/\mathbf{F}}$ is a smooth morphism whose source is a scheme, then $T \times_{\mathcal{Z}^\tau_{/\mathbf{F}}} \mathcal{U}_{/\mathbf{F}}$ is reduced, and is a dense open subscheme of T.*

Proof. Since $\mathcal{Z}^{\tau,1}_{/\mathbf{F}}$ is a Noetherian algebraic stack (being of finite presentation over $\operatorname{Spec} \mathbf{F}$), the open immersion

$$\mathcal{U}_{/\mathbf{F}} \to \mathcal{Z}^\tau_{/\mathbf{F}}$$

is quasi-compact ([Sta13, Tag 0CPM]). Since $T \to \mathcal{Z}^{\tau}_{/\mathbf{F}}$ is flat (being smooth, by assumption), the pullback $T \times_{\mathcal{Z}^{\tau}_{/\mathbf{F}}} \mathcal{U}_{/F} \to T$ is an open immersion with dense image; here we use the fact that for a quasi-compact morphism, the property of being scheme-theoretically dominant is preserved by flat base-change, together with the fact that an open immersion with dense image induces a scheme-theoretically dominant morphism after passing to underlying reduced substacks. Since the source of this morphism is smooth over the reduced algebraic stack $\mathcal{U}_{/\mathbf{F}}$, it is itself reduced. $\square$

The following result is standard, but we recall the proof for the sake of completeness.

Lemma 5.1.6. *Let T be a Noetherian scheme, all of whose local rings at finite type points are G-rings. If T is reduced (resp. generically reduced), then so are all of its complete local rings at finite type points.*

Proof. Let t be a finite type point of T, and write $A := \mathcal{O}_{T,t}$. Then A is a (generically) reduced local G-ring, and we need to show that its completion $\widehat{A}$ is also (generically) reduced. Let $\widehat{\mathfrak{p}}$ be a (minimal) prime of $\widehat{A}$; since $A \to \widehat{A}$ is (faithfully) flat, $\widehat{\mathfrak{p}}$ lies over a (minimal) prime $\mathfrak{p}$ of A by the going-down theorem.

Then $A_{\mathfrak{p}}$ is reduced by assumption, and we need to show that $\widehat{A}_{\widehat{\mathfrak{p}}}$ is reduced. By [Sta13, Tag 07QK], it is enough to show that the morphism $A \to \widehat{A}_{\widehat{\mathfrak{p}}}$ is regular. Both A and $\widehat{A}$ are G-rings (the latter by [Sta13, Tag 07PS]), so the composite

$$A \to \widehat{A} \to (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ }$$

is a composite of regular morphisms, and is thus a regular morphism by [Sta13, Tag 07QI].

This composite factors through the natural morphism $A_{\mathfrak{p}} \to (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ }$, so this morphism is also regular. Factoring it as the composite

$$A_{\mathfrak{p}} \to \widehat{A}_{\widehat{\mathfrak{p}}} \to (\widehat{A}_{\widehat{\mathfrak{p}}})\widehat{\ },$$

it follows from [Sta13, Tag 07NT] that $A_{\mathfrak{p}} \to \widehat{A}_{\widehat{\mathfrak{p}}}$ is regular, as required. $\square$

Proof of Proposition 5.1.1. By Corollary 3.10.18, we have a versal morphism

$$\operatorname{Spf} R^{\tau,\mathrm{BT}}_{\overline{r}}/\varpi \to \mathcal{Z}^{\tau}_{/\mathbf{F}}.$$

Since $\mathcal{Z}^{\tau}_{/\mathbf{F}}$ is an algebraic stack of finite presentation over $\mathbf{F}$ (as $\mathcal{Z}^{\tau}$ is a ϖ-adic formal algebraic stack of finite presentation over $\operatorname{Spf}\mathcal{O}$), we may apply [Sta13, Tag 0DR0] to this morphism so as to find a smooth morphism $V \to \mathcal{Z}^{\tau}_{/\mathbf{F}}$ with source a finite type $\mathcal{O}/\varpi$-scheme, and a point $v \in V$ with residue field $\mathbf{F}'$, such that there is an isomorphism $\widehat{\mathcal{O}}_{V,v} \cong R^{\tau,\mathrm{BT}}_{\overline{r}}/\varpi$, compatible with the given morphism to $\mathcal{Z}^{\tau}_{/\mathbf{F}}$. Proposition 5.1.2 and Lemma 5.1.5 taken together show that V is generically reduced, and so the result follows from Lemma 5.1.6. $\square$

5.2. **The geometric Breuil–Mézard conjecture.** We now study the irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$. We do this by a slightly indirect method, defining certain formal sums of these irreducible components which we then compute via the geometric Breuil–Mézard conjecture, and in particular the results of Appendix C.

By Lemma 3.9.8 and Proposition 3.10.19, $\mathcal{Z}^{\mathrm{dd},1}$ is reduced and equidimensional, and each $\mathcal{Z}^{\tau,1}$ is a union of some of its irreducible components. Let $K(\mathcal{Z}^{\mathrm{dd},1})$ be the free abelian group generated by the irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$. We say that an element of $K(\mathcal{Z}^{\mathrm{dd},1})$ is *effective* if the multiplicity of each irreducible component is nonnegative. We say that an element of $K(\mathcal{Z}^{\mathrm{dd},1})$ is *reduced and effective* if the multiplicity of each irreducible component is 0 or 1.

Let x be a finite type point of $\mathcal{Z}^{\mathrm{dd},1}$, corresponding to a representation $\overline{r}: G_K \to \mathrm{GL}_2(\mathbf{F}')$. As in Section 3.10, there is a quotient R_x of the framed deformation ring $R_{\overline{r}}^{[0,1]}/\varpi$ which is a versal ring to $\mathcal{Z}^{\mathrm{dd},1}$ at x; each $R^{\tau,1}$ is a quotient of R_x. Indeed, since $\mathcal{Z}^{\tau,1}$ is a union of irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$, $\operatorname{Spec} R^{\tau,1}$ is a union of irreducible components of $\operatorname{Spec} R_x$.

Let $K(R_x)$ be the free abelian group generated by the irreducible components of $\operatorname{Spec} R_x$. By [Sta13, Tag 0DRB,Tag 0DRD], there is a natural multiplicity-preserving surjection from the set of irreducible components of $\operatorname{Spec} R_x$ to the set of irreducible components of $\mathcal{Z}^{\mathrm{dd},1}$ which contain x. Using this surjection, we can define a group homomorphism

$$K(\mathcal{Z}^{\mathrm{dd},1}) \to K(R_x)$$

in the following way: we send any irreducible component $\mathcal{Z}$ of $\mathcal{Z}^{\mathrm{dd},1}$ which contains x to the formal sum of the irreducible components of $\operatorname{Spec} R_x$ in the preimage of $\mathcal{Z}$ under this surjection, and we send every other irreducible component to 0.

Lemma 5.2.1. *An element $\overline{\mathcal{T}}$ of $K(\mathcal{Z}^{\mathrm{dd},1})$ is effective if and only if for every finite type point x of $\mathcal{Z}^{\mathrm{dd},1}$, the image of $\overline{\mathcal{T}}$ in $K(R_x)$ is effective. We have $\overline{\mathcal{T}} = 0$ if and only if its image is 0 in every $K(R_x)$.*

Proof. The "only if" direction is trivial, so we need only consider the "if" implication. Write $\overline{\mathcal{T}} = \sum_{\overline{\mathcal{Z}}} a_{\overline{\mathcal{Z}}} \overline{\mathcal{Z}}$, where the sum runs over the irreducible components $\overline{\mathcal{Z}}$ of $\mathcal{Z}^{\mathrm{dd},1}$, and the $a_{\overline{\mathcal{Z}}}$ are integers.

Suppose first that the image of $\overline{\mathcal{T}}$ in $K(R_x)$ is effective; we then have to show that each $a_{\overline{\mathcal{Z}}}$ is nonnegative. To see this, fix an irreducible component $\overline{\mathcal{Z}}$, and choose x to be a finite type point of $\mathcal{Z}^{\mathrm{dd},1}$ which is contained in $\overline{\mathcal{Z}}$ and in no other irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$. Then the image of $\overline{\mathcal{T}}$ in $K(R_x)$ is equal to $a_{\overline{\mathcal{Z}}}$ times the sum of the irreducible components of $\operatorname{Spec} R_x$. By hypothesis, this must be effective, which implies that $a_{\overline{\mathcal{Z}}}$ is nonnegative, as required.

Finally, if the image of $\overline{\mathcal{T}}$ in $K(R_x)$ is 0, then $a_{\overline{\mathcal{Z}}} = 0$; so if this holds for all x, then $\overline{\mathcal{T}} = 0$. □

For each tame type τ, we let $\mathcal{Z}(\tau)$ denote the formal sum of the irreducible components of $\mathcal{Z}^{\tau,1}$, considered as an element of $K(\mathcal{Z}^{\mathrm{dd},1})$. By Lemma C.1, for each non-Steinberg Serre weight $\overline{\sigma}$ of $\mathrm{GL}_2(k)$, there are integers $n_\tau(\overline{\sigma})$ such that $\overline{\sigma} = \sum_\tau n_\tau(\overline{\sigma})\overline{\sigma}(\tau)$ in the Grothendieck group of mod p representations of $\mathrm{GL}_2(k)$, where the τ run over the tame types. We set

$$\mathcal{Z}(\overline{\sigma}) := \sum_\tau n_\tau(\overline{\sigma})\mathcal{Z}(\tau) \in K(\mathcal{Z}^{\mathrm{dd},1}).$$

The integers $n_\tau(\overline{\sigma})$ are not necessarily unique, but it follows from the following result that $\mathcal{Z}(\overline{\sigma})$ is independent of the choice of $n_\tau(\overline{\sigma})$, and is reduced and effective.

Theorem 5.2.2. *(1) Each $\mathcal{Z}(\overline{\sigma})$ is an irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$.*

(2) The finite type points of $\mathcal{Z}(\overline{\sigma})$ are precisely the representations $\overline{r}: G_K \to \mathrm{GL}_2(\mathbf{F}')$ having $\overline{\sigma}$ as a Serre weight.

(3) For each tame type τ, we have $\mathcal{Z}(\tau) = \sum_{\overline{\sigma} \in \mathrm{JH}(\overline{\sigma}(\tau))} \mathcal{Z}(\overline{\sigma})$.

(4) Every irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$ is of the form $\mathcal{Z}(\overline{\sigma})$ for some unique Serre weight $\overline{\sigma}$.

(5) For each tame type τ, and each $J \in \mathcal{P}_\tau$, we have $\mathcal{Z}(\overline{\sigma}_J) = \overline{\mathcal{Z}}(J)$.

Proof. Let x be a finite type point of $\mathcal{Z}^{\mathrm{dd},1}$ corresponding to $\overline{r} : G_K \to \mathrm{GL}_2(\mathbf{F}')$, and write $\mathcal{Z}(\overline{\sigma})_x$, $\mathcal{Z}(\tau)_x$ for the images in $K(R_x)$ of $\mathcal{Z}(\overline{\sigma})$ and $\mathcal{Z}(\tau)$ respectively. Each $\operatorname{Spec} R^{\tau,1}$ is a closed subscheme of $\operatorname{Spec} R^{\square}$, the universal framed deformation $\mathcal{O}_{E'}$-algebra for $\overline{r}$, so we may regard the $\mathcal{Z}(\tau)_x$ as formal sums (with multiplicities) of irreducible subschemes of $\operatorname{Spec} R^{\square}/\pi$.

By definition, $\mathcal{Z}(\tau)_x$ is just the underlying cycle of $\operatorname{Spec} R^{\tau,1}$. By Proposition 5.1.1, this is equal to the underlying cycle of $\operatorname{Spec} R_{\overline{r}}^{\tau,\mathrm{BT}}/\varpi$. Consequently, $\mathcal{Z}(\overline{\sigma})_x$ is the cycle denoted by $\mathcal{C}_{\overline{\sigma}}$ in Appendix C. It follows from Theorem C.4 that:

- $\mathcal{Z}(\overline{\sigma})_x$ is effective, and is nonzero precisely when $\overline{\sigma}$ is a Serre weight for $\overline{r}$.
- For each tame type τ, we have $\mathcal{Z}(\tau)_x = \sum_{\overline{\sigma} \in \mathrm{JH}(\overline{\sigma}(\tau))} \mathcal{Z}(\overline{\sigma})_x$.

Applying Lemma 5.2.1, we see that each $\mathcal{Z}(\overline{\sigma})$ is effective, and that (3) holds. Since $\mathcal{Z}^{\tau,1}$ is reduced, $\mathcal{Z}(\tau)$ is reduced and effective, so it follows from (3) that each $\mathcal{Z}(\overline{\sigma})$ is reduced and effective. Since x is a finite type point of $\mathcal{Z}(\overline{\sigma})$ if and only if $\mathcal{Z}(\overline{\sigma})_x \neq 0$, we have also proved (2).

Since every irreducible component of $\mathcal{Z}^{\mathrm{dd},1}$ is an irreducible component of some $\mathcal{Z}^{\tau,1}$, in order to prove (1) and (4) it suffices to show that for each τ, every irreducible component of $\mathcal{Z}^{\tau,1}$ is of the form $\mathcal{Z}(\overline{\sigma}_J)$ for some J, and that each $\mathcal{Z}(\overline{\sigma}_J)$ is irreducible. Now, by Corollary 4.8.3, we know that $\mathcal{Z}^{\tau,1}$ has exactly $\#\mathcal{P}_\tau$ irreducible components, namely the $\overline{\mathcal{Z}}(J')$ for $J' \in \mathcal{P}_\tau$. On the other hand, the $\mathcal{Z}(\overline{\sigma}_J)$ are reduced and effective, and since there certainly exist representations admitting $\overline{\sigma}_J$ as their unique Serre weight, it follows from (2) that for each J, there must be a $J' \in \mathcal{P}_\tau$ such that $\overline{\mathcal{Z}}(J')$ contributes to $\mathcal{Z}(\overline{\sigma}_J)$, but not to any $\mathcal{Z}(\overline{\sigma}_{J''})$ for $J'' \neq J$.

Since $\mathcal{Z}(\tau)$ is reduced and effective, and the sum in (3) is over $\#\mathcal{P}_\tau$ weights $\overline{\sigma}$, it follows that we in fact have $\mathcal{Z}(\overline{\sigma}_J) = \overline{\mathcal{Z}}(J')$. This proves (1) and (4), and to prove (5), it only remains to show that $J' = J$. To see this, note that by (2), $\mathcal{Z}(\overline{\sigma}_J) = \overline{\mathcal{Z}}(J')$ has a dense open substack whose finite type points have $\overline{\sigma}_J$ as their unique non-Steinberg Serre weight (namely the complement of the union of the $\mathcal{Z}(\overline{\sigma}')$ for all $\overline{\sigma}' \neq \overline{\sigma}_J$). By Theorem 4.6.17, it also has a dense open substack whose finite type points have $\overline{\sigma}_{J'}$ as a Serre weight. Considering any finite type point in the intersection of these dense open substacks, we see that $\overline{\sigma}_J = \overline{\sigma}_{J'}$, so that $J = J'$, as required. $\square$

Appendix A. Formal algebraic stacks

In this appendix we briefly recall some basic definitions and facts concerning formal algebraic stacks; our primary reference is [Eme]. We then develop some simple geometric lemmas which will be applied in the main body of the paper.

We first recall the definition of a formal algebraic stack [Eme, Def. 5.3].

Definition A.1. An *fppf* stack in groupoids $\mathcal{X}$ over a scheme S is called a *formal algebraic stack* if there is a morphism $U \to \mathcal{X}$, whose domain U is a formal algebraic space over S (in the sense of [Sta13, Tag 0AIL]), and which is representable by algebraic spaces, smooth, and surjective.

We will be primarily interested in the case when $S = \operatorname{Spec} \mathcal{O}$, where, as in the main body of the paper, $\mathcal{O}$ is the ring of integers in a finite extension E of $\mathbf{Q}_p$. We let ϖ denote a uniformiser of $\mathcal{O}$, and let $\operatorname{Spf}\mathcal{O}$ denote the affine formal scheme (or affine formal algebraic space, in the terminology of [Sta13]) obtained by ϖ-adically completing $\operatorname{Spec}\mathcal{O}$.

Among all the formal algebraic stacks over $\operatorname{Spec}\mathcal{O}$, we single out the ϖ-adic formal algebraic stacks as being of particular interest. The following definition is a particular case of [Eme, Def. 7.6].

Definition A.2. A formal algebraic stack $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}$ is called ϖ-adic if the canonical map $\mathcal{X} \to \operatorname{Spec}\mathcal{O}$ factors through $\operatorname{Spf}\mathcal{O}$, and if the induced map $\mathcal{X} \to \operatorname{Spf}\mathcal{O}$ is algebraic, i.e. representable by algebraic stacks (in the sense of [Sta13, Tag 06CF] and [Eme, Def. 3.1]).

We refer to [Eme] for the various other notions related to formal algebraic stacks that we employ. We also recall the following key lemma, which allows us to recognise certain Ind-algebraic stacks as being formal algebraic stacks [Eme, Lem. 6.3].

Lemma A.3. *If $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{X}_n \hookrightarrow \cdots$ is a sequence of finite order thickenings of algebraic stacks, then $\varinjlim_n \mathcal{X}_n$ is a formal algebraic stack.*

A.4. **Open and closed substacks.** Suppose first that $\mathcal{X}$ is an algebraic stack over some scheme S. An open substack of $\mathcal{X}$ is then, by definition ([Sta13, Tag 04YM]), a strictly full substack $\mathcal{X}'$ such that the natural morphism $\mathcal{X}' \to \mathcal{X}$ is an open immersion; that is, it is representable by algebraic spaces, and an open immersion on all pullbacks to algebraic spaces. A closed substack is defined in the analogous way. The stack $\mathcal{X}$ has an underlying topological space $|\mathcal{X}|$, and the open subsets of $|\mathcal{X}|$ are in natural bijection with the open substacks of $\mathcal{X}$ by [Sta13, Tag 06FJ].

We now note that the preceding definitions of open and closed substacks in fact apply perfectly well to a stack over S which is not assumed to be algebraic (see e.g. [Eme, Def. 3.26]). In particular, we can apply it in the case when $\mathcal{X}$ is a formal algebraic stack.

We make an important observation: if $\mathcal{X} \hookrightarrow \mathcal{X}'$ is a morphism of stacks over S which is representable by algebraic spaces and is a thickening (in the sense of [Sta13, Tag 0BPN]), then pull-back under this morphism induces a bijection between open substacks of $\mathcal{X}'$ and open substacks of $\mathcal{X}$. (If $\mathcal{U} \to \mathcal{X}$ is an open immersion, and T is an S-scheme, we define

$$\mathcal{U}'(T) := \{T \to \mathcal{X}' \mid \text{ the base-changed morphism } \mathcal{X} \times_{\mathcal{X}'} T \to \mathcal{X} \text{ factors through } \mathcal{U}\}.$$

We leave it to the reader to check that $\mathcal{U}'$ is an open substack of $\mathcal{X}'$, and that $\mathcal{U} \mapsto \mathcal{U}'$ and $\mathcal{U}' \mapsto \mathcal{X} \times_{\mathcal{X}'} \mathcal{U}'$ are mutually inverse; see also [Eme, Lem. 3.41], where this result is established in the context of a more general statement about the topological invariance of the étale site.)

Finally, we say that an open substack $\mathcal{U}$ of a formal algebraic stack $\mathcal{X}$ is *dense* if its underlying topological space $|\mathcal{U}|$ is dense in $|\mathcal{X}|$. Note that this need not imply that it is scheme-theoretically dense (as is already the case if $\mathcal{X}$ is a non-reduced scheme).

A.5. **A geometric situation.** We suppose given a commutative diagram of morphisms of formal algebraic stacks

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr] & \mathcal{Y} \arrow[d] \\ & \operatorname{Spf} \mathcal{O} \end{tikzcd}$$

We suppose that each of $\mathcal{X}$ and $\mathcal{Y}$ is quasi-compact and quasi-separated, and that the horizontal arrow is scheme-theoretically dominant, in the sense of [Eme, Def. 6.13]. We furthermore suppose that the morphism $\mathcal{X} \to \operatorname{Spf} \mathcal{O}$ realises $\mathcal{X}$ as a finite type ϖ-adic formal algebraic stack.

Concretely, if we write $\mathcal{X}^a := \mathcal{X} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$, then each $\mathcal{X}^a$ is an algebraic stack, locally of finite type over $\operatorname{Spec} \mathcal{O}/\varpi^a$, and there is an isomorphism $\varinjlim_a \mathcal{X}^a \xrightarrow{\sim} \mathcal{X}$. Furthermore, the assumption that the horizontal arrow is scheme-theoretically dominant means that we may find an isomorphism $\mathcal{Y} \cong \varinjlim_a \mathcal{Y}^a$, with each $\mathcal{Y}^a$ being a quasi-compact and quasi-separated algebraic stack, and with the transition morphisms being thickenings, such that the morphism $\mathcal{X} \to \mathcal{Y}$ is induced by a compatible family of morphisms $\mathcal{X}^a \to \mathcal{Y}^a$, each of which is scheme-theoretically dominant. (The $\mathcal{Y}^a$ are uniquely determined by the requirement that for all $b \geq a$ large enough so that the morphism $\mathcal{X}^a \to \mathcal{Y}$ factors through $\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^b$, $\mathcal{Y}^a$ is the scheme-theoretic image of the morphism $\mathcal{X}^a \to \mathcal{Y} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^b$. In particular, $\mathcal{Y}^a$ is a closed substack of $\mathcal{Y} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$.)

It is often the case, in the preceding situation, that $\mathcal{Y}$ is also a ϖ-adic formal algebraic stack. For example, we have the following result. (Note that the usual graph argument shows that the morphism $\mathcal{X} \to \mathcal{Y}$ is necessarily algebraic, i.e. representable by algebraic stacks, in the sense of [Sta13, Tag 06CF] and [Eme, Def. 3.1]. Thus it makes sense to speak of it being proper, following [Eme, Def. 3.11].)

Proposition A.6. *Suppose that the morphism $\mathcal{X} \to \mathcal{Y}$ is proper, and that $\mathcal{Y}$ is locally Ind-finite type over* $\operatorname{Spec} \mathcal{O}$ (*in the sense of* [Eme, Rem. 8.30]). *Then $\mathcal{Y}$ is a ϖ-adic formal algebraic stack.*

Proof. This is an application of [Eme, Prop. 10.5]. $\square$

A key point is that, because the formation of scheme-theoretic images is not generally compatible with non-flat base-change, the closed immersion

$$\mathcal{Y}^a \hookrightarrow \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}/\varpi^a \tag{A.7}$$

is typically *not* an isomorphism, even if $\mathcal{Y}$ is a ϖ-adic formal algebraic stack. Our goal in the remainder of this discussion is to give a criterion (involving the morphism $\mathcal{X} \to \mathcal{Y}$) on an open substack $\mathcal{U} \hookrightarrow \mathcal{Y}$ which guarantees that the closed immersion $\mathcal{U} \times_{\mathcal{Y}} \mathcal{Y}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ induced by (A.7) *is* an isomorphism.

We begin by establishing a simple lemma. For any $a \geq 1$, we have the 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}^a \arrow[r] \arrow[d] & \mathcal{Y}^a \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{A.8}$$

Similarly, if $b \geq a \geq 1$, then we have the 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}^a \arrow[r] \arrow[d] & \mathcal{Y}^a \arrow[d] \\ \mathcal{X}^b \arrow[r] & \mathcal{Y}^b \end{tikzcd} \tag{A.9}$$

Lemma A.10. *Each of the diagrams (A.8) and (A.9) is 2-Cartesian.*

Proof. We may embed the diagram (A.8) in the larger 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}^a \arrow[r] \arrow[d] & \mathcal{Y}^a \arrow[r, "(A.7)"] \arrow[d] & \mathcal{Y} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^a \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \arrow[r, equal] & \mathcal{Y} \end{tikzcd}$$

Since the outer rectangle is manifestly 2-Cartesian, and since (A.7) is a closed immersion (and thus a monomorphism), we conclude that (A.8) is indeed 2-Cartesian.

A similar argument shows that (A.9) is 2-Cartesian. □

We next note that, since each of the closed immersions $\mathcal{Y}^a \hookrightarrow \mathcal{Y}$ is a thickening, giving an open substack $\mathcal{U} \hookrightarrow \mathcal{Y}$ is equivalent to giving an open substack $\mathcal{U}^a \hookrightarrow \mathcal{Y}^a$ for some, or equivalently, every, choice of $a \geq 1$; the two pieces of data are related by the formulas $\mathcal{U}^a := \mathcal{U} \times_{\mathcal{Y}} \mathcal{Y}^a$ and $\varinjlim_a \mathcal{U}^a \xrightarrow{\sim} \mathcal{U}$.

Proposition A.11. *Suppose that $\mathcal{X} \to \mathcal{Y}$ is proper. If $\mathcal{U}$ is an open substack of $\mathcal{Y}$, then the following conditions are equivalent:*

1. *The morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is a monomorphism.*
2. *The morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is an isomorphism.*
3. *For every $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a monomorphism.*
4. *For every $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is an isomorphism.*
5. *For some $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a monomorphism.*
6. *For some $a \geq 1$, the morphism $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is an isomorphism.*

Furthermore, if these equivalent conditions hold, then the closed immersion $\mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ is an isomorphism, for each $a \geq 1$.

Proof. The key point is that Lemma A.10 implies that the diagram

$$\begin{tikzcd} \mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \arrow[r] \arrow[d] & \mathcal{U}^a \arrow[d] \\ \mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \arrow[r] & \mathcal{U} \end{tikzcd}$$

is 2-Cartesian, for any $a \geq 1$, and similarly, that if $b \geq a \geq 1$, then the diagram

$$\begin{tikzcd} \mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \arrow[r] \arrow[d] & \mathcal{U}^a \arrow[d] \\ \mathcal{X}^b \times_{\mathcal{Y}^b} \mathcal{U}^b \arrow[r] & \mathcal{U}^b \end{tikzcd}$$

is 2-Cartesian. Since the vertical arrows of this latter diagram are finite order thickenings, we find (by applying the analogue of [Sta13, Tag 09ZZ] for algebraic stacks, whose straightforward deduction from that result we leave to the reader) that the top horizontal arrow is a monomorphism if and only if the bottom horizontal arrow is. This shows the equivalence of (3) and (5). Since the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} \to \mathcal{U}$ is obtained as the inductive limit of the various morphisms $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$, we find that (3) implies (1) (by applying e.g. [Eme, Lem. 4.11 (1)], which shows that the inductive limit of monomorphisms is a monomorphism), and also that (4) implies (2) (the inductive limit of isomorphisms being again an isomorphism).

Conversely, if (1) holds, then the base-changed morphism

$$\mathcal{X} \times_{\mathcal{Y}} (\mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a) \to \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$$

is a monomorphism. The source of this morphism admits an alternative description as $\mathcal{X}^a \times_{\mathcal{Y}} \mathcal{U}$, which the 2-Cartesian diagram at the beginning of the proof allows us to identify with $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a$. Thus we obtain a monomorphism

$$\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}_{/}\varpi^a.$$

Since this monomorphism factors through the closed immersion $\mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$, we find that each of the morphisms of (3) is a monomorphism; thus (1) implies (3). Similarly, (2) implies (4), and also implies that the closed immersion $\mathcal{U}^a \hookrightarrow \mathcal{U} \times_{\mathcal{O}} \mathcal{O}/\varpi^a$ is an isomorphism, for each $a \geq 1$.

Since clearly (4) implies (6), while (6) implies (5), to complete the proof of the proposition, it suffices to show that (5) implies (6). Suppose then that $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a monomorphism. Since $\mathcal{U}^a \hookrightarrow \mathcal{Y}^a$ is an open immersion, it is in particular flat. Since $\mathcal{X}^a \to \mathcal{Y}^a$ is scheme-theoretically dominant and quasi-compact (being proper), any flat base-change of this morphism is again scheme-theoretically dominant, as well as being proper. Thus we see that $\mathcal{X}^a \times_{\mathcal{Y}^a} \mathcal{U}^a \to \mathcal{U}^a$ is a scheme-theoretically dominant proper monomorphism, i.e. a scheme-theoretically dominant closed immersion, i.e. an isomorphism, as required. $\square$

Appendix B. Serre weights and tame types

We begin by recalling some results from [Dia07] on the Jordan–Hölder factors of the reductions modulo p of lattices in principal series and cuspidal representations of $\mathrm{GL}_2(k)$, following [EGS15, §3] (but with slightly different normalisations than those of *loc. cit.*).

Let τ be a tame inertial type. Recall from Section 1.7 that we associate a representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$ to τ as follows: if $\tau \simeq \eta \oplus \eta'$ is a tame principal series type, then we set $\sigma(\tau) := \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \eta' \otimes \eta$, while if $\tau = \eta \oplus \eta^q$ is a tame cuspidal type, then $\sigma(\tau)$ is the inflation to $\mathrm{GL}_2(\mathcal{O}_K)$ of the cuspidal representation of $\mathrm{GL}_2(k)$ denoted by $\Theta(\eta)$ in [Dia07]. (Here we have identified η, η' with their composites with Art_K.)

Write $\overline{\sigma}(\tau)$ for the semisimplification of the reduction modulo p of (a $\mathrm{GL}_2(\mathcal{O}_K)$-stable $\mathcal{O}$-lattice in) $\sigma(\tau)$. The action of $\mathrm{GL}_2(\mathcal{O}_K)$ on $\overline{\sigma}(\tau)$ factors through $\mathrm{GL}_2(k)$, so the Jordan–Hölder factors $\mathrm{JH}(\overline{\sigma}(\tau))$ of $\overline{\sigma}(\tau)$ are Serre weights. By the results of [Dia07], these Jordan–Hölder factors of $\overline{\sigma}(\tau)$ are pairwise non-isomorphic, and are parametrised by the set $\mathcal{P}_\tau$ (defined in Section 4.5.6) in a fashion that we now recall.

Suppose first that $\tau = \eta \oplus \eta'$ is a tame principal series type. Set $f' = f$ in this case. We define $0 \leq \gamma_i \leq p-1$ (for $i \in \mathbf{Z}/f\mathbf{Z}$) to be the unique integers not all equal to $p-1$ such that $\eta(\eta')^{-1} = \prod_{i=0}^{f-1} \omega_{\sigma_i}^{\gamma_i}$. If instead $\tau = \eta \oplus \eta'$ is a cuspidal type, set $f' = 2f$. We define $0 \leq \gamma_i \leq p-1$ (for $i \in \mathbf{Z}/f'\mathbf{Z}$) to be the unique integers such that $\eta(\eta')^{-1} = \prod_{i=0}^{f'-1} \omega_{\sigma'_i}^{\gamma_i}$. Here the σ'_i are the embeddings $l \to \mathbf{F}$, where l is the quadratic extension of k, σ'_0 is a fixed choice of embedding extending σ_0, and $(\sigma'_{i+1})^p = \sigma'_i$ for all i.

If τ is scalar then we set $\mathcal{P}_\tau = \{\varnothing\}$. Otherwise we have $\eta \neq \eta'$, and as in Section 4.5.6 we let $\mathcal{P}_\tau$ be the collection of subsets $J \subset \mathbf{Z}/f'\mathbf{Z}$ satisfying the conditions:

- if $i-1 \in J$ and $i \notin J$ then $\gamma_i \neq p-1$, and
- if $i-1 \notin J$ and $i \in J$ then $\gamma_i \neq 0$

and, in the cuspidal case, satisfying the further condition that $i \in J$ if and only if $i+f \notin J$.

The Jordan–Hölder factors of $\overline{\sigma}(\tau)$ are by definition Serre weights, and are parametrised by $\mathcal{P}_\tau$ as follows (see [EGS15, §3.2, 3.3]). For any $J \subseteq \mathbf{Z}/f'\mathbf{Z}$, we let δ_J denote the characteristic function of J, and if $J \in \mathcal{P}_\tau$ we define $s_{J,i}$ by

$$s_{J,i} = \begin{cases} p-1-\gamma_i-\delta_{J^c}(i) & \text{if } i-1 \in J \\ \gamma_i - \delta_J(i) & \text{if } i-1 \notin J, \end{cases}$$

and we set $t_{J,i} = \gamma_i + \delta_{J^c}(i)$ if $i-1 \in J$ and 0 otherwise.

In the principal series case we let $\overline{\sigma}(\tau)_J := \overline{\sigma}_{\vec{t},\vec{s}} \otimes \eta' \circ \det$; the $\overline{\sigma}(\tau)_J$ are precisely the Jordan–Hölder factors of $\overline{\sigma}(\tau)$.

In the cuspidal case, one checks that $s_{J,i} = s_{J,i+f}$ for all i, and also that the character $\eta' \cdot \prod_{i=0}^{f'-1} (\sigma'_i)^{t_{J,i}} : l^\times \to \mathbf{F}^\times$ factors as $\theta \circ N_{l/k}$ where $N_{l/k}$ is the norm map. We let $\overline{\sigma}(\tau)_J := \overline{\sigma}_{0,\vec{s}} \otimes \theta \circ \det$; the $\overline{\sigma}(\tau)_J$ are precisely the Jordan–Hölder factors of $\overline{\sigma}(\tau)$.

Remark B.1. The parameterisations above are easily deduced from those given in [EGS15, §3.2, 3.3] for the Jordan–Hölder factors of the representations $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \eta' \otimes \eta$ and $\Theta(\eta)$. (Note that there is a minor mistake in [EGS15, §3.1]: since the conventions of [EGS15] regarding the inertial Langlands correspondence agree with those of [GK14], the explicit identification of $\sigma(\tau)$ with a principal series or cuspidal type in [EGS15, §3.1] is missing a dual. The explicit parameterisation we are using here is of course independent of this issue.

This mistake has the unfortunate effect that various explicit formulae in [EGS15, §7] need to be modified in a more or less obvious fashion; note that since $\sigma(\tau)$ is self dual up to twist, all formulae can be fixed by making twists and/or exchanging η and η'. In particular, the definition of the strongly divisible module before [EGS15, Rem. 7.3.2] is incorrect as written, and can be fixed by either reversing the roles of η, η' or changing the definition of the quantity $c^{(j)}$ defined there.)

Remark B.2. In the cuspidal case, write η in the form $(\sigma'_0)^{(q+1)b+1+c}$ where $0 \le b \le q-2$, $0 \le c \le q-1$. Set $t'_{J,i} = t_{J,i+f}$ for integers $1 \le i \le f$. Then one can check that $\overline{\sigma}(\tau)_J = \overline{\sigma}_{\vec{t}',\vec{s}} \otimes (\sigma_0^{(q+1)b+\delta_J(0)} \circ \det)$.

We now recall some facts about the set of Serre weights $W(\overline{r})$ associated to a representation $\overline{r} : G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$.

Definition B.3. *We say that a crystalline representation $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has* type $\overline{\sigma}_{\vec{t},\vec{s}}$ *provided that for each embedding $\sigma_j : k \hookrightarrow \mathbf{F}$ there is an embedding $\widetilde{\sigma}_j : K \hookrightarrow \overline{\mathbf{Q}}_p$ lifting σ_j such that the $\widetilde{\sigma}_j$-labeled Hodge–Tate weights of r are $\{-s_j - t_j, 1 - t_j\}$, and the remaining $(e-1)f$ pairs of Hodge–Tate weights of r are* *all* $\{0,1\}$. (*In particular the representations of type* $\overline{\sigma}_{\vec{0},\vec{0}}$ (*the trivial weight*) *are the same as those of Hodge type* 0.)

Definition B.4. *Given a representation* $\overline{r}: G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ *we define* $W(\overline{r})$ *to be the set of Serre weights* $\overline{\sigma}$ *such that* $\overline{r}$ *has a crystalline lift of type* $\overline{\sigma}$.

It follows easily from the formula $\overline{\sigma}^\vee_{\vec{t},\vec{s}} = \overline{\sigma}_{-\vec{s}-\vec{t},\vec{s}}$ that $\overline{\sigma} \in W(\overline{r})$ if and only if $\overline{\sigma}^\vee$ is in the set of Serre weights associated to $\overline{r}^\vee$ in [GLS15, Defn. 4.1.3].

There are several definitions of the set $W(\overline{r})$ in the literature, which by the papers [BLGG13, GK14, GLS15] are known to be equivalent (up to normalisation). While the preceding definition is perhaps the most compact, it is the description of $W(\overline{r})$ via the Breuil–Mézard conjecture that appears to be the most amenable to generalisation; see Theorem C.4 for an instance of this description, and [GHS18] for much more discussion.

Recall that $\overline{r}$ is *très ramifiée* if it is a twist of an extension of the trivial character by the mod p cyclotomic character, and if furthermore the splitting field of its projective image is *not* of the form $K(\alpha_1^{1/p},\dots,\alpha_s^{1/p})$ for some $\alpha_1,\dots,\alpha_s \in \mathcal{O}_K^\times$.

Lemma B.5. *(1) If τ is a tame type, then $\overline{r}$ has a potentially Barsotti–Tate lift of type τ if and only if $W(\overline{r}) \cap \mathrm{JH}(\overline{\sigma}(\tau)) \neq 0$.*

(2) The following conditions are equivalent:
- *(a) $\overline{r}$ admits a potentially Barsotti–Tate lift of some tame type.*
- *(b) $W(\overline{r})$ contains a non-Steinberg Serre weight.*
- *(c) $\overline{r}$ is not très ramifiée.*

Proof. (1) By the main result of [GLS15], and bearing in mind the differences between our conventions and those of [GK14] as recalled in Section 1.7, we have $\overline{\sigma} \in W(\overline{r})$ if and only if $\overline{\sigma}^\vee \in W^{\mathrm{BT}}(\overline{r})$, where $W^{\mathrm{BT}}(\overline{r})$ is the set of weights defined in [GK14, §3]. By [GK14, Cor. 3.5.6] (bearing in mind once again the differences between our conventions and those of [GK14]), it follows that we have $W(\overline{r}) \cap \mathrm{JH}(\overline{\sigma}(\tau)) \neq 0$ if and only if $e(R^{\square,0,\tau}/\pi) \neq 0$ in the notation of *loc. cit.*, and by definition $\overline{r}$ has a potentially Barsotti–Tate lift of type τ if and only if $R^{\square,0,\tau} \neq 0$. It follows from [EG14, Prop. 4.1.2] that $R^{\square,0,\tau} \neq 0$ if and only if $e(R^{\square,0,\tau}/\pi) \neq 0$, as required.

(2) By part (1), condition (a) is equivalent to $W(\overline{r})$ containing a Serre weight occurring as a Jordan–Hölder factor of $\overline{\sigma}(\tau)$ for some tame type τ. It is easily seen (either by inspection, or by Lemma C.1 below) that the Serre weights occurring as Jordan–Hölder factors of the $\overline{\sigma}(\tau)$ are precisely the non-Steinberg Serre weights, so (a) and (b) are equivalent.

Suppose that (a) holds; then $\overline{r}$ becomes finite flat over a tame extension. However the restriction to a tame extension of a très ramifiée representation is still très ramifiée, and therefore not finite flat, so (c) also holds. Conversely, suppose for the sake of contradiction that (c) holds, but that (b) does not hold, i.e. that $W(\overline{r})$ consists of a single Steinberg weight.

Twisting, we can without loss of generality assume that $W(\overline{r}) = \{\overline{\sigma}_{\vec{0},\vec{p-1}}\}$. By [GK14, Cor. A.5] we can globalise $\overline{r}$, and then the hypothesis that $W(\overline{r})$ contains $\overline{\sigma}_{\vec{0},\vec{p-1}}$ implies that it has a semistable lift of Hodge type 0. If this lift were in fact crystalline, then $W(\overline{r})$ would also contain the weight $\overline{\sigma}_{\vec{0},\vec{0}}$ by (1). So this lift is not crystalline, and in particular the monodromy operator N on the corresponding weakly admissible module is nonzero. But then $\ker(N)$ is a free filtered submodule of rank 1, and since the lift has Hodge type 0, $\ker(N)$ is in fact a weakly admissible submodule. It follows that the lift is an unramified twist of an extension of ε^{-1} by the trivial character, so that $\overline{r}$ is an unramified twist of an extension of $\overline{\varepsilon}^{-1}$ by the trivial character. But we are assuming that (c) holds, so $\overline{r}$ is finite flat, so that by (1), $W(\overline{r})$ contains the weight $\overline{\sigma}_{\vec{0},\vec{0}}$, a contradiction. □

Lemma B.6. *Suppose that $\overline{\sigma}_{\vec{t},\vec{s}}$ is a non-Steinberg Serre weight. Suppose that $\overline{r}: G_K \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is a reducible representation satisfying*

$$\overline{r}|_{I_K} \cong \begin{pmatrix} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{s_i+t_i} & * \\ 0 & \overline{\varepsilon}^{-1} \prod_{i=0}^{f-1} \omega_{\sigma_i}^{t_i} \end{pmatrix},$$

and that $\overline{r}$ is not très ramifiée. Then $\overline{\sigma}_{\vec{t},\vec{s}} \in W(\overline{r})$.

Proof. Write $\overline{r}$ as an extension of characters $\overline{\chi}$ by $\overline{\chi}'$. It is straightforward from the classification of crystalline characters as in [GHS18, Lem. 5.1.6] that there exist crystalline lifts χ, χ' of $\overline{\chi}, \overline{\chi}'$ so that χ, χ' have Hodge–Tate weights $1 - t_j$ and $-s_j - t_j$ respectively at one embedding lifting each σ_j and Hodge–Tate weights 1 and 0 respectively at the others. In the case that $\overline{r}$ is not the twist of an extension of $\overline{\varepsilon}^{-1}$ by 1 the result follows because the corresponding $H^1_f(G_K, \chi' \otimes \chi^{-1})$ agrees with the full $H^1(G_K, \chi' \otimes \chi^{-1})$ (as a consequence of the usual dimension formulas for H^1_f, [Nek93, Prop. 1.24]).

If $\overline{r}$ is twist of an extension of $\overline{\varepsilon}^{-1}$ by 1, the assumption that $\overline{\sigma}_{\vec{t},\vec{s}}$ is non-Steinberg implies $s_j = 0$ for all j. The hypothesis that $\overline{r}$ is not très ramifiée guarantees that $\overline{r} \otimes \prod_{i=0}^{f-1} \omega_{\sigma_i}^{-t_i}$ is finite flat, so has a Barsotti–Tate lift, and we deduce that $\overline{\sigma}_{\vec{t},\vec{0}} \in W(\overline{r})$. □

Appendix C. The geometric Breuil–Mézard Conjecture for potentially Barsotti–Tate representations

In this appendix, by combining the methods of [EG14] and [GK14] we prove a special case of the geometric Breuil–Mézard conjecture [EG14, Conj. 4.2.1]. This result is "globalised" in Section 5.

Let $K/\mathbf{Q}_p$ be a finite extension, and let $E/\mathbf{Q}_p$ be another finite extension, with ring of integers $\mathcal{O}$, uniformiser ϖ, and residue field $\mathbf{F}$. We assume that E is sufficiently large, and in particular that E contains K. Let $\overline{r}: G_K \to \mathrm{GL}_2(\mathbf{F})$ be a continuous representation, and let $R^{\square}_{\overline{r}}$ be the universal framed deformation $\mathcal{O}$-algebra for $\overline{r}$. For each tame type τ, let $R^{\square}_{\overline{r},0,\tau}$ be the reduced and p-torsion free quotient of $R^{\square}_{\overline{r}}$ whose $\overline{\mathbf{Q}}_p$-points correspond to the potentially Barsotti–Tate lifts of $\overline{r}$ of type τ. In Section 5 we denote this ring by $R^{\tau,\mathrm{BT}}_{\overline{r}}$, but we use the more cumbersome notation $R^{\square}_{\overline{r},0,\tau}$ here to make it easier for the reader to refer to [EG14] and [GK14].

By [EG14, Prop. 4.1.2], $R^{\square}_{\overline{r},0,\tau}/\varpi$ is zero if $\overline{r}$ has no potentially Barsotti–Tate lifts of type τ, and otherwise it is equidimensional of dimension $4 + [K : \mathbf{Q}_p]$. Each $\operatorname{Spec} R^{\square}_{\overline{r},0,\tau}/\varpi$ is a closed subscheme of $\operatorname{Spec} R^{\square}_{\overline{r}}/\varpi$, and we write $Z(R^{\square}_{\overline{r},0,\tau}/\varpi)$ for the corresponding cycle, as in [EG14, Defn. 2.2.5]. (This is a formal sum of the irreducible components of $\operatorname{Spec} R^{\square}_{\overline{r},0,\tau}/\varpi$, weighted by the multiplicities with which they occur.)

Lemma C.1. *If $\overline{\sigma}$ is a non-Steinberg Serre weight of $\mathrm{GL}_2(k)$, then there are integers $n_\tau(\overline{\sigma})$ such that $\overline{\sigma} = \sum_\tau n_\tau(\overline{\sigma})\overline{\sigma}(\tau)$ in the Grothendieck group of mod p representations of $\mathrm{GL}_2(k)$, where the τ run over the tame types.*

Proof. This is an immediate consequence of the surjectivity of the natural map from the Grothendieck group of $\overline{\mathbf{Q}}_p$-representations of $\mathrm{GL}_2(k)$ to the Grothendieck group of $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_2(k)$ [Ser77, §III, Thm. 33], together with the observation that the reduction of the Steinberg representation of $\mathrm{GL}_2(k)$ is precisely $\overline{\sigma}_{\vec{0},\vec{p-1}}$. ☐

Let $\overline{\sigma}$ be a non-Steinberg Serre weight of $\mathrm{GL}_2(k)$, so that by Lemma C.1 we can write

$$\overline{\sigma} = \sum_\tau n_\tau(\overline{\sigma})\overline{\sigma}(\tau) \tag{C.2}$$

in the Grothendieck group of mod p representations of $\mathrm{GL}_2(k)$. Note that the integers $n_\tau(\overline{\sigma})$ are not uniquely determined; however, all our constructions elsewhere in this paper will be (non-obviously!) independent of the choice of the $n_\tau(\overline{\sigma})$. We also write

$$\overline{\sigma}(\tau) = \sum_{\overline{\sigma}} m_{\overline{\sigma}}(\tau)\overline{\sigma};$$

since $\overline{\sigma}(\tau)$ is multiplicity-free, each $m_{\overline{\sigma}}(\tau)$ is equal to 0 or 1. Then

$$\overline{\sigma} = \sum_{\overline{\sigma}'}\left(\sum_\tau n_\tau(\overline{\sigma})m_{\overline{\sigma}'}(\tau)\right)\overline{\sigma}',$$

and therefore

$$\sum_\tau n_\tau(\overline{\sigma})m_{\overline{\sigma}'}(\tau) = \delta_{\overline{\sigma},\overline{\sigma}'}. \tag{C.3}$$

For each non-Steinberg Serre weight $\overline{\sigma}$, we set

$$\mathcal{C}_{\overline{\sigma}} := \sum_\tau n_\tau(\overline{\sigma})Z(R^{\square}_{\overline{r},0,\tau}/\varpi),$$

where the sum ranges over the tame types τ, and the integers $n_\tau(\overline{\sigma})$ are as in (C.2). By definition this is a formal sum with (possibly negative) multiplicities of irreducible subschemes of $\operatorname{Spec} R^{\square}_{\overline{r}}/\varpi$; recall that we say that it is *effective* if all of the multiplicities are non-negative.

Theorem C.4. *Let $\overline{\sigma}$ be a non-Steinberg Serre weight. Then the cycle $\mathcal{C}_{\overline{\sigma}}$ is effective, and is nonzero precisely when $\overline{\sigma} \in W(\overline{r})$. It is independent of the choice of integers $n_\tau(\overline{\sigma})$ satisfying (C.2). For each tame type τ, we have*

$$Z(R^{\square}_{\overline{r},0,\tau}/\varpi) = \sum_{\overline{\sigma}\in\mathrm{JH}(\overline{\sigma}(\tau))} \mathcal{C}_{\overline{\sigma}}.$$

Proof. We will argue exactly as in the proof of [EG14, Thm. 5.5.2] (taking $n = 2$), and we freely use the notation and definitions of [EG14]. Since $p > 2$, we have $p \nmid n$ and thus a suitable globalisation $\overline{\rho}$ exists provided that [EG14, Conj. A.3] holds for $\overline{r}$. Exactly as in the proof of [EG14, Thm. 5.5.4], this follows from the proof of Theorem A.1.2 of [GK14] (which shows that $\overline{r}$ has a potentially Barsotti–Tate lift) and Lemma 4.4.1 of *op.cit.* (which shows that any potentially Barsotti–Tate representation is potentially diagonalizable). These same results also show that the equivalent conditions of [EG14, Lem. 5.5.1] hold in the case that $\lambda_v = 0$ for all v, and in particular in the case that $\lambda_v = 0$ and τ_v is tame for all v, which is all that we will require.

By [EG14, Lem. 5.5.1(5)], we see that for each choice of tame types τ_v, we have

$$Z(\overline{R}_\infty/\varpi) = \sum_{\otimes_{v|p}\overline{\sigma}_v} \prod_{v|p} m_{\overline{\sigma}_v}(\tau_v) Z'_{\otimes_{v|p}\overline{\sigma}_v}(\overline{\rho}). \tag{C.5}$$

Now, by definition we have

$$Z(\bar{R}_\infty/\varpi) = \prod_{v|p} Z(R^{\square}_{\overline{r},0,\tau_v}/\varpi) \times Z(\mathbf{F}[[[x_1,\dots,x_{q-[F^+:\mathbf{Q}]n(n-1)/2},t_1,\dots,t_{n^2}]]). \tag{C.6}$$

Fix a non-Steinberg Serre weight $\overline{\sigma} = \otimes_v \overline{\sigma}_v$, and sum over all choices of types τ_v, weighted by $\prod_{v|p} n_{\tau_v}(\overline{\sigma}_v)$. We obtain

$$\sum_\tau \prod_{v|p} n_{\tau_v}(\overline{\sigma}_v) \prod_{v|p} Z(R^{\square}_{\overline{r},0,\tau_v}/\varpi) \times Z(\mathbf{F}[[[x_1,\dots,x_{q-[F^+:\mathbf{Q}]n(n-1)/2},t_1,\dots,t_{n^2}]])$$

$$= \sum_\tau \prod_{v|p} n_{\tau_v}(\overline{\sigma}_v) \sum_{\otimes_{v|p}\overline{\sigma}'_v} \prod_{v|p} m_{\overline{\sigma}'_v}(\tau_v) Z'_{\otimes_{v|p}\overline{\sigma}'_v}(\overline{\rho})$$

which by (C.3) simplifies to

$$\prod_{v|p} \mathcal{C}_{\overline{\sigma}_v} \times Z(\mathbf{F}[[[x_1,\dots,x_{q-[F^+:\mathbf{Q}]n(n-1)/2},t_1,\dots,t_{n^2}]]) = Z'_{\otimes_{v|p}\overline{\sigma}_v}(\overline{\rho}). \tag{C.7}$$

Since $Z'_{\otimes_{v|p}\overline{\sigma}_v}(\overline{\rho})$ is effective by definition (as it is defined as a positive multiple of the support cycle of a patched module), this shows that every $\prod_{v|p} \mathcal{C}_{\overline{\sigma}_v}$ is effective. We conclude that either every $\mathcal{C}_{\overline{\sigma}}$ is effective, or that every $-\mathcal{C}_{\overline{\sigma}}$ is effective.

Substituting (C.7) and (C.6) into (C.5), we see that

$$\prod_{v|p} Z(R^{\square}_{\overline{r},0,\tau_v}/\varpi) \times Z(\mathbf{F}[[[x_1,\dots,x_{q-[F^+:\mathbf{Q}]n(n-1)/2},t_1,\dots,t_{n^2}]]) \tag{C.8}$$

$$= \prod_{v|p} \left(\sum_{\overline{\sigma}\in \mathrm{JH}(\overline{\sigma}(\tau))} \mathcal{C}_{\overline{\sigma}_v} \right) \times Z(\mathbf{F}[[[x_1,\dots,x_{q-[F^+:\mathbf{Q}]n(n-1)/2},t_1,\dots,t_{n^2}]],$$

and we deduce that either $Z(R^{\square}_{\overline{r},0,\tau}/\varpi) = \sum_{\overline{\sigma}} m_{\overline{\sigma}}(\tau)\mathcal{C}_{\overline{\sigma}}$ for all τ, or $Z(R^{\square}_{\overline{r},0,\tau}/\varpi) = -\sum_{\overline{\sigma}} m_{\overline{\sigma}}(\tau)\mathcal{C}_{\overline{\sigma}}$ for all τ.

Since each $Z(R^{\square}_{\overline{r},0,\tau}/\varpi)$ is effective, the second possibility holds if and only if every $-\mathcal{C}_{\overline{\sigma}}$ is effective (since either all the $-\mathcal{C}_{\overline{\sigma}}$ are effective, or all the $\mathcal{C}_{\overline{\sigma}}$ are effective). It remains to show that this possibility leads to a contradiction. Now, if $Z(R^{\square}_{\overline{r},0,\tau}/\varpi) = -\sum_{\overline{\sigma}} m_{\overline{\sigma}}(\tau)\mathcal{C}_{\overline{\sigma}}$ for all τ, then substituting into the definition $\mathcal{C}_{\overline{\sigma}} = \sum_\tau n_\tau(\overline{\sigma}) Z(R^{\square}_{\overline{r},0,\tau}/\varpi)$, we obtain

$$\mathcal{C}_{\overline{\sigma}} = \sum_{\overline{\sigma}'} \left(\sum_\tau n_\tau(\overline{\sigma}) m_{\overline{\sigma}'}(\tau) \right) (-\mathcal{C}_{\overline{\sigma}'}),$$

and applying (C.3), we obtain $\mathcal{C}_{\overline{\sigma}} = -\mathcal{C}_{\overline{\sigma}}$, so that $\mathcal{C}_{\overline{\sigma}} = 0$ for all $\overline{\sigma}$. Thus all the $\mathcal{C}_{\overline{\sigma}}$ are effective, as claimed.

Since $Z'_{\otimes_{v|p}\overline{\sigma}_v}(\overline{\rho})$ by definition depends only on (the global choices in the Taylor–Wiles method, and) $\otimes_{v|p}\overline{\sigma}_v$, and *not* on the particular choice of the $n_\tau(\overline{\sigma})$, it follows from (C.7) that $\mathcal{C}_{\overline{\sigma}}$ is also independent of this choice.

Finally, note that by definition $Z'_{\otimes_{v|p}\overline{\sigma}_v}(\overline{\rho})$ is nonzero precisely when $\overline{\sigma}_v$ is in the set $W^{\mathrm{BT}}(\overline{r})$ defined in [GK14, §3]; but by the main result of [GLS15], this is precisely the set $W(\overline{r})$. □

Remark C.8.1. As we do not use wildly ramified types elsewhere in the paper, we have restricted the statement of Theorem C.4 to the case of tame types; but the statement admits a natural extension to the case of wildly ramified inertial types (with some components now occurring with multiplicity greater than one), and the proof goes through unchanged in this more general setting.

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