Examples of reducible curves with 27 points

We give examples to show that each of the possibilities enumerated in Section 5.2 of [Sav] do indeed occur. Recall that $\eta \in \mathbb{F}_8$ is a chosen root of $\eta^3 + \eta + 1 = 0$. Let β be a generator of $\mathbb{F}_{64}^\times$ such that $\beta^9 = \eta$.

Each intersection described below has exactly 27 points over F_8, and each was found in the computer calculations whose output is available at

http://www.math.mcgill.ca/∼dsavitt/curves/ .

- The intersection of $XY + ZW = 0$ with the cubic $(Z + W)(X + \eta^3 Y + \eta^{-2} Z)(X + \eta^{-3} Y + \eta^{-3} W) = 0$ consists of three geometrically irreducible conics. Let the conics be C_1, C_2, and C_3 respectively (in the order of the factors of the cubic). Then C_1 and C_2 meet in the two $\mathbb{F}_{64}$-points $[\beta^{24} : \beta^{-24} : 1 : 1]$ and $[\beta^3 : \beta^{-3} : 1 : 1]$; C_1 and C_3 meet in the two $\mathbb{F}_{64}$-points $[\beta^{32} : \beta^{-32} : 1 : 1]$ and $[\beta^4 : \beta^{-4} : 1 : 1]$; and C_2 and C_3 meet in the two $\mathbb{F}_{64}$-points $[\beta^2 : \beta^{44} : \beta^{46} : 1]$ and $[\beta^{16} : \beta^{37} : \beta^{53} : 1]$. The quadric and the cubic intersect in 189 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $(X+Y)(Y+Z+W)(X+Z+W) = 0$ consists of three geometrically irreducible conics. The three conics have no intersection over $\mathbb{F}_8$, and over $\mathbb{F}_{64}$ all three conics pass through the conjugate points $[\beta^{21} : \beta^{21} : \beta^{42} : 1]$ and $[\beta^{42} : \beta^{42} : \beta^{21} : 1]$. The quadric and the cubic intersect in 191 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $X^2W + \eta XYW + \eta^{-1} XZW + \eta^{-3} XW^2 + \eta Y^2 Z + Y^2 W + \eta^{-2} YZ^2 + \eta^{-1} YZW + YW^2 = 0$ contains the line $[X : 0 : Z : 0]$ and a component of degree 5, and has 119 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $X^2W + XYZ + XYW + XW^2 + \eta Y^2 Z + Y^2 W + \eta YZ^2 + YW^2 = 0$ contains the two non-intersecting lines $[X : 0 : Z : 0]$ and $[X : Y : Y : X]$ and a component of degree 4, and has 195 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $W(X^2 + Z^2 + \eta XY + \eta^{-3} XZ + \eta^{-1} XW + Y^2 + YZ + YW) = 0$ contains the two intersecting lines $[0 : Y : Z : 0]$ and $[X : 0 : Z : 0]$. The intersection of $XY + ZW = 0$ and $X^2 + Z^2 + \eta XY + \eta^{-3} XZ + \eta^{-1} XW + Y^2 + YZ + YW = 0$ is a curve of arithmetic genus 1 with 10 $\mathbb{F}_8$-points and 64 $\mathbb{F}_{64}$-points, and is singular at $[\eta : \eta^{-3} : \eta^{-2} : 1]$. The curve of genus 1 meets the line $[0 : Y : Z : 0]$ at $[0 : 1 : \beta^{21} : 0]$ and $[0 : 1 : \beta^{42} : 0]$ and the line $[X : 0 : Z : 0]$ at $[1 : 0 : \beta^{28} : 0]$ and $[1 : 0 : \beta^{35} : 0]$. The intersection of the quadric and the cubic has 189 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $W(X^2 + Z^2 + \eta^2 XY + XZ + \eta^2 XW + Y^2 + YZ + YW) = 0$ contains the two intersecting lines $[0 : Y : Z : 0]$ and $[X : 0 : Z : 0]$. The intersection of $XY + ZW = 0$ and $X^2 + Z^2 + \eta^2 XY + XZ + \eta^2 XW + Y^2 + YZ + YW = 0$ is an elliptic curve with 10 $\mathbb{F}_8$-points and 80 $\mathbb{F}_{64}$-points. The curve of genus 1 meets the line $[0 : Y : Z : 0]$ at $[0 : 1 : \beta^{21} : 0]$ and $[0 : 1 : \beta^{42} : 0]$ and the line $[X : 0 : Z : 0]$ at $[1 : 0 : \beta^{21} : 0]$ and $[1 : 0 : \beta^{42} : 0]$. The intersection of the quadric and the cubic has 205 points over $\mathbb{F}_{64}$.

- The intersection of $XY + ZW = 0$ with the cubic $Y(X^2 + \eta YZ + YW + \eta^{-1}Z^2 + \eta^{-3}ZW + W^2) = 0$ contains the two intersecting lines $[X : 0 : Z : 0]$ and $[X : 0 : 0 : W]$. The intersection of $XY + ZW = 0$ and $X^2 + \eta YZ + YW + \eta^{-1}Z^2 + \eta^{-3}ZW + W^2 = 0$ is an elliptic curve with 12 $\mathbb{F}_8$-points and 72 $\mathbb{F}_{64}$-points. This elliptic curve intersects the line $[X : 0 : Z : 0]$ in a double-point $[\eta^3 : 0 : 1 : 0]$ and the line $[X : 0 : 0 : W]$ in a double-point $[1 : 0 : 0 : 1]$, and intersection of the quadric and the cubic has 199 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $Y(\eta^3XW + \eta^2YZ + \eta YW + Z^2 + \eta^{-3}ZW + W^2) = 0$ contains the two intersecting lines $[X : 0 : Z : 0]$ and $[X : 0 : 0 : W]$. The intersection of $XY + ZW = 0$ and $\eta^3XW + \eta^2YZ + \eta YW + Z^2 + \eta^{-3}ZW + W^2$ is an elliptic curve with 12 $\mathbb{F}_8$-points and 72 $\mathbb{F}_{64}$-points. This elliptic curve intersects the line $[X : 0 : Z : 0]$ in a double-point $[1 : 0 : 0 : 0]$ and meets the line $[X : 0 : 0 : W]$ singly at $[1 : 0 : 0 : 0]$ and $[1 : 0 : 0 : \eta^3]$. The intersection of the quadric and the cubic has 199 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $XYW + XZ^2 + \eta XZW + \eta Y^2Z + Y^2W + YZ^2 + YZW + YW^2 = 0$ contains the lines $[\eta W : Y : \eta Y : W]$ and $[X : 0 : 0 : W]$. The cubic is $(XY + ZW)(\eta^{-1}W + \eta Z) + (\eta Y + Z)(YZ + XZ + \eta XW + \eta^{-1}W^2 + \eta ZW + \eta^{-1}YW)$. The intersection of $XY + ZW = 0$ with $YZ + XZ + \eta XW + \eta^{-1}W^2 + \eta ZW + \eta^{-1}YW = 0$ is an elliptic curve with 12 $\mathbb{F}_8$-points and 72 $\mathbb{F}_{64}$-points. It meets the line $[X : 0 : 0 : W]$ at the two points $[1 : 0 : 0 : 0]$ and $[1 : 0 : 0 : \eta^2]$, and meets the line $[\eta W : Y : \eta Y : W]$ at the two Galois-conjugate points $[\beta^{59} : 1 : \beta^9 : \beta^{50}]$ and $[\beta^{31} : 1 : \beta^9 : \beta^{22}]$. The intersection of the quadric and the cubic has 197 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $W(XZ + \eta XW + Y^2 + \eta^3YZ + YW) = 0$ contains the two intersecting lines $[X : 0 : Z : 0]$ and $[0 : Y : Z : 0]$. The intersection of $XY + ZW = 0$ and $XZ + \eta XW + Y^2 + \eta^3YZ + YW = 0$ is an elliptic curve with 13 $\mathbb{F}_8$-points and 65 $\mathbb{F}_{64}$-points. The elliptic curve meets both lines at $[0 : 0 : 1 : 0]$, and also meets $[X : 0 : Z : 0]$ and $[0 : Y : Z : 0]$ at $[1 : 0 : 0 : 0]$ and $[0 : \eta^3 : 1 : 0]$ respectively. The intersection of the quadric and the cubic has 191 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $Y(XZ + \eta XW + \eta^{-3}X^2 + \eta YZ + YW + \eta^{-3}Z^2 + ZW + W^2) = 0$ contains the two lines $[X : 0 : Z : 0]$ and $[X : 0 : 0 : W]$. The intersection of $XY + ZW = 0$ and $XZ + \eta XW + \eta^{-3}X^2 + \eta YZ + YW + \eta^{-3}Z^2 + ZW + W^2$ is an elliptic curve with 14 $\mathbb{F}_8$-points and 56 $\mathbb{F}_{64}$-points. The elliptic curve meets the line $[X : 0 : Z : 0]$ at the points $[1 : 0 : \eta^2 : 0]$ and $[1 : 0 : \eta^{-2} : 0]$, and the line $[X : 0 : 0 : W]$ at the points $[1 : 0 : 0 : \eta]$ and $[1 : 0 : 0 : \eta^3]$. The intersection of the quadric and the cubic has 181 points over $\mathbb{F}_{64}$.
- The intersection of $XY + ZW = 0$ with the cubic $\eta^{-2}X^2Z + \eta^3XYZ + \eta^3XYW + \eta^{-2}XZ^2 + \eta^3XZW + Y^2W + \eta^3YZW + YW^2 = 0$ contains the three non-intersecting lines $[0 : Y : Z : 0]$, $[X : 0 : 0 : W]$, and $[X : Y : X : Y]$ and three lines defined over $\mathbb{F}_{512}$. The intersection has 195 points over $\mathbb{F}_{64}$.

References

[Sav] Savitt, David. *The maximum number of points on a curve of genus* 4 *over* $\mathbb{F}_8$ *is* 25. With an appendix by Kristin Lauter. To appear, Can. J. Math.