CORRIGENDUM TO "ON A CONJECTURE OF CONRAD, DIAMOND, AND TAYLOR"

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We take this opportunity to correct an error [Sav05], as a consequence of which there is one more family of strongly divisible modules that we must study by the methods of [Sav05]. Once this is done, the remaining claims of [Sav05] are unaffected. We adopt the notation of [Sav05] without further comment, and all numbered references are to that paper.

The mistake is in the statement and proof of Theorem 6.12(4). In the situation of that item, if $m = 1 + (p+1)j$ — i.e., if $i = 1$ — then the two characters ω_2^{m+p} and ω_2^{pm+1} are both characters of niveau one, and are equal; hence in this case the proof of Theorem 6.12(4) does *not* show that $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)\,|_{I_p}$ decomposes as a sum of two conjugate characters. In fact, for each choice c of square root of $\overline{w}$, the map $\mathcal{M}'_2 \to \mathcal{M}_E(F_2/\mathbb{Q}_{p^2}, e_2, c, m-1)$ extends to a map $\mathcal{M}' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, c, j)$; by Proposition 5.4(1), we conclude when $i = 1$ that

$$T_{st,2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E) \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \cong \lambda_{c^{-1}}\omega^{1+j} \oplus \lambda_{-c^{-1}}\omega^{1+j}\,.$$

This means that when $i = 1$ and $\mathrm{val}(b) > 0$ we still need to construct a strongly divisible lattice in $\mathcal{D}_{m,[1:b]}$ whose reduction mod p has trivial endomorphisms; or, conversely, we need to study deformations of type $\tilde{\omega}_2^m \oplus \tilde{\omega}_2^{pm}$ (with $i = 1$) of non-split residual representations of the form

$$\begin{pmatrix} \lambda_{c^{-1}}\omega^{1+j} & * \\ 0 & \lambda_{-c^{-1}}\omega^{1+j} \end{pmatrix}.$$

We rectify this omission now. Our statements are numbered to mesh with the original article.

Lemma 6.7. (2) *If $i = 1$, $\mathrm{val}_p(b) > 0$, and w is a square in E, then there is $X \in S_{F_2,\mathcal{O}_E}^{\times}$ satisfying*

$$X(1 \otimes wb) = 1 \otimes w - \left(1 + \frac{u^{pe_2}}{p}\right) X\phi(X).$$

Proof. The constant term of X may be taken to be $1 \otimes x_0$ where x_0 is either root of $x_0^2 + wbx_0 - w$ in $\mathcal{O}_E^{\times}$. The recursion for the coefficient x_n of u^n is $x_n(x_0 + wb) =$ lower terms, and so the recursion can be solved to obtain $X \in S_{F_2,\mathcal{O}_E}^{\times}$. □

Moreover, since $\mathrm{val}_p(b) > 0$, by putting the variable B for b we obtain an element X_B of $S_{F_2,\mathcal{O}_E[[B]]}$ which specializes to X under the map $\mathcal{O}_E[[B]] \to \mathcal{O}_E$ sending $B \mapsto b$. Note that the image of X in $(\mathbb{F}_{p^2} \otimes \mathbf{k}_E)[u]/u^{e_2 p}$ is $1 \otimes c$ with c a square root of $\overline{w}$. Assume henceforth that the coefficient field E contains a square root of w. Now Proposition 6.10 is modified as follows.

Proposition 6.10. *In the case $i = 1$ and $\mathrm{val}_p(b) > 0$, we instead define*

$$\mathcal{M}_{m,[1:b]} = S_{F_2,\mathcal{O}_E} \cdot g_1 + S_{F_2,\mathcal{O}_E} \cdot g_2$$

$$\begin{aligned} g_1 &= \mathbf{e}_1 + \frac{X}{pw} u^{p(p-1)} \mathbf{e}_2 \\ g_2 &= \mathbf{e}_2, \end{aligned}$$

and this is a strongly divisible $\mathcal{O}_E$-module with descent data inside $\mathcal{D}_{m,[1:b]}$.

Proof. Put $\mathcal{M} = \mathcal{M}_{m,[1:b]}$. Observe that $h := u^{p-1} g_1 + \left(\frac{X}{w} + (1 \otimes b)\right) g_2$ lies in $\mathrm{Fil}^1 \mathcal{M}$. Since $\frac{X}{w} + (1 \otimes b)$ is a unit in $S_{F_2,\mathcal{O}_E}$ and g_1 does not lie in $\mathrm{Fil}^1 \mathcal{M}$, we deduce that $\mathrm{Fil}^1 \mathcal{M} = S_{F_2,\mathcal{O}_E} \cdot h + (\mathrm{Fil}^1 S_{F_2,\mathcal{O}_E})\mathcal{M}$. From this it is easy to check that $I\mathcal{M} \cap \mathrm{Fil}^1 \mathcal{M} = I \mathrm{Fil}^1 \mathcal{M}$. Finally, we compute that

$$\begin{aligned} \phi(g_1) &= \phi(X) u^{p^2(p-1)} g_1 + \left(1 - X\phi(X) \frac{u^{pe_2}}{pw}\right) g_2 \\ \phi(g_2) &= pw g_1 - X u^{p(p-1)} g_2 \end{aligned}$$

both lie in $\mathcal{M}$; using the defining relation for X we find $\phi_1(h) = (1 \otimes w) X^{-1} g_1 \in \mathcal{M}$ and conclude that $\mathcal{M}$ is a strongly divisible module. □

Now amend Theorem 6.12(4) so that it applies only to that case $i > 1$, and add the following.

Theorem 6.12. (5) *If $i = 1$ and $\mathrm{val}_p(b) > 0$, then $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)$ is independent of b and*

$$T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E) \cong \begin{pmatrix} \lambda_{-c^{-1}} \omega^{1+j} & * \\ 0 & \lambda_{c^{-1}} \omega^{1+j} \end{pmatrix}$$

with $* \neq 0$.

Proof. Write $\mathcal{M}' = T_0(\mathcal{M}/\mathfrak{m}_E)$. Then $\mathrm{Fil}^1 \mathcal{M}'$ is generated by $u^{p-1} g_1 + c^{-1} g_2$ and $u^{e_2} g_1$, with $\phi_1(u^{p-1} g_1 + c^{-1} g_2) = c g_1$ and $\phi_1(u^{e_2} g_1) = u^{p^2(p-1)} c g_1 + g_2$. Note that $\phi_1(u^{p(p-1)} g_2) = -c g_2$. There is evidently a nontrivial map $\mathcal{M}' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, c, j)$ sending $g_2 \mapsto 0$ and $g_1 \mapsto u^{p^2} \mathbf{e}$. On the other hand if $f : \mathcal{M}' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, d, n)$ is a nontrivial map sending $g_1 \mapsto \alpha \mathbf{e}$ and $g_2 \mapsto \beta \mathbf{e}$, then α, β must both be polynomials in u^p since g_1, g_2 are in the image of ϕ_1. On the other hand if $\beta \neq 0$ then the relation $f \circ \phi_1 = \phi_1 \circ f$ on $u^{p(p-1)} g_2$ implies that β is a unit times u^p; but then $f(u^{p-1} g_1 + c^{-1} g_2) \in \langle u^{e_2} \mathbf{e} \rangle$ implies that α has a linear term, a contradiction. Therefore $\beta = 0$, and then it is easy to check that $c = d$ and $j = n$. It follows that $* \neq 0$. □

(We also note the following typos in the published version of the proof of Theorem 6.12(4): in the first sentence, the expression $\phi_1(u^{e_2})$ should be $\phi_1(u^{e_2} g_2)$; in the last sentence, the characters λ_c should both be $\lambda_{c^{-1}}$.)

The proof of Corollary 6.15(2) should then invoke Theorem 6.12(5) in lieu of Theorem 6.12(4) in the case of representations ρ to which Theorem 6.12(5) applies, noting that the two choices for x_0 lead to different reductions of ρ.

We now turn to deformation spaces of strongly divisible modules. The proof of the following proposition is identical to the proof that the corresponding module $\mathcal{M}_{m,[1:b]}$ of Proposition 6.10 is a strongly divisible module. As noted in Remark 6.20, we omit the description of N in the strongly divisible module below.

Proposition 6.21. *There exists a strongly divisible module with descent data and $\mathcal{O}_E[[B]]$-coefficients as follows.*

(6) *If $i=1$ and assuming that w is a square in E,*

$$\mathcal{M}_X = (S_{F_2,\mathcal{O}_E[[B]]}) \cdot g_1 \oplus (S_{F_2,\mathcal{O}_E[[B]]}) \cdot g_2,$$

$$\mathrm{Fil}^1\mathcal{M}_X = S_{F_2,\mathcal{O}_E[[B]]} \cdot (u^{p-1}g_1 + (w^{-1}X_B + (1\otimes B))g_2) + (\mathrm{Fil}^1 S_{F_2,\mathcal{O}_E[[B]]})\mathcal{M}_X,$$

$$\begin{aligned}\phi(g_1) &= \phi(X_B)u^{p^2(p-1)}g_1 + \left(1 - X_B\phi(X_B)\frac{u^{pe_2}}{pw}\right)g_2,\\ \phi(g_2) &= pwg_1 - X_Bu^{p(p-1)}g_2,\end{aligned}$$

$$\widehat{g}(g_1) = (\tilde{\omega}_2^m \otimes 1)g_1, \qquad \widehat{g}(g_2) = (\tilde{\omega}_2^{pm}\otimes 1)g_2.$$

Finally, one must amend the proof of Theorem 6.24 to include a proof that the canonical injection

$$R(2,\tau(\mathcal{M}_X),\overline{\rho}(\mathcal{M}_X))_{\mathcal{O}_E} \to R(\mathcal{M}_X)$$

is a surjection; this proceeds exactly along the strategy outlined in the proof of Theorem 6.24. Indeed, let $\mathcal{M}''$ denote the minimal Breuil module with descent data from F_2 to $\mathbb{Q}_p$ associated to the character $\lambda_{-c^{-1}}\omega^{1+j}$, with generator h such that $\phi_1(h) = -c^{-1}h$. Then a map $f : \mathcal{M}'' \to T_0(\mathcal{M}_X/(\mathfrak{m}_E, B^2))$ must send h to an element of the form $\alpha u^{e_2}g_1 + \beta(u^{p-1}g_1 + (w^{-1}X_B + B)g_2)$ (where, abusing notation, we identify elements of $S_{F_2,\mathcal{O}_E[[B]]}$ with their images in $(\mathbb{F}_{p^2}\otimes \mathbf{k}_E[B]/(B^2))[u]/u^{e_2p}$). Write $\alpha = \alpha_0 + B\alpha_1$ and $\beta = \beta_0 + B\beta_1$ to separate out the terms involving B. The relation $f(\phi_1(h)) = \phi_1(f(h))$ shows first that $\alpha_0 = au^p$, $\beta_0 = -au^{p^2}$ for some $a \in \mathbf{k}_E$, by considering the relation mod B; then, after some algebra, the full relation eventually implies $a = 0$. Thus the image of f lies in $B \cdot T_0(\mathcal{M}_X/(\mathfrak{m}_E, B^2))$, as desired.

References

[Sav05] David Savitt, *On a conjecture of Conrad, Diamond, and Taylor*, Duke Math. J. **128** (2005), no. 1, 141–197.