ON A CONJECTURE OF CONRAD, DIAMOND, AND TAYLOR

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Abstract. We prove a conjecture of Conrad, Diamond, and Taylor on the size of certain deformation rings parametrizing potentially Barsotti-Tate Galois representations. To achieve this, we extend results of Breuil and Mézard (classifying Galois lattices in semistable representations in terms of "strongly divisible modules") to the potentially crystalline case in Hodge-Tate weights $(0,1)$. We then use these strongly divisible modules to compute the desired deformation rings. As a corollary, we obtain new results on the modularity of potentially Barsotti-Tate representations.

1. Introduction

In their paper [CDT99], Conrad, Diamond, and Taylor conjectured that certain deformation rings parametrizing potentially Barsotti-Tate Galois representations are sufficiently small for the methods of Taylor-Wiles to yield a modularity result. Breuil and Mézard [BM02] reformulated and vastly generalized these conjectures, and proved their new conjectures for semistable Galois representations in even weight. In this article, essentially a sequel to [BM02], we prove the conjectures of Breuil and Mézard in the cases originally conjectured by Conrad, Diamond, and Taylor.

We now describe these conjectures. Fix p an odd prime, and let E be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbf{k}_E$. To each potentially crystalline Galois representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$, we attach a representation $\mathrm{WD}(\rho)$ of the Weil group $W_{\mathbb{Q}_p}$ (see Def. 2.15), and hence a Galois type $\tau(\rho) = \mathrm{WD}(\rho)\,|_{I_p}$.

Suppose that $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbf{k}_E)$ is such that $\mathrm{End}_{\,\mathbf{k}_E[G_{\mathbb{Q}_p}]}\,\overline{\rho} = \mathbf{k}_E$; we shall say that $\overline{\rho}$ has trivial endomorphisms. Let $R^{\mathrm{univ}}_{\mathcal{O}_E}(\overline{\rho})$ be the universal deformation ring parametrizing deformations of $\overline{\rho}$ over complete local noetherian $\mathcal{O}_E$-algebras. If $2 \le k < p$ and if $\mathcal{O}_{E'}$ are the integers in a finite extension of E, we say that a deformation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathcal{O}_{E'})$ of $\overline{\rho}$ has type (k,τ) if

- ρ is potentially semi-stable and $\tau(\rho) \cong \tau$,
- ρ has Hodge-Tate weights $(0, k-1)$, and
- $\det(\rho)$ is a fixed lift of $\det(\overline{\rho})$ of the following form: the $(k-1)$st power of the p-adic cyclotomic character times a finite character of order prime to p.

Date: April 19, 2004. *Revised:* September 3, 2004.
2000 Mathematics Subject Classification: Primary 11F80; Secondary 14L15.
Author's work supported by a National Science Foundation (NSF) International Research Fellowship and by the Office of Multidisciplinary Activities under the NSF Mathematics and Physical Sciences Directorate Program. This work also supported by a Natural Sciences and Engineering Research Council of Canada (NSERC) postdoctoral fellowship.

The kernel $\mathfrak{p}$ of the corresponding map $R^{\mathrm{univ}}_{\mathcal{O}_E}(\overline{\rho}) \to \mathcal{O}_{E'}$ is also said to have type (k,τ), and we define

$$R(k,\tau,\overline{\rho})_{\mathcal{O}_E} = R^{\mathrm{univ}}_{\mathcal{O}_E}(\overline{\rho}) / \bigcap_{\mathfrak{p} \text{ type } (k,\tau)} \mathfrak{p}.$$

The first part of the conjectures of Breuil and Mézard (see [BM02, Conj. 2.2.2.4]) posits that $R(k,\tau,\overline{\rho})_{\mathcal{O}_E}$ should be equidimensional of Krull dimension 2, and that $R(k,\tau,\overline{\rho})_{\mathcal{O}_E} \otimes E$ should be regular. Let $\mu_{\mathrm{gal}}(k,\tau,\overline{\rho})$ be the Samuel multiplicity of $\overline{R} = R(k,\tau,\overline{\rho})_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} \mathbf{k}_E$; so conjecturally, this is $\dim_{\mathbf{k}_E} \mathfrak{m}^n_{\overline{R}}/\mathfrak{m}^{n+1}_{\overline{R}}$ for n sufficiently large. Via a recipe on the automorphic side, Breuil and Mézard also define an integer $\mu_{\mathrm{aut}}(k,\tau,\overline{\rho})$ (see [BM02, Sec 2.1] for the details). We then have the following.

Conjecture 1.1 ([BM02], Conj. 2.3.1.1)**.** *If* $\det(\tau)$ *is tame, then*

$$\mu_{\mathrm{gal}}(k,\tau,\overline{\rho}) = \mu_{\mathrm{aut}}(k,\tau,\overline{\rho}).$$

The conjectures of Conrad, Diamond, and Taylor to which we have referred (see [CDT99, Conjs. 1.2.2, 1.2.3]) are, more or less, the case $k=2$ and τ tamely ramified in Conjecture 1.1. Our main theorem, then, is the following.

Theorem 1.2. *Conjecture 1.1 holds when* $k=2$ *and* τ *is tamely ramified.*

Indeed, we show the following (see Exams. 2.13, 2.14, 2.16 for notation and Ths. 6.22 and 6.23 for more precise statements).

Theorem 1.3. *Suppose that* $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbf{k}_E)$ *has trivial endomorphisms. Suppose that* $\tau \cong \tilde{\omega}^i \oplus \tilde{\omega}^j$ *with* $i \not\equiv j \pmod{p-1}$*. Then we have the following.*

(1) $\mu_{\mathrm{gal}}(2,\tau,\overline{\rho}) = 0$ *if* $\overline{\rho}|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \notin \left\{ \begin{pmatrix} \omega^{1+i} & * \\ 0 & \omega^j \end{pmatrix}, \begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^i \end{pmatrix}, \omega_2^k \oplus \omega_2^{pk} \right\}$ *with* $k = 1+\{j-i\}+(p+1)i$*, where* $\{a\}$ *is the unique integer in* $\{0,\ldots,p-2\}$ *which is congruent to* $a \pmod{p-1}$*;*

(2) $\mu_{\mathrm{gal}}(2,\tau,\overline{\rho}) = 1$ *if* $\overline{\rho}|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \in \left\{ \begin{pmatrix} \omega^{1+i} & * \\ 0 & \omega^j \end{pmatrix}, \begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^i \end{pmatrix} \right\}$,

(3) $\mu_{\mathrm{gal}}(2,\tau,\overline{\rho}) = 2$ *if* $\overline{\rho}|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \cong \omega_2^k \oplus \omega_2^{pk}$ *with* $k = 1+\{j-i\}+(p+1)i$.

Theorem 1.4. *Suppose that* $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbf{k}_E)$ *has trivial endomorphisms. Suppose that* $\tau \cong \tilde{\omega}_2^m \oplus \tilde{\omega}_2^{pm}$ *with* $p+1 \nmid m$*. Write* $m = i+(p+1)j$ *with* $i \in \{1,\ldots,p\}$ *and* $j \in \mathbb{Z}/(p-1)\mathbb{Z}$*. Then we have the following.*

(1) $\mu_{\mathrm{gal}}(2,\tau,\overline{\rho}) = 1$ *if* $\overline{\rho}|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \in \left\{ \begin{pmatrix} \omega^{i+j} & * \\ 0 & \omega^{1+j} \end{pmatrix}, \begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^{i+j} \end{pmatrix} \right\}$*, the first* $*$ *peu ramifié when* $i=2$ *and the second when* $i = p-1$*;*

(2) $\mu_{\mathrm{gal}}(2,\tau,\overline{\rho}) = 1$ *if* $\overline{\rho}|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \in \left\{ \omega_2^{p+m} \oplus \omega_2^{1+pm}, \omega_2^{1+m} \oplus \omega_2^{p(1+m)} \right\}$*;*

(3) $\mu_{\mathrm{gal}}(2,\tau,\overline{\rho}) = 0$ *otherwise.*

We note an important consequence of these results. The method of Taylor-Wiles, as utilized in [BCDT01], may be reformulated as follows.

Theorem 1.5 ([BCDT01], Th. 1.4.1)**.** *Let* $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ *be an odd continuous representation ramified at only finitely many primes. Assume that its reduction* $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbf{k}_E)$ *is modular and is absolutely irreducible after restriction to* $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$*. Further, suppose that*

- $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ *has trivial endomorphisms,*

- $\rho_p = \rho|_{G_{\mathbb{Q}_p}}$ *is potentially Barsotti-Tate, and*
- $\mu_{\mathrm{gal}}(2, \tau(\rho_p), \overline{\rho}) \leq 1 \leq \mu_{\mathrm{aut}}(2, \tau(\rho_p), \overline{\rho})$.

Then ρ is modular.

The import of Conjecture 1.1 is that if it is true, then the last condition of Theorem 1.5 may be replaced with $\mu_{\mathrm{aut}}(2, \tau(\rho_p), \overline{\rho}) \leq 1$, removing the irksome hypothesis involving μ_{gal} from the theorem. In particular, we obtain the following immediate corollary of Theorems 1.3 and 1.4 (together with the $k = 2$, τ scalar case of Conj. 1.1, proved in [BM02]).

Theorem 1.6. *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ be an odd continuous representation ramified at only finitely many primes. Assume that its reduction $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbf{k}_E)$ is modular and is absolutely irreducible after restriction to $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$. Further, suppose that*

- $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ *has trivial endomorphisms,*
- $\rho_p = \rho|_{G_{\mathbb{Q}_p}}$ *is potentially Barsotti-Tate,*
- $\tau(\rho)$ *is tamely ramified, and*
- *if* $\tau(\rho) \cong \tilde{\omega}^i \oplus \tilde{\omega}^j$ *with* $i \not\cong j \pmod{p-1}$ *then* $\overline{\rho}|_{G_{\mathbb{Q}_p}} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p$ *is reducible.*

Then ρ is modular.

This is a significant improvement on the main results in [Sav04], where $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ had to be reducible and defined over $\mathbb{F}_p$. It would be of interest to know whether the methods of Taylor-Wiles could be extended to handle cases where $\mu_{\mathrm{gal}}(2, \tau, \overline{\rho}) = \mu_{\mathrm{aut}}(2, \tau, \overline{\rho}) = 2$, in order to remove the last hypothesis from this theorem.[1]

We give a brief outline of this paper. We follow the same strategy established by Breuil and Mézard to prove Conjecture 1.1 in the case τ scalar, k even. To achieve this, we must provide (as best we can) "potential" versions, when $k = 2$, of the machinery of [BM02] classifying lattices in semi-stable Galois representations by means of strongly divisible modules. We begin in Section 2 by recalling Fontaine's filtered modules with coefficients and descent data and computing the particular filtered modules that arise in the proofs of our main theorems.

Sections 3 and 4 contain the bulk of the technicalities: in the former, we use the equivalence between p-divisible groups and lattices in potentially Barsotti-Tate representations to add (tame) descent data to the strongly divisible modules of [BM02] when $k = 2$; in the latter, we introduce coefficients into the mix. Since we are working over a base ring that may be highly ramified, the results of [BM02] do not entirely go over to our situation, and so in some cases we must scrape by with weaker results.

Finally, we perform the calculations using strongly divisible modules (with coefficients and descent data) necessary to prove our main theorems. In Section 5 we perform calculations with characters, and use these results repeatedly in Section 6, which contains the bulk of our calculations.

We remark that, in the course of our work, we completely determine (Ths. 6.11 and 6.12) the reductions (mod p) of 2-dimensional potentially Barsotti-Tate Galois representations that become crystalline over a tamely ramified extension of $\mathbb{Q}_p$. In Section 6.4, we apply these results to re-prove an old result on the (mod p) representations attached to modular forms, and to suggest a first step towards a new one.

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[1] This question is resolved in considerable generality in a new preprint of Mark Kisin [Kis].

Remark 1.7. The current version of this paper fixes an error in the published version, as a consequence of which there is one more family of strongly divisible modules that we must study by the methods of this paper than was studied in the published version. The main results of this paper are unaffected.

The mistake is in the statement and proof of Theorem 6.12(4) of the published version. Let $T := T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M})$ be the p-adic Galois representation considered there. In the notation of that item, if $m = 1 + (p+1)j$ — i.e., if $i = 1$ — then the two characters ω_2^{m+p} and ω_2^{pm+1} are both characters of niveau one, and are equal. Hence the proof of Theorem 6.12(4) does *not* show that the reduction mod p of T has niveau two in this case, and indeed that proof can be modified to check that the reduction mod p of T is split. This means that when $i = 1$ we still need to construct a strongly divisible lattice in $T[1/p]$ whose reduction mod p has trivial endomorphisms. That construction is contained in this version of the paper.

For further details, see the corrigendum on the author's website. Numbering in this version is consistent with the published version.

2. Filtered modules with coefficients and descent data

The purpose of this section is to provide "potential" versions of the results in [BM02, Sec. 3.1].

2.1. **Weil-Deligne representations.** Suppose that E/K is an extension of fields, and suppose that F/K is a finite Galois extension. Endow $F \otimes_K E$ with an action of $G = \mathrm{Gal}(F/K)$ by letting G act naturally on the first factor and trivially on the second. Let g denote an element of G. In this section, we examine the structure of $(F \otimes_K E)$-modules with equivariant G-actions, which we dub (F, E, G)-modules, for short. By a map of (F, E, G)-modules, we mean an $(F \otimes_K E)$-module homomorphism that is also a G-homomorphism.

Lemma 2.1. *Every (F, E, G)-module is free.*

Proof. Let M be an (F, E, G)-module. Let $V = M^G$, the G-invariants of M. By Galois descent, we have $M = F \otimes_K V$ as F-vector spaces with an action of G. But since G acts trivially on E we find that V is actually an E-vector space; since the actions of F and E on M commute, M is a free (F, E, G)-module. □

For the remainder of this section, we consider what happens when F is actually contained inside E.

Lemma 2.2. *If E contains F, then the map θ taking $f \otimes e \mapsto (\sigma(f)e)_\sigma$, and extended by linearity, is an isomorphism*

$$\theta : F \otimes_K E \to \coprod_{\sigma : F \underset{K}{\hookrightarrow} E} E \tag{2.3}$$

of (F, E, G)-modules where, on the right-hand side, if $(e_\sigma)_\sigma$ denotes the vector that has e_σ in the σ-component then $(f \otimes e) \cdot (e_\sigma)_\sigma = (\sigma(f) e_\sigma e)_\sigma$ and $g \cdot (e_\sigma)_\sigma = (e_\sigma)_{\sigma \circ g^{-1}}$.

Proof. To begin, we note that right-hand side of (2.3) is indeed an (F, E, G)-module and that the map θ is well-defined, after which it is easy to see that θ is a map of (F, E, G)-modules. But θ is surjective, since the elements of G are linearly independent over E, and so by a dimension count θ is an isomorphism. □

Proposition 2.4. *If E contains F, any (F, E, G)-module M is isomorphic to one of the form*

$$M \cong \coprod_{\sigma : F \underset{K}{\hookrightarrow} E} V$$

for some E-vector space V, with the (F, E, G)-module structure on the right-hand side defined as in Lemma 2.2.

Proof. Let E_σ be the $(F \otimes_K E)$-submodule of $\coprod_\sigma E$ consisting of elements that are nonzero at most in the position corresponding to σ. Let I_σ be the ideal $\theta^{-1}(E_\sigma)$ in $F \otimes_K E$, and put $M_\sigma = I_\sigma M$; if $\tau = \sigma \circ g^{-1}$, then g induces E-linear maps $E_\sigma \to E_\tau$, $I_\sigma \to I_\tau$, and $\mu_{\sigma,\tau} : M_\sigma \to M_\tau$. By definition, $\mu_{\sigma,\tau}$ and $\mu_{\tau,\sigma}$ must be inverses of one another, and hence they are isomorphisms of E-vector spaces.

Now, the summation map $\coprod M_\sigma \to M$ is evidently surjective. To prove injectivity, suppose that we have a relation $\sum_\sigma m_\sigma = 0$ with each $m_\sigma \in M_\sigma$. Note that $(f \otimes 1)m_\sigma = (1 \otimes \sigma f)m_\sigma$ follows from the analogous relation in I_σ, and so

$$\sum_\sigma (1 \otimes \sigma f)m_\sigma = 0$$

for all $f \in F$. It follows from the linear independence of the elements of G that $m_\sigma = 0$ for all σ, and so $M = \coprod_\sigma M_\sigma$.

Fix any $\tau : F \hookrightarrow E$. We map M bijectively to $\coprod_\sigma M_\tau$ via the map $\coprod \mu_{\sigma,\tau}$. One checks without difficulty that, with the desired (F, E, G)-module structure on $\coprod_\sigma M_\tau$, this map is an isomorphism of (F, E, G)-modules. For example, $gm_\sigma = \mu_{\sigma,\sigma\circ g^{-1}} m_\sigma$ is mapped to the element that is equal to

$$\mu_{\sigma\circ g^{-1},\tau}\mu_{\sigma,\sigma\circ g^{-1}} m_\sigma = \mu_{\sigma,\tau} m_\tau$$

in the $(\sigma \circ g^{-1})$-position and zero elsewhere. □

Remark 2.5. Essentially the same argument shows that each (F, E, G)-submodule of $\coprod_\sigma V$ is equal to $\coprod_\sigma W$ for some sub-E-vector space $W \subset V$.

Now fix a group H and a surjection $\phi : H \twoheadrightarrow G$. Suppose that M is an $F \otimes_K E$-module endowed with two ϕ-semilinear, E-linear actions $\cdot_1$ and $\cdot_2$ of H: that is, if $m \in M$, $f \otimes e \in F \otimes_K E$, and $h \in H$, we ask that $h \cdot_i (f \otimes e)m = (\phi(h)f \otimes e)(h \cdot_i m)$ for $i = 1, 2$. Moreover, assume that the two actions of H commute with one another, and that the second action factors through an abelian quotient of H.

As in the proof of Proposition 2.4, M decomposes as a coproduct $\coprod M_\sigma$ of E-vector spaces, where $M_\sigma = I_\sigma M$. The preceeding hypotheses allow us to define a representation of H on each M_σ. Indeed, both actions of an element $h \in H$ induce a map $M_\sigma \to M_{\sigma\circ\phi(h^{-1})}$, and so we obtain an E-linear map $\rho_\sigma(h) : M_\sigma \to M_\sigma$ by setting

$$\mathrm{WD}_\sigma(h)(m_\sigma) = h^{-1} \cdot_2 h \cdot_1 m_\sigma.$$

The commutativity hypotheses on the two actions guarantee that WD_σ is a representation. Moreover, each h induces an isomorphism $\mathrm{WD}_\sigma \to \mathrm{WD}_{\sigma\circ\phi(h)^{-1}}$ via the second action; since ϕ is surjective, all of the WD_σ are isomorphic.

Definition 2.6. The isomorphism class of the WD_σ is called the *Weil-Deligne representation of H attached to M*, and is denoted $\mathrm{WD}(M)$.

2.2. **Weakly admissible filtered modules.** Let p be an odd prime. Choose an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, let E and F be finite extensions of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$, and let F' be a field lying between $\mathbb{Q}_p$ and F such that F/F' is Galois. Fix the uniformizer $p \in \mathbb{Q}_p$, thereby fixing an inclusion $B_{\mathrm{st}} \to B_{\mathrm{dR}}$. Let F_0 denote the maximal unramified extension of $\mathbb{Q}_p$ contained in F. We retain this notation for the remainder of the paper.

Definition 2.7. A *filtered* $(\varphi, N, F/F', E)$*-module* of rank n is a free $(F_0 \otimes_{\mathbb{Q}_p} E)$-module D of rank n equipped with

- an F_0-semilinear, E-linear automorphism φ,
- a nilpotent $(F_0 \otimes_{\mathbb{Q}_p} E)$-linear endomorphism N such that $N\varphi = p\varphi N$,
- a decreasing filtration on $D_F = F \otimes_{F_0} D$ such that $\mathrm{Fil}^i D_F$ is zero if $i \gg 0$ and is equal to D_F if $i \ll 0$, and
- an F_0-semilinear, E-linear action of $\mathrm{Gal}(F/F')$ commuting with φ and N and preserving the filtration.

Suppose that $\rho : G_{F'} \to \mathrm{GL}(V)$ is a potentially semistable representation of $G_{F'}$ on an n-dimensional E-vector space V, such that $\rho|_{G_F}$ is semistable. Then

$$D_{\mathrm{st}}^F(V) = (B_{\mathrm{st}} \otimes V)^{G_F}$$

is an example of a filtered $(\varphi, N, F/F', E)$-module of rank n. For instance, to see that the action of $\mathrm{Gal}(F/F')$ preserves the filtration, we note that the filtration is induced from the map

$$F \otimes_{F_0} D_{\mathrm{st}}^F(V) \to B_{\mathrm{dR}} \otimes V.$$

This map is Galois-equivariant because the inclusion $B_{\mathrm{st}} \to B_{\mathrm{dR}}$ is; since the action of Galois preserves the filtration on B_{dR}, it also preserves the filtration on $F \otimes_{F_0} D_{\mathrm{st}}^F(V)$. Note that in the special case $F' = \mathbb{Q}_p$, Lemma 2.1 implies that the filtration consists of free $(F \otimes_{\mathbb{Q}_p} E)$-modules; this is not always the case (see [BM02, Rem 3.1.1.4]).

Definition 2.8. A filtered $(\varphi, N, F/F', E)$-module is said to be *weakly admissible* if the underlying (φ, N, F, E)-module is weakly admissible in the sense of [BM02, Déf 3.1.1.1(ii)] (i.e., if one forgets about the $\mathrm{Gal}(F/F')$-action).

Therefore D_{st}^F is a functor from the category of E-representations of $G_{F'}$ which become semistable when restricted to G_F, to the category of weakly admissible $(\varphi, N, F/F', E)$-modules. Conversely, if D is a weakly admissible $(\varphi, N, F/F', E)$-module, define

$$V_{\mathrm{st}}^{F'}(D) = (B_{\mathrm{st}} \otimes_{F_0} D)_{N=0}^{\varphi=1} \cap \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_F (F \otimes_{F_0} D)).$$

This is an E-representation of $G_{F'}$, where $G_{F'}$ acts as usual on B_{st} and through $\mathrm{Gal}(F/F')$ on D; moreover, by the results of [CF00] we know that the restriction of $V_{\mathrm{st}}^{F'}(D)$ to G_F is a semi-stable representation of dimension (over E) equal to $\mathrm{rk}_{F_0 \otimes_{\mathbb{Q}_p} E} D$.

Proposition 2.9. *D_{st}^F and $V_{\mathrm{st}}^{F'}$ are quasi-inverses.*

Proof. Consider the natural $G_{F'}$-homomorphism

$$B_{\mathrm{st}} \otimes V_{\mathrm{st}}^{F'}(D) \to B_{\mathrm{st}} \otimes D.$$

Taking G_F-invariants yields a map

$$D_{\mathrm{st}}^F(V_{\mathrm{st}}^{F'}(D)) \to (B_{\mathrm{st}} \otimes D)^{G_F} = D,$$

which we know must be an isomorphism of underlying (φ, N, F, E)-modules. Since our first map was actually a $G_{F'}$-homomorphism, this isomorphism respects the action of $\mathrm{Gal}(F/F')$ and so is an isomorphism of $(\varphi, N, F/F', E)$-modules as well.

The argument for the map $V \to V_{\mathrm{st}}^{F'}(D_{\mathrm{st}}^F(V))$ is analogous. □

Corollary 2.10. *The category of E-representations of $G_{F'}$ which become semistable when restricted to G_F and the category of weakly admissible $(\varphi, N, F/F', E)$-modules are equivalent.*

Following [BM02], we make use of functors $D_{\mathrm{st},k}^F$ and $V_{\mathrm{st},k}^{F'}$, defined as follows:

$$V_{\mathrm{st},k}^{F'}(D) = (B_{\mathrm{st}} \otimes_{F_0} D)_{N=0}^{\varphi=p^{k-1}} \cap \mathrm{Fil}^{k-1}(B_{\mathrm{dR}} \otimes_F (F \otimes_{F_0} D))$$

and

$$D_{\mathrm{st},k}^F(V) = D_{\mathrm{st}}^F(V(1-k)),$$

where $V(1-k)$ denotes the Tate twist $V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1-k)$. That these functors are quasi-inverse to one another follows from the next lemma.

Lemma 2.11. *For all filtered $(\varphi, N, F/F', E)$-modules D, there is an isomorphism $V_{\mathrm{st},k}^{F'}(D) \cong V_{\mathrm{st}}^{F'}(D)(k-1)$.*

The proof of this lemma is exactly the same as the proof of [BM02, Lem. 3.1.1.2], and similarly we have the following immediate corollary.

Corollary 2.12. *The functor $V_{\mathrm{st},k}^{F'}$ is an equivalence of categories between the category of weakly admissible filtered $(\varphi, N, F/F', E)$-modules D such that $\mathrm{Fil}^0(F \otimes_{F_0} D) = F \otimes_{F_0} D$ and $\mathrm{Fil}^k(F \otimes_{F_0} D) = 0$, and the category of E-representations of $G_{F'}$ which are semistable when restricted to G_F and have Hodge-Tate weights in the range $\{0, \dots, k-1\}$.*

Example 2.13. Let ϵ denote the cyclotomic character of $G_{\mathbb{Q}_p}$, let $\tilde{\omega}$ denote the Teichmüller lift of the mod p reduction of ϵ, and for $a \in \mathcal{O}_E^\times$ let λ_a denote the unramified character of $G_{\mathbb{Q}_p}$ sending arithmetic Frobenius to a. Set $F_1 = \mathbb{Q}_p(\zeta_p)$. Then $\epsilon^i \tilde{\omega}^j \lambda_a$ becomes semistable when restricted to G_{F_1}, and $D_{\mathrm{st},k}^{F_1}(\epsilon^i \tilde{\omega}^j \lambda_a)$ is a 1-dimensional filtered module $E \cdot \mathbf{e}$ satisfying $N = 0$,

$$\varphi(\mathbf{e}) = p^{k-i-1} a^{-1} \mathbf{e},$$

and, for $g \in \mathrm{Gal}(F_1/\mathbb{Q}_p)$,

$$g(\mathbf{e}) = \tilde{\omega}^j(g)(\mathbf{e}).$$

Indeed, this follows directly from the result in the special cases ϵ, $\tilde{\omega}$, and λ_a. For the first two, use that the element $t \in B_{\mathrm{st}}$ is a period for ϵ, and that $\tilde{\omega}\,|_{G_{F_1}}$ is trivial, respectively. For λ_a, one can use Hilbert's Theorem 90 for $\overline{\mathbb{F}}_p/\mathbb{F}_p$ and a Hensel-like approximation argument to show that the p-adic completion $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ of the maximal unramified extension of $\mathbb{Q}_p$ is a period ring for unramified representations. Then if $\mathbf{e} = \sum x_i \otimes e_i \in (B_{\mathrm{st}} \otimes E)^{G_{F_1}}$ with the $x_i \in \widehat{\mathbb{Q}_p^{\mathrm{un}}}$ we have

$$\varphi(\mathbf{e}) = \sum \mathrm{Frob}(x_i) \otimes e_i = a^{-1} \sum \mathrm{Frob}(x_i) \otimes a e_i = a^{-1} \mathrm{Frob}(\mathbf{e}) = a^{-1} \mathbf{e},$$

where Frob is any representative of arithmetic Frobenius in $G_{\mathbb{Q}_p}$.

Example 2.14. Similarly, let ϖ be a choice of $(-p)^{1/(p^2-1)}$, set $F_2 = \mathbb{Q}_{p^2}(\varpi)$, and suppose E is a finite extension of $\mathbb{Q}_{p^2}$. Let $\tilde{\omega}_2 : G_{\mathbb{Q}_{p^2}} \to \mathcal{O}_E^\times$ be the character $\tilde{\omega}_2(g) = (g\varpi)/\varpi$. Then the character $\tilde{\omega}_2^m(\epsilon^i\lambda_a)|_{G_{\mathbb{Q}_{p^2}}}$ becomes semistable when restricted to G_{F_2}, and $D^{F_2}_{\mathrm{st},k}(\tilde{\omega}_2^m(\epsilon^i\lambda_a)|_{G_{\mathbb{Q}_{p^2}}})$ is a module $(\mathbb{Q}_{p^2} \otimes_{\mathbb{Q}_p} E) \cdot \mathbf{e}$ satisfying $N = 0$,

$$\varphi(\mathbf{e}) = p^{k-i-1}(1 \otimes a^{-1})\mathbf{e},$$

and, for $g \in \mathrm{Gal}(F_2/\mathbb{Q}_{p^2})$,

$$g(\mathbf{e}) = (1 \otimes \tilde{\omega}_2^m(g))(\mathbf{e}).$$

Finally, the character $\epsilon^i\tilde{\omega}^j\lambda_a$ of $G_{\mathbb{Q}_p}$ also becomes semistable over F_2, and $D^{F_2}_{\mathrm{st},k}(\epsilon^i\tilde{\omega}^j\lambda_a)$ is a module $(\mathbb{Q}_{p^2} \otimes_{\mathbb{Q}_p} E) \cdot \mathbf{e}$ satisfying $N = 0$,

$$\varphi(\mathbf{e}) = p^{k-i-1}(1 \otimes a^{-1})\mathbf{e},$$

and, for $g \in \mathrm{Gal}(F/\mathbb{Q}_p)$,

$$g(\mathbf{e}) = (1 \otimes \tilde{\omega}^j(g))(\mathbf{e}).$$

Let $W_{F'}$ denote the Weil subgroup $W_{F'}$ of $G_{F'}$; if $\mathbb{F}'$ is the residue field of F', recall that there is a map $\alpha : W_{F'} \to \mathbb{Z} \subset \mathrm{Gal}(\overline{\mathbb{F}'}/\mathbb{F}')$ sending arithmetic Frobenius to 1. Now, if D is a filtered $(\varphi, N, F/F', E)$-module, observe that $W_{F'}$ acts in two different ways on D: by restriction to $\mathrm{Gal}(F/F')$, but also by letting $g \in W_{F'}$ act as $\varphi^{\alpha(g)}$. These two actions satisfy the compatibilities necessary to attach a Weil-Deligne representation of $W_{F'}$ to D.

Definition 2.15. Suppose that E contains F_0. If V is an E-representation of $G_{\mathbb{Q}_p}$ which is semistable when restricted to G_F, then the Weil-Deligne representation $\mathrm{WD}(V)$ attached to V is $\mathrm{WD}(D^F_{\mathrm{st}}(V))$. The *Galois type* (or *type*) $\tau(V)$ of V is defined to be $\mathrm{WD}(V)|_{I_p}$.

From [CDT99, App. B.2] we recall several properties of $\mathrm{WD}(V)$:

- $\mathrm{WD}(V)$ does not depend on the choice of F,
- $\mathrm{WD}(V_1 \otimes V_2) \cong \mathrm{WD}(V_1) \otimes \mathrm{WD}(V_2)$, and
- $\mathrm{WD}(\epsilon_F)$ is unramified, where ϵ_F denotes the cyclotomic character of G_F.

In particular, these facts imply that we could equivalently have defined $\tau(V)$ using $\mathrm{WD}(D^F_{\mathrm{st},k}(V))$ instead of $\mathrm{WD}(D^F_{\mathrm{st}}(V))$

Example 2.16. Let ω and ω_2 denote the mod p reductions of $\tilde{\omega}$ and $\tilde{\omega}_2$. By abuse of notation, we often refer to the restrictions $\omega|_{I_p}, \omega_2|_{I_p}, \tilde{\omega}|_{I_p}, \tilde{\omega}_2|_{I_p}$ simply as $\omega, \omega_2, \tilde{\omega}, \tilde{\omega}_2$. Suppose that V is a 2-dimensional potentially semistable E-representation of $G_{\mathbb{Q}_p}$ and suppose that $\tau(V)$ is tamely ramified. Then either $\tau(V) = \tilde{\omega}^i \oplus \tilde{\omega}^j$ for integers i and j (the "principal series" case) or else $\tau(V) = \tilde{\omega}_2^m \oplus \tilde{\omega}_2^{pm}$ for an integer m not divisible by $p+1$ (the "supercuspidal" case).

Proposition 2.17. *Suppose that V is an indecomposable 2-dimensional potentially semistable E-representation of $G_{\mathbb{Q}_p}$ with*

- *Hodge-Tate weights $(0, 1)$, and*
- *type $\tau(V) = \tilde{\omega}^i \oplus \tilde{\omega}^j$ for $i \not\equiv j \pmod{p-1}$.*

Let π be a choice of $(-p)^{1/(p-1)}$ and set $F_1 = \mathbb{Q}_p(\pi)$. Then V is crystalline over F_1 and $D^{F_1}_{\mathrm{st},2}(V)$ is of the form

$$D = E \cdot \mathbf{e}_1 \oplus E \cdot \mathbf{e}_2,$$

$$\varphi(\mathbf{e}_1) = x_1\mathbf{e}_1\ , \quad \varphi(\mathbf{e}_2) = x_2\mathbf{e}_2\ , \quad N = 0,$$
$$\mathrm{Fil}^1(F_1 \otimes_{\mathbb{Q}_p} D) = (F_1 \otimes_{\mathbb{Q}_p} E)(\pi^{j-i}\mathbf{e}_1 + \mathbf{e}_2),$$
$$g\cdot\mathbf{e}_1 = \tilde{\omega}(g)^i\mathbf{e}_1\ , \quad g\cdot\mathbf{e}_2 = \tilde{\omega}(g)^j\mathbf{e}_2 \text{ for } g \in \mathrm{Gal}(F_1/\mathbb{Q}_p),$$

with $x_1, x_2 \in \mathcal{O}_E$ *and* $\mathrm{val}_p(x_1x_2) = 1$.

Proof. Since $\tau(V)$ is nonscalar, V is potentially crystalline by [BM02, Lem. 2.2.2.2]; moreover, $\tau(V)\,|_{I_{F_1}}$ is trivial and so V becomes crystalline over F_1. Hence $N = 0$.

From the construction of $\mathrm{WD}(V)$, it is easy to see that D must have a basis $\mathbf{e}_1, \mathbf{e}_2$ on which $\mathrm{Gal}(F_1/\mathbb{Q}_p)$ acts via $g\cdot\mathbf{e}_1 = \tilde{\omega}(g)^i\mathbf{e}_1$ and $g\cdot\mathbf{e}_2 = \tilde{\omega}(g)^j\mathbf{e}_2$. Since φ and g commute, and again using the fact that $\tau(V)$ is nonscalar, it follows that $\varphi(\mathbf{e}_1) = x_1\mathbf{e}_1$ and $\varphi(\mathbf{e}_2) = x_2\mathbf{e}_2$ for some x_1 and x_2.

Using the fact that $\mathrm{Gal}(F_1/\mathbb{Q}_p)$ preserves the filtration, we find $\mathrm{Fil}^1(F_1 \otimes_{\mathbb{Q}_p} D)$ to be of the form $(F_1 \otimes_{\mathbb{Q}_p} E)(\pi^{j-i}a\mathbf{e}_1 + b\mathbf{e}_2)$ for $a, b \in E$. Both a and b must be nonzero: otherwise, the resulting $(\varphi, N, F_1/\mathbb{Q}_p, E)$-module would be a direct sum of two 1-dimensional $(\varphi, N, F_1/\mathbb{Q}_p, E)$-modules, contradicting the indecomposability of V. Replacing $\mathbf{e}_1$ by $a\mathbf{e}_1$ and $\mathbf{e}_2$ by $b\mathbf{e}_2$ in our basis for D, we see that Fil^1 may be taken to have the desired form. Finally, the weak admissibility of D implies that $x_1, x_2 \in \mathcal{O}_E$ and that $\mathrm{val}_p(x_1x_2) = 1$. □

We denote the filtered modules of the preceding Proposition by D_{x_1,x_2}.

Proposition 2.18. *Suppose that* E *contains* $\mathbb{Q}_{p^2}$*, and suppose that* V *is a 2-dimensional potentially semistable* E*-representation of* $G_{\mathbb{Q}_p}$ *with*

- *Hodge-Tate weights* $(0, 1)$*, and*
- *type* $\tau(V) = \tilde{\omega}_2^m \oplus \tilde{\omega}_2^{pm}$ *with* $p + 1 \nmid m$.

Write $m = i + (p+1)j$ *with* $i \in \{1, \ldots, p\}$ *and* $j \in \mathbb{Z}/(p-1)\mathbb{Z}$*. Let* ϖ *be a choice of* $(-p)^{1/(p^2-1)}$*, set* $F_2 = \mathbb{Q}_{p^2}(\varpi)$*, and let* g_φ *denote the element of* $\mathrm{Gal}(F_2/\mathbb{Q}_p)$ *which fixes* ϖ *and is nontrivial on* $\mathbb{Q}_{p^2}$*. Then* V *is crystalline over* F_2 *and* $D^{F_2}_{\mathrm{st},2}(V)$ *is of the form*

$$D = (\mathbb{Q}_{p^2} \otimes E)\cdot\mathbf{e}_1 \oplus (\mathbb{Q}_{p^2} \otimes E)\cdot\mathbf{e}_2,$$
$$\varphi(\mathbf{e}_1) = \mathbf{e}_2\ , \quad \varphi(\mathbf{e}_2) = (1 \otimes x)\mathbf{e}_1\ , \quad N = 0,$$
$$\mathrm{Fil}^1(F_2 \otimes_{\mathbb{Q}_{p^2}} D) = (F_2 \otimes_{\mathbb{Q}_p} E)((\varpi^{(p-1)i} \otimes a)\mathbf{e}_1 + (1 \otimes b)\mathbf{e}_2),$$
$$g\cdot\mathbf{e}_1 = (\tilde{\omega}_2(g)^m \otimes 1)\mathbf{e}_1\ , \quad g\cdot\mathbf{e}_2 = (\tilde{\omega}_2(g)^{pm} \otimes 1)\mathbf{e}_2 \text{ for } g \in \mathrm{Gal}(F_2/\mathbb{Q}_{p^2}),$$
$$g_\varphi\cdot\mathbf{e}_1 = \mathbf{e}_1\ , \quad g_\varphi\cdot\mathbf{e}_2 = \mathbf{e}_2,$$

with $(a, b) \in E^2 \setminus (0, 0)$*,* $x \in \mathcal{O}_E$ *and* $\mathrm{val}_p(x) = 1$.

Proof. Exactly as in Proposition 2.17, V becomes crystalline over F_2 and $N = 0$. Let σ_1 and σ_2 denote the two embeddings of $\mathbb{Q}_{p^2}$ into E. For each $\mu = 1, 2$, the construction of $\mathrm{WD}(V)$ implies that D_{σ_μ} has an E-basis $v_{\mu 1}, v_{\mu 2}$ on which $g \in \mathrm{Gal}(F_2/\mathbb{Q}_{p^2})$ acts as

$$g\cdot v_{\mu 1} = \sigma_\mu(\tilde{\omega}_2^m(g))v_{\mu 1} \text{ and } g\cdot v_{\mu 2} = \sigma_\mu(\tilde{\omega}_2^{pm}(g))v_{\mu 2}.$$

Since g_φ is an E-linear map on D which swaps the two subspaces D_{σ_μ} and satisfies the relation $g_\varphi g g_\varphi = g^p$ for all $g \in \mathrm{Gal}(F_2/\mathbb{Q}_{p^2})$, it follows (possibly after multiplying some $v_{\mu\nu}$ by constants in E) that $g_\varphi\cdot v_{\mu\nu} = v_{(3-\mu)\nu}$. Similarly, since φ swaps the D_{σ_μ} and commutes with the action of $\mathrm{Gal}(F_2/\mathbb{Q}_{p^2})$, there exist $c, d \in E$ such that $\varphi(v_{11}) = cv_{22}$, $\varphi(v_{21}) = cv_{12}$, $\varphi(v_{12}) = dv_{21}$, and $\varphi(v_{22}) = dv_{11}$.

Taking $\mathbf{e}_1 = v_{11} + v_{21}$, $\mathbf{e}_2 = c(v_{12} + v_{22})$, and $x = cd$, and using the fact that Fil^1 must be preserved by $\mathrm{Gal}(F_2/\mathbb{Q}_p)$, we see without difficulty that in this basis D has the desired form. □

We denote the filtered modules of Proposition 2.18 by $D_{m,[a:b]}$.

Remark 2.19. It is not difficult to see that these filtered modules match those of "type IV" in [FM95, §11] (which deals only with the case $E = \mathbb{Q}_p$).

By a similar argument, we also find the following.

Proposition 2.20. *Suppose that E contains $\mathbb{Q}_{p^2}$, and suppose that V is an indecomposable 2-dimensional potentially semistable E-representation of $G_{\mathbb{Q}_p}$ with*

- *Hodge-Tate weights $(0,1)$, and*
- *type $\tau(V) = \tilde{\omega}^i \oplus \tilde{\omega}^j$ for $i \not\equiv j \pmod{p-1}$.*

Let ϖ, F_2, and the elements of $\mathrm{Gal}(F_2/\mathbb{Q}_p)$ be as in Proposition 2.18, and set $\pi = \varpi^{p+1}$. Then V is crystalline over F_2 and $D^{F_2}_{\mathrm{st},2}(V)$ is of the form

$$D = (\mathbb{Q}_{p^2} \otimes E) \cdot \mathbf{e}_1 \oplus (\mathbb{Q}_{p^2} \otimes E) \cdot \mathbf{e}_2,$$

$$\varphi(\mathbf{e}_1) = (1 \otimes x_1)\mathbf{e}_1 \ , \quad \varphi(\mathbf{e}_2) = (1 \otimes x_2)\mathbf{e}_2 \ , \quad N = 0,$$

$$\mathrm{Fil}^1(F_2 \otimes_{\mathbb{Q}_{p^2}} D) = (F_2 \otimes_{\mathbb{Q}_p} E)((\pi^{j-i} \otimes 1)\mathbf{e}_1 + \mathbf{e}_2),$$

$$g \cdot \mathbf{e}_1 = (\tilde{\omega}(g)^i \otimes 1)\mathbf{e}_1 \ , \quad g \cdot \mathbf{e}_2 = (\tilde{\omega}(g)^j \otimes 1)\mathbf{e}_2 \ \textit{for } g \in \mathrm{Gal}(F_2/\mathbb{Q}_{p^2}),$$

$$g_\varphi \cdot \mathbf{e}_1 = \mathbf{e}_1 \ , \quad g_\varphi \cdot \mathbf{e}_2 = \mathbf{e}_2,$$

with $x_1, x_2 \in \mathcal{O}_E$ and $\mathrm{val}_p(x_1 x_2) = 1$.

Denote this filtered module by D'_{x_1,x_2}, and note that $(D'_{x_1,x_2})^{\mathrm{Gal}(F_2/F_1)} = D_{x_1,x_2}$. When $0 < \mathrm{val}_p(x_1), \mathrm{val}_p(x_2) < 1$, we actually need to work with D'_{x_1,x_2} instead of D_{x_1,x_2}.

Finally, we conclude this section by checking the following.

Proposition 2.21. *The filtered modules listed in Propositions 2.17, 2.18, 2.20 are weakly admissible.*

Proof. To verify the weak admissibility of one of these filtered modules D, by [BM02, Prop. 3.1.1.5] one needs to check that any $(F_0 \otimes E)$-submodule of D that is preserved by φ satisfies $t_H \le t_N$ (in the notation of Fontaine [Fon94]). For the filtered modules of Proposition 2.17, this is evident. We give the details for the filtered modules of Proposition 2.18; the details for Proposition 2.20 are similar.

Suppose that $\mathbb{Q}_{p^2} = \mathbb{Q}_p(\tau)$ with $\tau^2 \in \mathbb{Q}_p$. Put $u = \frac{1}{2}(1 \otimes 1 - \tau \otimes \tau^{-1}) \in \mathbb{Q}_{p^2} \otimes E$ and set $v = 1 \otimes 1 - u$, so that u, v are idempotents satisfying $\varphi(u) = v$, $\varphi(v) = u$. It then easy to see that any nonzero proper $\mathbb{Q}_{p^2} \otimes E$-submodule D_0 of D which is preserved by φ must be generated by a pair of elements $u(a\mathbf{e}_1 + b\mathbf{e}_2)$, $v(bx\mathbf{e}_1 + a\mathbf{e}_2)$ with $a, b \in E$: if D_0 contains two E-linearly independent elements of uD, then it contains all of uD; hence it is equal to D. One checks that $t_H(D_0) = 0$ for such D_0. □

3. Strongly divisible modules with tame descent data

In [Bre00], C. Breuil constructed a category of "strongly divisible modules" over a local field F', and he proved that it is (anti-)equivalent to the category of Galois lattices inside Barsotti-Tate representations of $G_{F'}$. Following the strategy of [BCDT01, Sec. 5.4], we formulate descent data on strongly divisible modules, thereby extending Breuil's antiequivalence to Galois lattices inside potentially Barsotti-Tate representations. This is essentially formal, but for simplicity we work exclusively with descent data for tame extensions F/F'.

3.1. **Galois lattices and p-divisible groups.** In this section, we review the relation between Galois lattices and p-divisible groups. Let $\rho : G_{F'} \to \mathrm{GL}(V)$ be a p-adic representation, and let T be a Galois $\mathbb{Z}_p$-lattice inside V. To T we may associate a p-divisible group over F', as follows: each T/p^n is a finite representation of $G_{F'}$, hence corresponds to a finite flat group scheme $\Gamma(n)$ over F'. Then the p-divisible group associated to T is $\Gamma = \cup\Gamma(n)$. Conversely, given a p-divisible group Γ over F' we may recover the Galois lattice

$$\varprojlim_n \Gamma(n)(\overline{\mathbb{Q}}_p),$$

and these two operations are readily seen to be inverse to one another.

Breuil shows ([Bre00, Th. 5.3.2]) that Γ extends to a p-divisible group $\mathcal{G}$ over the integers $\mathcal{O}_{F'}$ if and only if the representation ρ is crystalline with Hodge-Tate weights in $\{0, 1\}$. More precisely, Breuil shows that if ρ is crystalline with Hodge-Tate weights in $\{0, 1\}$, then there exists some lattice inside V for which the associated p-divisible group over F' extends; but then by a scheme-theoretic closure argument (see [Ray74, Secs. 2.2, 2.3]), for any lattice $T \subset V$ the p-divisible group over F' associated to T extends to a p-divisible group over $\mathcal{O}_{F'}$. Tate's full faithfulness theorem guarantees that this extension is unique up to isomorphism.

Suppose now that ρ is merely *potentially* crystalline with Hodge-Tate weights in $\{0, 1\}$, and, more precisely, that ρ becomes crystalline over F. Let $T \subset V$ be a Galois lattice. Then T regarded as a G_F-lattice does correspond to a p-divisible group Γ over F which, as above, extends to a p-divisible group $\mathcal{G}$ over $\mathcal{O}_F$. However, the restriction from F' to F also induces descent data on Γ. Indeed, recall that

$$\Gamma(n) = \mathrm{Spec}(\mathrm{Maps}_{G_F}(T/p^n, \overline{\mathbb{Q}}_p)).$$

The algebra on the right-hand side carries an action of $\mathrm{Gal}(F/F')$: if $g \in \mathrm{Gal}(F/F')$, let $\tilde{g}$ be any extension of g to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F')$, and for $f \in \mathrm{Maps}_{G_{F'}}(T/p^n, \overline{\mathbb{Q}}_p))$, we set $g \cdot f = \tilde{g} \circ f \circ \tilde{g}^{-1}$. This is easily seen to be well-defined and compatible among different values of n, so that we obtain a g-semilinear map $\langle g \rangle : \Gamma \to \Gamma$. It is convenient to factor $\langle g \rangle$ as

$$\begin{array}{ccccc} \Gamma & \xrightarrow{[g]} & {}^g\Gamma & \longrightarrow & \Gamma \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \longrightarrow & \mathrm{Spec}(F) & \xrightarrow{g} & \mathrm{Spec}(F) \end{array}$$

where the right-hand square is Cartesian, so that the $[g]$ are maps of p-divisible groups over F satisfying the compatibility $[gh] = ({}^g[h]) \circ [g]$. (Here and henceforth, the superscript ^{g} denotes base change by g.) Finally, by Tate's full faithfulness theorem, each $[g]$ extends to a map $\mathcal{G} \to {}^g\mathcal{G}$. We again denote this by $[g]$, and we note that the compatibility relation is automatically still satisfied.

Definition 3.1. If $\mathcal{G}$ is a p-divisible group over $\mathcal{O}_F$, then *descent data relative to* F' is a collection of maps $[g] : \mathcal{G} \to {}^g\mathcal{G}$ for each $g \in \mathrm{Gal}(F'/F)$ satisfying $[gh] = ({}^g[h]) \circ [g]$.

In the reverse direction, if $\mathcal{G} = \cup\mathcal{G}(n)$ is a p-divisible group over $\mathcal{O}_F$ with descent data relative to F', we can construct a $G_{F'}$-lattice. Writing $\mathcal{G}(n) = \mathrm{Spec}(R_n)$, the descent data comes from a compatible collection of $\mathrm{Gal}(F/F')$-actions on the R_n. If $\sigma \in G_{F'}$, and $f \in \mathcal{G}(n)(\mathcal{O}_{\overline{\mathbb{Q}}_p}) = \mathrm{Hom}(R_n, \mathcal{O}_{\overline{\mathbb{Q}}_p})$, we set $\sigma \cdot f = \sigma \circ f \circ (\sigma^{-1}|_F)$. Since the descent data is actually descent data on the whole p-divisible group, these $G_{F'}$-actions are compatible for varying n and yield a $G_{F'}$-action on

$$\varprojlim_n \mathcal{G}(n)(\mathcal{O}_{\overline{\mathbb{Q}}_p}).$$

Unsurprisingly, this construction is inverse to the construction from a $G_{F'}$-lattice of a p-divisible group over $\mathcal{O}_F$ with descent data relative to F'. As a consequence, we have the following.

Proposition 3.2. *The above constructions describe an equivalence between the category of Galois lattices inside potentially crystalline $G_{F'}$-representations that become crystalline over F and have Hodge-Tate weights inside $\{0,1\}$, and the category of p-divisible groups over $\mathcal{O}_F$ with descent data relative to F'.*

3.2. **Big rings and categories of filtered modules without descent data.** In this section we review the definitions of various big rings and categories of filtered modules from [Bre00] and [Bre99].

Let R be a complete discrete valuation ring of characteristic zero, absolute ramification index e, and perfect residue field k of characteristic p. Fix a uniformizer π of R. Let $S = S_R$ be the p-adic completion of $W(k)[u, \frac{u^{ie}}{i!}]_{i\in\mathbb{N}}$, and let $S_n = S/p^nS$. The map $\phi : S \to S$ is the unique Frobenius-semilinear map sending $\phi(u) = u^p$ and $\phi(u^{ie}/i!) = u^{iep}/i!$; we also use ϕ to denote the map $S_n \to S_n$ induced by ϕ. Let N denote the unique $W(k)$-linear derivation such that $N(u) = -u$ and $N(u^{ie}/i!) = -ieu^{ie}/i!$, so that $N\phi = p\phi N$. Let $E(u) \in S$ denote the minimal polynomial of π over $W(k)$, and if $k \geq 1$, let $\mathrm{Fil}^{k-1}S$ be the p-adic completion of the ideal of S generated by $E(u)^i/i!$ for $i \geq k-1$. (Note that we are using k to denote both the residue field and the weight; it should not be possible to confuse the two uses with one another.) Then $\phi(\mathrm{Fil}^{k-1}S) \subset p^{k-1}S$ for $k \leq p$, and so for $k \leq p$, we let ϕ_{k-1} denote ϕ/p^{k-1} on $\mathrm{Fil}^{k-1}S$. Finally, let c denote $\phi_1(E(u))$.

We now repeat (essentially verbatim) some notation and definitions of [BCDT01] and [Bre00]; we refer the reader to [BCDT01, Secs. 5.3, 5.4] for details.

Let $\mathrm{Spf}(R)_{\mathrm{syn}}$ be the small p-adic formal syntomic site over R, and let (Ab/R) denote the category of abelian sheaves on $\mathrm{Spf}(R)_{\mathrm{syn}}$.

If $\mathfrak{X} \in \mathrm{Spf}(R)_{\mathrm{syn}}$, set $\mathfrak{X}_n = \mathfrak{X} \times_R R/p^n$. The sheaf $\mathcal{O}^{\mathrm{cris}}_{n,\pi}$ is the sheaf of S_n-modules on $\mathrm{Spf}(R)_{\mathrm{syn}}$ associated to the presheaf

$$\mathfrak{X} \mapsto \left(W_n(k)[u] \otimes_{\phi^n, W_n(k)} W_n(\Gamma(\mathfrak{X}_1, \mathcal{O}_{\mathfrak{X}_1}))\right)^{DP},$$

where ϕ is Frobenius on $W_n(k)$, where "DP" means that we take the divided power envelope with respect to the kernel of the map

$$W_n(k)[u] \otimes_{\phi^n, W_n(k)} W_n(\Gamma(\mathfrak{X}_1, \mathcal{O}_{\mathfrak{X}_1})) \to \Gamma(\mathfrak{X}_n, \mathcal{O}_{\mathfrak{X}_n})$$

$$s(u) \otimes (w_0, \dots, w_{n-1}) \mapsto s(\pi)(\hat{w}_0^{p^n} + \dots + p^{n-1}\hat{w}_{n-1}^p)$$

and relative to the usual divided power structure on the maximal ideal of $W_n(k)$, and where $\hat{w}_i$ is a local lifting of w_i. If $\mathcal{O}_n \in (\mathrm{Ab}/R)$ is the sheaf $\mathcal{O}_n(\mathfrak{X}) = \Gamma(\mathfrak{X}_n, \mathcal{O}_{\mathfrak{X}_n})$, then the above map induces a morphism $\mathcal{O}_{n,\pi}^{\mathrm{cris}} \to \mathcal{O}_n$, and we denote its kernel by $\mathcal{J}_{n,\pi}^{\mathrm{cris}}$. The map $\phi : \mathcal{O}_{n,\pi}^{\mathrm{cris}} \to \mathcal{O}_{n,\pi}^{\mathrm{cris}}$ induced by crystalline Frobenius satisfies $\phi(\mathcal{J}_{n,\pi}^{\mathrm{cris}}) \subset p\mathcal{O}_{n,\pi}^{\mathrm{cris}}$, and there exists $\phi_1 : \mathcal{J}_{n,\pi}^{\mathrm{cris}} \to \mathcal{O}_{n,\pi}^{\mathrm{cris}}$ which may be thought of as ϕ/p. Define $\mathcal{O}_{\infty,\pi}^{\mathrm{cris}} = \varinjlim_n \mathcal{O}_{n,\pi}^{\mathrm{cris}}$ and $\mathcal{J}_{\infty,\pi}^{\mathrm{cris}} = \varinjlim_n \mathcal{J}_{n,\pi}^{\mathrm{cris}}$; these limits are taken over the multiplication-by-p inclusions $\mathcal{O}_{n,\pi}^{\mathrm{cris}} \to \mathcal{O}_{n+1,\pi}^{\mathrm{cris}}$. See [Bre00, Sec 2.3] for further details regarding these sheaves.

If $R = \mathcal{O}_F$ is the ring of integers in a finite extension F over $\mathbb{Q}_p$, recall from [Bre00, Sec 5.3] that we define $A_{\mathrm{cris}} = \varprojlim W_n(\mathcal{O}_{\overline{\mathbb{Q}}_p}/p\mathcal{O}_{\overline{\mathbb{Q}}_p})^{DP}$. Fix a system of roots $(\pi_n)_{n\geq 0}$ in $\overline{\mathbb{Q}}_p$ such that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$, from which we construct an element $\underline{\pi} \in A_{\mathrm{cris}}$ (see [Bre99, Sec. 2.2.2]). Then $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}} \otimes_{W(k)} F_0$, where F_0 is the fraction field of $W(k)$, and $\widehat{A}_{\mathrm{cris}} = \varprojlim \mathcal{O}_{n,\pi}^{\mathrm{cris}}(\mathcal{O}_{\overline{\mathbb{Q}}_p})$ is isomorphic to the p-adic completion of $A_{\mathrm{cris}}[\frac{(u-\underline{\pi})^i}{i!}]_{i\in\mathbb{N}}$.

We refer the reader to [Bre99, Sec. 2.2.2] for the construction of the ring $\widehat{A}_{\mathrm{st}}$. The ring $\widehat{A}_{\mathrm{st}}$ has a filtration $\mathrm{Fil}^\bullet \widehat{A}_{\mathrm{st}}$, a Frobenius ϕ, and a monodromy operator N which is the unique A_{cris}-linear derivation such that $N(X) = 1 + X$. If $k \leq p$, the Frobenius satisfies $\phi(\mathrm{Fil}^{k-1}\widehat{A}_{\mathrm{st}}) \subset p^{k-1}\widehat{A}_{\mathrm{st}}$, so we let ϕ_{k-1} be ϕ/p^{k-1} on $\mathrm{Fil}^{k-1}\widehat{A}_{\mathrm{st}}$. The choice of $\underline{\pi}$ fixes an S-module structure on $\widehat{A}_{\mathrm{st}}$ and an embedding $\widehat{A}_{\mathrm{cris}} \to \widehat{A}_{\mathrm{st}}$ by sending $u \mapsto \underline{\pi}(1+X)^{-1}$, and this embedding induces a filtration, Frobenius, and monodromy operator on $\widehat{A}_{\mathrm{cris}}$ and a filtration on A_{cris}. Set $A_{\mathrm{cris},\infty} = A_{\mathrm{cris}} \otimes_{W(k)} F_0/W(k)$, $\widehat{A}_{\mathrm{cris},\infty} = \widehat{A}_{\mathrm{cris}} \otimes_{W(k)} F_0/W(k)$, and $\widehat{A}_{\mathrm{st},\infty} = \widehat{A}_{\mathrm{st}} \otimes_{W(k)} F_0/W(k)$ with the induced Frobenius, filtration, and monodromy operators (e.g., $\mathrm{Fil}^{k-1}\widehat{A}_{\mathrm{st},\infty} = (\mathrm{Fil}^{k-1}\widehat{A}_{\mathrm{st}}) \otimes F_0/W(k)$).

Let $k \in \{1, \dots, p-1\}$. Recalling [Bre99, Sec. 2.2.1], we let $'\underline{\mathrm{Mod}}^{k-1}$ denote the category of quadruples consisting of

- an S-module $\mathcal{M}$,
- an S-submodule $\mathrm{Fil}^{k-1}\mathcal{M}$ of $\mathcal{M}$ containing $(\mathrm{Fil}^{k-1}S)\mathcal{M}$,
- a ϕ-semilinear map $\phi_{k-1} : \mathrm{Fil}^{k-1}\mathcal{M} \to \mathcal{M}$ such that for all $s \in \mathrm{Fil}^{k-1}S$ and $x \in \mathcal{M}$ we have $\phi_{k-1}(sx) = \phi_{k-1}(s)\phi(x)$ with $\phi(x) = \frac{1}{c^{k-1}}\phi_{k-1}(E(u)^{k-1}x)$, and
- a $W(k)$-linear map $N : \mathcal{M} \to \mathcal{M}$ satisfying: $N(sx) = N(s)x + sN(x)$ for $s \in S, x \in \mathcal{M}$; and $E(u)N(\mathrm{Fil}^{k-1}\mathcal{M}) \subset \mathrm{Fil}^{k-1}\mathcal{M}$ and $\phi_{k-1}(E(u)N(x)) = cN(\phi_{k-1}(x))$ for $x \in \mathrm{Fil}^{k-1}\mathcal{M}$.

Morphisms in $'\underline{\mathrm{Mod}}^{k-1}$ are the S-linear maps preserving Fil^{k-1} and commuting with ϕ_{k-1} and N. We define six additional categories as follows: $'\underline{\mathrm{Mod}}_0^{k-1}$ is the category obtained by omitting N in the definition of $'\underline{\mathrm{Mod}}^{k-1}$, while $\underline{\mathrm{Mod}}^{k-1}$ and $\underline{\mathrm{Mod}}_0^{k-1}$ are the full subcategories of $'\underline{\mathrm{Mod}}^{k-1}$ and $'\underline{\mathrm{Mod}}_0^{k-1}$ with the following extra conditions:

- $\mathcal{M}$ is of the form $\oplus_i S_{n_i}$ for some finite list of positive integers n_i, and
- $\phi_{k-1}(\mathrm{Fil}^{k-1}\mathcal{M})$ generates $\mathcal{M}$ over S.

Next, Mod^{k-1} and Mod_0^{k-1} are the full subcategories of $'\underline{\mathrm{Mod}}^{k-1}$ and $'\underline{\mathrm{Mod}}_0^{k-1}$ with the following extra conditions:

- $\mathcal{M}$ is a free S-module and $\mathrm{Fil}^{k-1}\mathcal{M} \cap p\mathcal{M} = p\mathrm{Fil}^{k-1}\mathcal{M}$, and
- $\phi_{k-1}(\mathrm{Fil}^{k-1}\mathcal{M})$ generates $\mathcal{M}$ over S.

Finally, let $\mathrm{Mod}_{\mathrm{cris}}^{k-1}$ be the full subcategory of objects of Mod^{k-1} with the property that $N(\mathcal{M}) \subset I\mathcal{M}$, where I is the ideal $\sum_{i\geq 1} \frac{u^i}{\lfloor i/e \rfloor !} S$ in S.

The category Mod_0^{k-1} is called the category of strongly divisible modules (of weight k). Let $R = \mathcal{O}_F$ be the integers in a finite extension F of $\mathbb{Q}_p$, and $\mathcal{M}$ be a strongly divisible module of weight 2 for R. By [Bre00, Prop. 5.1.3(1)], there exists a unique $W(k)$-linear endomorphism N of $\mathcal{M}$ such that:

- $N(sx) = N(s)x + sN(x)$ for $s \in S$ and $x \in \mathcal{M}$,
- $N\phi_1 = \phi N$, and
- $N(\mathcal{M}) \subset I\mathcal{M}$, where I is the ideal $\sum_{i\geq 1} \frac{u^i}{\lfloor i/e \rfloor !} S$ in S.

Thus for $R = \mathcal{O}_F$, the categories Mod_0^1 and $\mathrm{Mod}_{\mathrm{cris}}^1$ are equivalent. Before proceeding to the next section, we note the following examples:

- S is an object of $\mathrm{Mod}_{\mathrm{cris}}^{k-1}$,
- each S_n is an object of $\underline{\mathrm{Mod}}^{k-1}$,
- $\widehat{A}_{\mathrm{cris}}$, $\widehat{A}_{\mathrm{cris},\infty}$, $\widehat{A}_{\mathrm{st}}$, and $\widehat{A}_{\mathrm{st},\infty}$ are objects of $'\underline{\mathrm{Mod}}^{k-1}$,
- $\mathcal{O}_{n,\pi}^{\mathrm{cris}}(\mathcal{O}_{\overline{\mathbb{Q}}_p})$ is an object of $'\underline{\mathrm{Mod}}_0^1$, and
- regarding A_{cris} as an S-module via $u \cdot x = \underline{\pi} x$ we can make A_{cris} and $A_{\mathrm{cris},\infty}$ into objects of $'\underline{\mathrm{Mod}}_0^{k-1}$; then the maps $\widehat{A}_{\mathrm{st}} \to A_{\mathrm{cris}}$ and $\widehat{A}_{\mathrm{st},\infty} \to A_{\mathrm{cris},\infty}$ sending $X \mapsto 0$ are morphisms in $'\underline{\mathrm{Mod}}_0^{k-1}$.

3.3. **p-divisible groups and strongly divisible modules with tame descent data.** Let $\mathcal{G} = \cup \mathcal{G}(n)$ be a p-divisible group over R. Then for each n we may regard $\mathcal{G}(n)$ as a sheaf on $\mathrm{Spf}(R)_{\mathrm{syn}}$ and we define

$$\mathcal{M}_\pi(\mathcal{G}(n)) = \mathrm{Hom}_{(\mathrm{Ab}/R)}(\mathcal{G}(n), \mathcal{O}_{\infty,\pi}^{\mathrm{cris}}) = \mathrm{Hom}_{(\mathrm{Ab}/R)}(\mathcal{G}(n), \mathcal{O}_{n,\pi}^{\mathrm{cris}}),$$

$$\mathrm{Fil}^1\mathcal{M}_\pi(\mathcal{G}(n)) = \mathrm{Hom}_{(\mathrm{Ab}/R)}(\mathcal{G}(n), \mathcal{J}_{\infty,\pi}^{\mathrm{cris}}) = \mathrm{Hom}_{(\mathrm{Ab}/R)}(\mathcal{G}(n), \mathcal{J}_{n,\pi}^{\mathrm{cris}}),$$

and

$$\phi_1 : \mathrm{Fil}^1\mathcal{M}_\pi(\mathcal{G}(n)) \to \mathcal{M}_\pi(\mathcal{G}(n))$$

induced by $\phi_1 : \mathcal{J}_{\infty,\pi}^{\mathrm{cris}} \to \mathcal{O}_{\infty,\pi}^{\mathrm{cris}}$. Next, define

$$(\mathcal{M}_\pi(\mathcal{G}), \mathrm{Fil}^1\mathcal{M}_\pi(\mathcal{G}), \phi_1) = (\varprojlim_n \mathcal{M}_\pi(\mathcal{G}(n)), \varprojlim_n \mathrm{Fil}^1\mathcal{M}_\pi(\mathcal{G}(n)), \varprojlim_n \phi_1).$$

We often denote the triples $(\mathcal{M}_\pi(\cdot), \mathrm{Fil}^1\mathcal{M}_\pi(\cdot), \phi_1)$ by $\mathcal{M}_\pi(\cdot)$ or, suppressing the fixed uniformizer π, by $\mathcal{M}(\cdot)$. By [Bre00, Cor. 4.2.2.7, Lem. 4.2.2.8], the $\mathcal{M}(\mathcal{G}(n))$ are objects in $\underline{\mathrm{Mod}}_0^1$, while $\mathcal{M}(\mathcal{G})$ is an object in the category Mod_0^1 of strongly divisible modules.

Now suppose that $g : R \to R$ is a continuous automorphism of R. For simplicity of notation, we assume that $g(\pi) = h_g\pi$ with $h_g \in W(k)$. This assumption is not strictly necessary until Corollary 3.6, but we do not need the extra generality for our applications. Define $\widehat{g} : W(k)[[u]] \to W(k)[[u]]$ by $\widehat{g}(\sum w_i u^i) = \sum g(w_i) h_g^i u^i$, and similarly, let $\widehat{g} : S \to S$ be the unique ring isomorphism such that $\widehat{g}\left(w_i \frac{u^i}{\lfloor i/e \rfloor !}\right) = g(w_i) \frac{u^i}{\lfloor i/e \rfloor !} h_g^i$. We also let $\widehat{g}$ denote the isomorphism induced on S_n.

If $\mathfrak{X} \in \mathrm{Spf}(R)_{\mathrm{syn}}$, let ${}^g\mathfrak{X} = \mathrm{Spf}(R) \times_{g^*,\mathrm{Spf(R)}} \mathfrak{X}$, and as in [BCDT01, Sec. 5.4], we define

$$\mathcal{O}_{n,\pi}^{\mathrm{cris},(g)}(\mathfrak{X}) = \mathcal{O}_{n,\pi}^{\mathrm{cris}}({}^g\mathfrak{X}), \quad \mathcal{J}_{n,\pi}^{\mathrm{cris},(g)}(\mathfrak{X}) = \mathcal{J}_{n,\pi}^{\mathrm{cris}}({}^g\mathfrak{X}),$$

so that $\mathcal{O}_{n,\pi}^{\mathrm{cris},(g)} \in (\mathrm{Ab}/R)$ is the sheaf associated the the presheaf

$$\begin{aligned} \mathfrak{X} &\mapsto \left(W_n(k)[u] \otimes_{\phi^n, W_n(k)} W_n(\Gamma({}^g\mathfrak{X}_1, \mathcal{O}_{{}^g\mathfrak{X}_1}))\right)^{DP} \\ &= \left(W_n(k)[u] \otimes_{\phi^n, W_n(k)} W_n(R \otimes_{g,R} \Gamma(\mathfrak{X}_1, \mathcal{O}_{\mathfrak{X}_1}))\right)^{DP}. \end{aligned}$$

Then there is a canonical isomorphism

$$\mathcal{O}_{n,\pi}^{\mathrm{cris}} \otimes_{S_n, \widehat{g}} S_n \xrightarrow{\sim} \mathcal{O}_{n,\pi}^{\mathrm{cris},(g)}$$

coming from the $\widehat{g}$-semilinear map from $W_n(k)[u] \otimes_{\phi^n, W_n(k)} W_n(\Gamma(\mathfrak{X}_1, \mathcal{O}_{\mathfrak{X}_1}))$ to $W_n(k)[u] \otimes_{\phi^n, W_n(k)} W_n(R \otimes_{g,R} \Gamma(\mathfrak{X}_1, \mathcal{O}_{\mathfrak{X}_1}))$ sending

$$s \otimes (w_0, \ldots, w_{n-1}) \mapsto \widehat{g}(s) \otimes (1 \otimes w_0, \ldots, 1 \otimes w_{n-1}),$$

and this induces

$$\mathcal{J}_{n,\pi}^{\mathrm{cris}} \otimes_{S_n, \widehat{g}} S_n \xrightarrow{\sim} \mathcal{J}_{n,\pi}^{\mathrm{cris},(g)}.$$

Lemma 5.4.4 of [BCDT01] tells us that the diagram

$$\begin{array}{ccc} \mathcal{J}_{n,\pi}^{\mathrm{cris}} \otimes_{S_n, \widehat{g}} S_n & \xrightarrow{\sim} & \mathcal{J}_{n,\pi}^{\mathrm{cris},(g)} \\ {\scriptstyle \phi_1 \otimes \phi} \downarrow & & {\scriptstyle \phi_1} \downarrow \\ \mathcal{O}_{n,\pi}^{\mathrm{cris}} \otimes_{S_n, \widehat{g}} S_n & \xrightarrow{\sim} & \mathcal{O}_{n,\pi}^{\mathrm{cris},(g)} \end{array}$$

is commutative. Moreover, looking at the presheaves, it is evident that the above diagrams for n and $n+1$ are compatible under the multiplication-by-p inclusion (i.e., we have a commutative cube, where the front and back faces are the above diagrams for n and $n+1$, and the four front-to-back maps are induced by $\mathcal{O}_{n,\pi}^{\mathrm{cris}} \to \mathcal{O}_{n+1,\pi}^{\mathrm{cris}}$). We have the following proposition, which is an analogue of [BCDT01, Cor. 5.4.5] and is proved in essentially the same manner.

Proposition 3.3. *Let $g : R \to R$ be a continuous automorphism such that $g\pi = h_g \pi$ with $h_g \in W(k)$.*

(1) *Let $\mathcal{G}$ be a p-divisible group over R. Then there are canonical isomorphisms in Mod_0^1:*

$$(\mathcal{M}_\pi(\mathcal{G}) \otimes_{\widehat{g}} S, \mathrm{Fil}^1 \mathcal{M}_\pi(\mathcal{G}) \otimes_{\widehat{g}} S, \phi_1 \otimes \phi) \xrightarrow{\sim} (\mathcal{M}_\pi({}^g\mathcal{G}), \mathrm{Fil}^1 \mathcal{M}_\pi({}^g\mathcal{G}), \phi_1).$$

(2) *If $f : \mathcal{G} \to \mathcal{G}'$ is a morphism of p-divisible groups over R, then there is a commutative diagram in Mod_0^1:*

$$\begin{array}{ccc} \mathcal{M}_\pi(\mathcal{G}') \otimes_{\widehat{g}} S & \xrightarrow{\mathcal{M}_\pi(f)} & \mathcal{M}_\pi(\mathcal{G}) \otimes_{\widehat{g}} S \\ \downarrow & & \downarrow \\ \mathcal{M}_\pi({}^g\mathcal{G}') & \xrightarrow{\mathcal{M}_\pi({}^g f)} & \mathcal{M}_\pi({}^g\mathcal{G}) \end{array}.$$

(3) *If g_1, g_2 are two continuous automorphisms such that $g_i \pi = h_{g_i} \pi$ with $h_{g_i} \in W(k)$ for $i = 1, 2$, then on*

$$(\mathcal{M}_\pi(\mathcal{G}) \otimes_{\widehat{g}_1} S) \otimes_{\widehat{g}_2} S \cong \mathcal{M}_\pi(\mathcal{G}) \otimes_{\widehat{g_2 g_1}} S$$

one has $(\phi_1 \otimes \phi) \otimes \phi = \phi_1 \otimes \phi$.

Proof. First, prove the p^n-torsion analogue of this proposition; more precisely, prove the same statements for $\mathcal{G}(n)$ in $\underline{\mathrm{Mod}}_0^1$, and note that they are compatible under the inclusions $\mathcal{G}(n) \to \mathcal{G}(n+1)$. Then pass to the inverse limit. ☐

We then have the following analogue of [BCDT01, Cor. 5.4.6].

Corollary 3.4. *Let $\mathcal{G}$ be a p-divisible group over R. To give a morphism $\langle g\rangle : \mathcal{G} \to \mathcal{G}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\langle g\rangle} & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{\mathrm{Spec}(g)} & \mathrm{Spec}(R) \end{array}$$

is commutative and the induced morphism $[g] : \mathcal{G} \to {}^g\mathcal{G}$ is a morphism of p-divisible groups over R is equivalent to giving an additive map $\widehat{g} : \mathcal{M}_\pi(\mathcal{G}) \to \mathcal{M}_\pi(\mathcal{G})$ such that

- *for all $s \in S$ and $x \in \mathcal{M}_\pi(\mathcal{G})$, $\widehat{g}(sx) = \widehat{g}(s)\widehat{g}(x)$;*
- *$\widehat{g}(\mathrm{Fil}^1\mathcal{M}(\mathcal{G})) \subset \mathrm{Fil}^1\mathcal{M}(\mathcal{G})$ and $\phi_1 \circ \widehat{g} = \widehat{g} \circ \phi_1$.*

Proof. As in the proof of [BCDT01, Cor. 5.4.6], the map $\widehat{g}$ is the composition

$$\mathcal{M}_\pi(\mathcal{G}) \to \mathcal{M}_\pi(\mathcal{G}) \otimes_{\widehat{g}} S \to \mathcal{M}_\pi({}^g\mathcal{G}) \to \mathcal{M}_\pi(\mathcal{G})$$

where the leftmost map is $x \mapsto x \otimes 1$ and the rightmost map is $\mathcal{M}_\pi([g])$. □

We now specify hypotheses that we frquently need to assume.

Hypotheses 3.5. *Suppose $R = \mathcal{O}_F$, and suppose that F/F' is a tamely ramified Galois extension with ramification index $e(F/F')$. Fix a uniformizer $\pi \in F$ such that $\pi^{e(F/F')} \in F'$, and write $g(\pi) = h_g\pi$ for each $g \in \mathrm{Gal}(F/F')$. Let $\widehat{g} : S \to S$ be defined as before.*

Then Corollary 3.4 and parts (2) and (3) of Proposition 3.3 together imply the following.

Corollary 3.6. *Under Hypotheses 3.5, let $\mathcal{G}$ be a p-divisible group over $\mathcal{O}_F$. Giving descent data on $\mathcal{G}$ relative to F' is equivalent to giving additive bijections $\widehat{g} : \mathcal{M}_\pi(\mathcal{G}) \to \mathcal{M}_\pi(\mathcal{G})$ for all $g \in \mathrm{Gal}(F/F')$ such that*

- *$\widehat{g}(sx) = \widehat{g}(s)\widehat{g}(x)$ for $s \in S$, $x \in \mathcal{M}_\pi(\mathcal{G})$, and $g \in \mathrm{Gal}(F/F')$,*
- *$\widehat{g}(\mathrm{Fil}^1\mathcal{M}(\mathcal{G})) \subset \mathrm{Fil}^1\mathcal{M}(\mathcal{G})$ and $\phi_1 \circ \widehat{g} = \widehat{g} \circ \phi_1$ for all $g \in \mathrm{Gal}(F/F')$, and*
- *$\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \circ g_2}$ for $g_1, g_2 \in \mathrm{Gal}(F/F')$.*

This motivates the following definition.

Definition 3.7. Assume Hypotheses 3.5. If Mod is any one of the categories of Section 3.2 (${}'\mathrm{Mod}^{k-1}$, etc.), then the category $\mathrm{Mod}_{\mathrm{dd}}$ consists of objects $\mathcal{M}$ of Mod together with additive bijections $\widehat{g} : \mathcal{M} \to \mathcal{M}$ for each g in $\mathrm{Gal}(F/F')$ and satisfying the following compatibilities:

- $\widehat{g}(sx) = \widehat{g}(s)\widehat{g}(x)$ for $s \in S$, $x \in \mathcal{M}$, $g \in \mathrm{Gal}(F/F')$,
- $\widehat{g}(\mathrm{Fil}^{k-1}\mathcal{M}) \subset \mathrm{Fil}^{k-1}(\mathcal{M})$ and $\phi_{k-1} \circ \widehat{g} = \widehat{g} \circ \phi_{k-1}$ for each $g \in \mathrm{Gal}(F/F')$,
- $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \circ g_2}$ for $g_1, g_2 \in \mathrm{Gal}(F/F')$,
- $N \circ \widehat{g} = \widehat{g} \circ N$ if the category Mod is equipped with an N.

Morphisms in $\mathrm{Mod}_{\mathrm{dd}}$ are those of Mod which commute with $\widehat{g}$ for all $g \in \mathrm{Gal}(F/F')$.

So we may rephrase Corollary 3.6 as follows: under Hypotheses 3.5, the category of p-divisible groups over $\mathcal{O}_F$ with descent data relative to F' is equivalent to the category $\mathrm{Mod}^1_{0,\mathrm{dd}}$.

Definition 3.8. Under Hypotheses 3.5, we refer to $\underline{\mathrm{Mod}}^1_{0,\mathrm{dd}}$ as the category of *strongly divisible modules with tame descent data* (of weight 2).

Remark 3.9. Retain Hypotheses 3.5, and let $\mathcal{G}$ be a p-divisible group over $\mathcal{O}_F$ with descent data relative to F'. Then $\mathcal{G}(1)$ is a finite flat group scheme killed by p with descent data relative to F', and the filtered ϕ_1-module $\mathcal{M}_\pi(\mathcal{G}(1))$ obtains descent data in the sense of [BCDT01, Th. 5.6.1]. By construction, this descent data is exactly the collection of maps induced on

$$\mathcal{M}(\mathcal{G})/p\mathcal{M}(\mathcal{G}) \otimes_{S_1} k[u]/u^{ep}$$

by the descent data on $\mathcal{M}(\mathcal{G})$.

3.4. **Galois lattices and strongly divisible modules with descent data.** In the two preceding sections, we have seen how to pass between Galois lattices inside potentially crystalline Galois representations of $G_{F'}$ with Hodge-Tate weights in $\{0, 1\}$ and p-divisible groups over $\mathcal{O}_F$ with descent data relative to F', and between these and strongly divisible modules with descent data. We now describe how to pass directly to Galois lattices from strongly divisible modules with descent data.

We may extend the natural action of G_F on $\widehat{A}_{\mathrm{cris}}$ to an action of $G_{F'}$. In fact, more generally if A is a syntomic $\mathcal{O}_F$-algebra with a $\mathrm{Gal}(F/F')$-semilinear action of $G_{F'}$, then $G_{F'}$ acts on $\mathcal{O}^{\mathrm{cris}}_{n,\pi}(A)$ as follows: if $g \in G_{F'}$, then $x \in \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A)$ maps to $(g \cdot x) \otimes 1$ under the composition

$$\mathcal{O}^{\mathrm{cris}}_{n,\pi}(A) \to \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A \otimes_{g^{-1}} \mathcal{O}_F) \xrightarrow{\sim} \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A) \otimes_{g^{-1}} S_n,$$

where the first map is induced by the $\mathcal{O}_F$-algebra map $A \to A \otimes_{g^{-1}} \mathcal{O}_F$, $a \mapsto g(a) \otimes 1$. Under Hypotheses 3.5, it is not difficult to see (by checking on presheaves) that $g \cdot u = h_g u$ for the element $u \in \widehat{A}_{\mathrm{cris}}$. Therefore g preserves the filtration and commutes with the ϕ induced on $\widehat{A}_{\mathrm{cris}}$ from the ring $\widehat{A}_{\mathrm{st}}$ as defined in [Bre99, Sec. 2.2.2], and furthermore the A_{cris}-linear map $f_\pi : \widehat{A}_{\mathrm{cris}} \to B^+_{\mathrm{dR}}$ sending u to π is actually a $G_{F'}$-morphism. Thus we may regard $\mathcal{O}^{\mathrm{cris}}_{n,\pi}(\mathcal{O}_{\overline{\mathbb{Q}}_p})$ and $\widehat{A}_{\mathrm{cris}}$ as objects of $\underline{\mathrm{Mod}}^1_{0,\mathrm{dd}}$ and $'\underline{\mathrm{Mod}}^1$ respectively.

Let $\mathcal{M}$ be a strongly divisible module with tame descent data, and let $\mathcal{G} = \cup \mathcal{G}(n)$ be the p-divisible group over $\mathcal{O}_F$ with descent data relative to F' such that $\mathcal{M} \cong \mathcal{M}_\pi(\mathcal{G})$. Forgetting the descent data momentarily, by [Bre00, Th. 4.2.2.9] and the construction in [Bre00, Sec. 4.2.1] we know that

$$\mathcal{G}(n)(\mathcal{O}_{\overline{\mathbb{Q}}_p}) = \mathrm{Hom}_{'\underline{\mathrm{Mod}}^1_0}(\mathcal{M}/p^n\mathcal{M}, \mathcal{O}^{\mathrm{cris}}_{n,\pi}(\mathcal{O}_{\overline{\mathbb{Q}}_p}))$$

is an isomorphism of G_F-modules. The crucial point is that it is actually an isomorphism of $G_{F'}$-modules, where $\tilde{g} \in G_{F'}$ acts on the right-hand side via $f \mapsto \tilde{g} \cdot (f \circ \widehat{g}^{-1})$. (To simplify notation, in this section we use $\tilde{g}$ to denote an element of $G_{F'}$, and g to denote its restriction $\tilde{g}|_F$.) More generally, if $\mathcal{G}(n) = \mathrm{Spec}(R_n)$ and A is a syntomic $\mathcal{O}_F$-algebra with a $\mathrm{Gal}(F/F')$-semilinear action of $G_{F'}$, we show that the canonical bijection

$$\mathcal{G}(n)(A) = \mathrm{Hom}_{\mathcal{O}_F}(R_n, A) \xrightarrow{\sim} \mathrm{Hom}_{(\phi_1, \mathrm{Fil}^1)}(\mathcal{M}(\mathcal{G}(n)), \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A))$$

is a $G_{F'}$-module isomorphism. Here, the subscript (ϕ_1, Fil^1) denotes morphisms that commute with ϕ_1 and preserve Fil^1, with $\mathrm{Fil}^1(\mathcal{O}^{\mathrm{cris}}_{n,\pi}(A)) = \mathcal{J}^{\mathrm{cris}}_{n,\pi}(A)$. Indeed, consider the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_F}(R_n, A) & \longrightarrow & \mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{M}(\mathcal{G}(n)), \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{O}_F}(R_n \otimes_{g^{-1}} \mathcal{O}_F, A) & \longrightarrow & \mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{M}({}^{g^{-1}}\mathcal{G}(n)), \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{O}_F}(R_n \otimes_{g^{-1}} \mathcal{O}_F, A \otimes_{g^{-1}} \mathcal{O}_F) & \longrightarrow & \mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{M}({}^{g^{-1}}\mathcal{G}(n)), \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A \otimes_{g^{-1}} \mathcal{O}_F)) \\
\Big\downarrow & & \downarrow \\
 & & \mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{M}(\mathcal{G}(n)) \otimes_{\widehat{g}^{-1}} S_n, \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A) \otimes_{\widehat{g}^{-1}} S_n) \\
\Big\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{O}_F}(R_n, A) & \longrightarrow & \mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{M}(\mathcal{G}(n)), \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A))
\end{array}$$

in which

- the top square is functorial, induced by $[g^{-1}] : \mathcal{G}(n) \to {}^{g^{-1}}\mathcal{G}(n)$, and hence commutes;
- the middle square is functorial, induced by $A \to A \otimes_{g^{-1}} \mathcal{O}_F$, $a \mapsto \tilde{g}(a) \otimes 1$, and hence commutes;
- the left-hand vertical map in the bottom square is "untwisting", that is, takes a map sending $r \otimes 1 \mapsto a \otimes 1$ to a map sending $r \mapsto a$; the first right-hand vertical map in the bottom square is induced by the isomorphism $\mathcal{O}^{\mathrm{cris}}_{n,\pi} \otimes_{\widehat{g}^{-1}} S_n \xrightarrow{\sim} \mathcal{O}^{\mathrm{cris},(g)}_{n,\pi}$; and the second right-hand vertical map is again untwisting.

The actions of $\tilde{g}$ on $\mathrm{Hom}_{\mathcal{O}_F}(R_n, A)$ and $\mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{M}(\mathcal{G}(n)), \mathcal{O}^{\mathrm{cris}}_{n,\pi}(A))$ are the composites of the left-hand and right-hand vertical maps in the above diagram, respectively; hence it suffices to verify that the bottom square commutes. Indeed, if $\mathcal{A}, \mathcal{B} \in (\mathrm{Ab}/\mathcal{O}_F)$ are any two abelian sheaves, then one checks locally on sections that the composition

$$\begin{aligned}
\mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_F)}(\mathcal{A}, \mathcal{B}) \quad &\to \quad \mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{O}^{\mathrm{cris},(g^{-1})}_{n,\pi}(\mathcal{B}), \mathcal{O}^{\mathrm{cris},(g^{-1})}_{n,\pi}(\mathcal{A})) \\
&\xrightarrow{\sim} \quad \mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{O}^{\mathrm{cris}}_{n,\pi}(\mathcal{B}) \otimes_{\widehat{g}^{-1}} S_n, \mathcal{O}^{\mathrm{cris}}_{n,\pi}(\mathcal{A}) \otimes_{\widehat{g}^{-1}} S_n) \\
&\xrightarrow{\sim} \quad \mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{O}^{\mathrm{cris}}_{n,\pi}(\mathcal{B}), \mathcal{O}^{\mathrm{cris}}_{n,\pi}(\mathcal{A}))
\end{aligned}$$

is just the natural map $\mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_F)}(\mathcal{A}, \mathcal{B}) \to \mathrm{Hom}_{(\phi_1,\mathrm{Fil}^1)}(\mathcal{O}^{\mathrm{cris}}_{n,\pi}(\mathcal{B}), \mathcal{O}^{\mathrm{cris}}_{n,\pi}(\mathcal{A}))$. This yields the conclusion. Passing to the inverse limit over n, we obtain the following.

Theorem 3.10. *Assume Hypotheses 3.5. Suppose that $\mathcal{G}$ is a p-divisible group over $\mathcal{O}_F$ with descent data relative to F', and let $\mathcal{M}$ be the corresponding strongly divisible module with descent data. Then there is an isomorphism of $G_{F'}$-lattices*

$$T_p(\mathcal{G}) = \varprojlim \mathcal{G}(n)(\mathcal{O}_{\overline{\mathbb{Q}}_p}) \cong \mathrm{Hom}_{'\underline{\mathrm{Mod}}^1_0}(\mathcal{M}, \widehat{A}_{\mathrm{cris}}).$$

3.5. **From $\widehat{A}_{\mathrm{cris}}$ to $\widehat{A}_{\mathrm{st}}$.** Assume Hypotheses 3.5, and let $\mathcal{M}$ be a strongly divisible module with tame descent data. Recall that $\phi : \mathcal{M} \to \mathcal{M}$ is defined as $\phi(x) = \frac{1}{c}\phi_1(E(u)x)$.

From the equivalence of categories between Mod_0^1 and $\mathrm{Mod}_{\mathrm{cris}}^1$, our strongly divisible module $\mathcal{M}$ obtains a monodromy operator N. Because our fixed uniformizer satisfies $\pi^{e(F/F')} \in F'$, it follows that $\widehat{g}(E(u)) = E(u)$. Since $\widehat{g}$ commutes with ϕ_1, it also commutes with ϕ, and so with N as well: indeed, for the latter, note that $\widehat{g}^{-1}N\widehat{g}$ satisfies the three properties of [Bre00, Prop 5.1.3(1)], and then invoke the uniqueness of N.

Moreover, any S-linear map from $\mathcal{M}$ to $\widehat{A}_{\mathrm{cris}}$, or to another strongly divisible module, that preserves Fil^1 and commutes with ϕ_1, automatically commutes with N. (If $f : \mathcal{M} \to \mathcal{M}'$ is such a map, one sees iteratively that the S-linear map $\Delta = f \circ N - N \circ f$ has $\Delta(\mathcal{M}) \subset \phi^m(I)\mathcal{M}'$ for all m.) Thus the equivalence of categories between Mod_0^1 and $\mathrm{Mod}_{\mathrm{cris}}^1$ extends to an equivalence of $\mathrm{Mod}_{0,\mathrm{dd}}^1$ and $\mathrm{Mod}_{\mathrm{cris},\mathrm{dd}}^1$, and

$$\mathrm{Hom}_{'\underline{\mathrm{Mod}}_0^1}(\mathcal{M}, \widehat{A}_{\mathrm{cris}}) = \mathrm{Hom}_{'\underline{\mathrm{Mod}}^1}(\mathcal{M}, \widehat{A}_{\mathrm{cris}}).$$

Henceforth, when we refer to a strongly divisible module with tame descent data (of weight 2), we are typically referring to an object of $\mathrm{Mod}_{\mathrm{cris},\mathrm{dd}}^1$ (i.e., the corresponding object of $\mathrm{Mod}_{0,\mathrm{dd}}^1$ endowed with its canonical N).

It is not difficult to see that our action of $G_{F'}$ on $\widehat{A}_{\mathrm{cris}}$ extends uniquely to $\widehat{A}_{\mathrm{st}}$. We then have the following.

Proposition 3.11. *The embedding $\widehat{A}_{\mathrm{cris}} \to \widehat{A}_{\mathrm{st}}$ induces an isomorphism of $G_{F'}$-lattices*

$$\mathrm{Hom}_{'\underline{\mathrm{Mod}}^1}(\mathcal{M}, \widehat{A}_{\mathrm{cris}}) = \mathrm{Hom}_{'\underline{\mathrm{Mod}}^1}(\mathcal{M}, \widehat{A}_{\mathrm{st}}).$$

Proof. The induced map is evidently injective, so we need to prove surjectivity. If $\gamma \in \mathrm{Hom}_{'\underline{\mathrm{Mod}}^1}(\mathcal{M}, \widehat{A}_{\mathrm{st}})$ it is not difficult to see that $\gamma(\mathcal{M}) \subset \widehat{A}_{\mathrm{cris}}[\frac{1}{p}]$, for example using [Bre00, Prop. 5.1.3] and the fact that $\{x \in \widehat{A}_{\mathrm{st}} \mid Nx = 0\} = A_{\mathrm{cris}}$. Therefore given $\gamma \in \mathrm{Hom}_{'\underline{\mathrm{Mod}}^1}(\mathcal{M}, \widehat{A}_{\mathrm{cris}})$ such that $\gamma(\mathcal{M}) \subset p\widehat{A}_{\mathrm{st}}$, we need only show that $\gamma(\mathcal{M}) \subset p\widehat{A}_{\mathrm{cris}}$.

We remark first that p divides $\phi_1\left(\frac{(u-\underline{\pi})^i}{i!}\right)$ in $\widehat{A}_{\mathrm{cris}}$ if $i \geq 2$. Indeed,

$$\phi\left(\frac{(u-\underline{\pi})^i}{i!}\right) = \frac{1}{i!}\left(p!\sum_{j=1}^{p}\frac{(u-\underline{\pi})^j}{j!}\frac{\underline{\pi}^{p-j}}{(p-j)!}\right)^i$$

and $\frac{p!^i}{i!p}$ is divisible by p for $i \geq 2$ since p is odd. Now suppose $x \in \mathrm{Fil}^1\mathcal{M}$, and write

$$\gamma(x) = \sum_{i\geq 0} a_i \frac{(u-\underline{\pi})^i}{i!} \in \mathrm{Fil}^1\widehat{A}_{\mathrm{cris}},$$

with each a_i in A_{cris}. (In particular, $a_0 \in A_{\mathrm{cris}} \cap \mathrm{Fil}^1\widehat{A}_{\mathrm{cris}}$.) Then

$$\phi_1(\gamma(x)) \in \phi_1(a_0) + \phi(a_1)\left(\frac{u^p - \underline{\pi}^p}{p}\right) + p\widehat{A}_{\mathrm{cris}}.$$

Since $\phi_1(\gamma(x)) = \gamma(\phi_1(x)) \in \gamma(\mathcal{M}) \subset p\widehat{A}_{\mathrm{st}}$ it follows that p divides $\phi_1(a_0)$ in A_{cris} (use that the map $\widehat{A}_{\mathrm{st}} \to A_{\mathrm{cris}}, X \mapsto 0$ sends u to $\underline{\pi}$), and therefore

$$\phi_1(\gamma(x)) \in \phi(a_1)\frac{u^p - \underline{\pi}^p}{p} + p\widehat{A}_{\mathrm{cris}} \subset A$$

where $A \subset \widehat{A}_{\mathrm{cris}}$ is the subset of elements of the form

$$pb + \sum_{i=1}^{p} b_i \frac{\underline{\pi}^{p-i}}{(p-i)!} \frac{(u-\underline{\pi})^i}{i!}$$

with $b \in \widehat{A}_{\mathrm{cris}}$ and each $b_i \in A_{\mathrm{cris}}$. Since $\mathcal{M}$ is generated over S by $\phi_1(\mathrm{Fil}^1\mathcal{M})$, it follows that $\gamma(\mathcal{M})$ is contained in the subset of $\widehat{A}_{\mathrm{cris}}$ generated over S by A, and we deduce that every element of $\gamma(\mathcal{M})$ is of the form $\sum_{i\geq 0} b_i \frac{(u-\underline{\pi})^i}{i!}$ with $b_1 = \underline{\pi}^{p-1}c_1 + pc_2$ with $c_1, c_2 \in \widehat{A}_{\mathrm{cris}}$. In particular this applies to a_1, so that $\phi(a_1)$ is divisible by p and $\gamma(\mathcal{M}) \subset p\widehat{A}_{\mathrm{cris}}$. □

Remark 3.12. Note that on the level of rings, it is not true that $\widehat{A}_{\mathrm{cris}} \cap p\widehat{A}_{\mathrm{st}} = p\widehat{A}_{\mathrm{cris}}$.

We note for future reference that this $G_{F'}$-lattice may, by the proof of [Bre99, Prop. 2.3.2.4], be written as

$$\varprojlim_n \mathrm{Hom}_{'\underline{\mathrm{Mod}}^1}(\mathcal{M}/p^n\mathcal{M}, \widehat{A}_{\mathrm{st},\infty}).$$

If $\mathcal{M}$ is a strongly divisible module with descent data, we define $G_{F'}$-modules

$$V_{\mathrm{st},2}^{F'}(\mathcal{M}/p^n\mathcal{M}) = \mathrm{Hom}_{'\underline{\mathrm{Mod}}^1}(\mathcal{M}/p^n\mathcal{M}, \widehat{A}_{\mathrm{st},\infty})$$

and

$$T_{\mathrm{st},2}^{F'}(\mathcal{M}) = \varprojlim_n V_{\mathrm{st},2}^{F'}(\mathcal{M}/p^n\mathcal{M})\widehat{\ }(1),$$

where $\widehat{\ }$ denotes the $(\mathbb{Q}_p/\mathbb{Z}_p)$-dual and where the (1) is a twist by the cyclotomic character.

If D is a filtered $(\varphi, N, F/F', \mathbb{Z}_p)$-module, we say $\mathcal{M}$ is contained in $S[1/p] \otimes_{F_0} D$ if $\mathcal{M} \otimes_{W(k)} F_0 \cong S[1/p] \otimes_{F_0} D$, the isomorphism respecting N, ϕ, the filtration, and descent data (which acts on $S[1/p] \otimes_{F_0} D$ in the obvious manner). We recall (see, e.g., [BM02, Sec. 3.2.3]) that

$$\mathrm{Fil}^1(S \otimes_{F_0} D) = \left\{ \sum s_i(u) \otimes d_i \mid \sum s_i(\pi) d_i \in \mathrm{Fil}^1 D_F \right\}.$$

(Recall that D_F denotes $F \otimes_{F_0} D$.)

Lemma 3.13. *If $\mathcal{M}$ is a strongly divisible module with tame descent data, then there exists a filtered $(\varphi, N, F/F', \mathbb{Q}_p)$-module such that*

- *$\mathcal{M}$ is contained in $S[1/p] \otimes_{F_0} D$,*
- *$N = 0$ on D, and*
- *$\mathrm{Fil}^i D_F = D_F$ if $i \leq 0$, and $\mathrm{Fil}^i D_F = 0$ if $i \geq 2$.*

Proof. Forgetting descent data momentarily, by [Bre00, Prop. 5.1.3(2)] we obtain a filtered $(\varphi, N, \mathbb{Q}_p)$-module D satisfying the above conditions on N and $\mathrm{Fil}^i D_F$ and such that $\mathcal{M} \otimes_{W(k)} F_0 \cong S[1/p] \otimes_{F_0} D$, the isomorphism respecting N, ϕ, and the filtration. However, since this isomorphism identifies D with $\ker(N)$ on $\mathcal{M} \otimes_{W(k)} F_0$, and since each $\widehat{g}$ commutes with N, it follows that each $\widehat{g}$ acts on D. Thus D is actually a filtered $(\varphi, N, F/F', \mathbb{Z}_p)$-module. □

Finally, we have the following.

Theorem 3.14. *Retain the hypotheses of Corollary 3.6. Suppose that $\rho : G_{F'} \to \mathrm{GL}(V)$ becomes crystalline over F and has Hodge-Tate weights in $\{0,1\}$. The functor $T^{F'}_{\mathrm{st},2}$ is an equivalence between the category of strongly divisible modules with tame descent data contained in $S[1/p] \otimes_{F_0} D^F_{\mathrm{st},2}(V)$ and the category of $G_{F'}$-lattices in ρ.*

Proof. By Lemma 3.13, Theorem 3.10, Corollary 3.6, and Propositions 3.2 and 3.11, it suffices to prove that if $\mathcal{M}$ is contained in $S[1/p] \otimes_{F_0} D^F_{\mathrm{st},2}(V)$, then $T^{F'}_{\mathrm{st},2}(\mathcal{M})$ is a $G_{F'}$-lattice in ρ. This follows by the same proof as [BM02, Lem. 3.2.3.1]: one simply notes that each map in that proof is now a $G_{F'}$-map, and not just a G_F-map. □

4. Coefficients

Throughout this section, we assume Hypotheses 3.5.

We now wish to add coefficients to our theory of strongly divisible modules. Specifically, let E be a finite extension of $\mathbb{Q}_p$, let $\mathcal{O}_E$ be its ring of integers, and let R be a complete local noetherian flat $\mathcal{O}_E$-algebra with maximal ideal $\mathfrak{m}_R$ and residue field a finite extension of the residue field $\mathbf{k}_E$ of $\mathcal{O}_E$. Let $\mathbf{k}_F$ be the residue field of F. We construct a category $R-\mathrm{Mod}^{k-1}_{\mathrm{cris,dd}}$, the category of strongly divisible R-modules with tame descent data, having roughly the following properties:

- there is a functor $T_{\mathrm{st},k}$ from $R-\mathrm{Mod}^{k-1}_{\mathrm{cris,dd}}$ to R-representations of $G_{F'}$ for each R, compatible with base change $R \to R'$, and
- when $k = 2$ and $R = \mathcal{O}_E$, the functor $T_{\mathrm{st},2}$ is an equivalence of categories between $R - \mathrm{Mod}^1_{\mathrm{cris,dd}}$ and the category of $\mathcal{O}_E$-lattices inside representations of $G_{F'}$ with Hodge-Tate weights $\{0,1\}$ and becoming crystalline over F, coinciding with $T^{F'}_{\mathrm{st},2}$ when $E = \mathbb{Q}_p$.

Our exposition follows that of [BM02, Sec. 3.2] as closely as possible (verbatim in many places), but some changes are forced by the lack of any restrictions on $e = e(F)$.

Set $S_{F,R}$ to be the ring

$$\left\{ \sum_{j=0}^{\infty} r_j \frac{u^j}{\lfloor j/e \rfloor !}, \text{ where } r_j \in W(\mathbf{k}_F) \otimes_{\mathbb{Z}_p} R,\ r_j \to 0 \ \mathfrak{m}_R\text{-adically as } j \to \infty \right\}.$$

Extend the definitions of $\mathrm{Fil}, \phi, \phi_k, N, \widehat{g}$ to $S_{F,R}$ in the evident (R-linear) manner; for example, $\mathrm{Fil}^{k-1} S_{F,R}$ is the $\mathfrak{m}_R$-adic completion of the ideal generated by the $E(u)^j/j!$ for $j \geq k-1$.

We remark that if I is any ideal of R, then

$$I S_{F,R} \cap \mathrm{Fil}^{k-1} S_{F,R} = I \mathrm{Fil}^{k-1} S_{F,R}.$$

Indeed, every element of $S_{F,R}$ may be written uniquely in the form

$$\sum_{j \geq 0} r_j(u)(E(u)^j/j!)$$

with $r_j(u)$ a polynomial of degree less than $e(F)$ over $W(\mathbf{k}_F) \otimes R$. For an element of $I S_{F,R} \cap \mathrm{Fil}^{k-1} S_{F,R}$, it follows (by uniqueness) that $r_j(u) = 0$ for $j < k-1$ and the coefficients of $r_j(u)$ lie in $W(\mathbf{k}_F) \otimes I$ for $j \geq k-1$. Since R is noetherian, such an element is actually in $I \mathrm{Fil}^{k-1} S_{F,R}$.

Note that if R is the ring of integers in a local field, then we actually have $S_{F,R} = R \otimes_{\mathbb{Z}_p} S_F$. We often abbreviate $S_{F,R}$ by S_R.

Definition 4.1. A strongly divisible R-module with tame descent data is a finitely generated free S_R-module $\mathcal{M}$, together with a sub-S_R-module $\mathrm{Fil}^{k-1}\mathcal{M}$, maps $\phi, N : \mathcal{M} \to \mathcal{M}$, and additive bijections $\widehat{g} : \mathcal{M} \to \mathcal{M}$ for each $g \in \mathrm{Gal}(F/F')$, satisfying the following conditions:

(1) $\mathrm{Fil}^{k-1}\mathcal{M}$ contains $(\mathrm{Fil}^{k-1}S_R)\mathcal{M}$,
(2) $\mathrm{Fil}^{k-1}\mathcal{M} \cap I\mathcal{M} = I\mathrm{Fil}^{k-1}\mathcal{M}$ for all ideals I in R,
(3) $\phi(sx) = \phi(s)\phi(x)$ for $s \in S_R$ and $x \in \mathcal{M}$,
(4) $\phi(\mathrm{Fil}^{k-1}\mathcal{M})$ is contained in $p^{k-1}\mathcal{M}$ and generates it over S_R,
(5) $N(sx) = N(s)x + sN(x)$ for $s \in S_R$ and $x \in \mathcal{M}$,
(6) $N\phi = p\phi N$,
(7) $E(u)N(\mathrm{Fil}^{k-1}\mathcal{M}) \subset \mathrm{Fil}^{k-1}\mathcal{M}$,
(8) $N(\mathcal{M}) \subset J\mathcal{M}$ where J is the ideal $\sum_{j \geq 1} \frac{u^j}{\lfloor j/e \rfloor !} S_R$ in S_R,
(9) $\widehat{g}(sx) = \widehat{g}(s)\widehat{g}(x)$ for all $s \in S_R$, $x \in \mathcal{M}$, $g \in \mathrm{Gal}(F/F')$,
(10) $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(F/F')$,
(11) $\widehat{g}(\mathrm{Fil}^{k-1}\mathcal{M}) \subset \mathrm{Fil}^{k-1}\mathcal{M}$ for all $g \in \mathrm{Gal}(F/F')$, and
(12) ϕ and N commute with $\widehat{g}$ for all $g \in \mathrm{Gal}(F/F')$.

The category $R - \mathrm{Mod}^{k-1}_{\mathrm{cris,dd}}$ consists of strongly divisible R-modules with tame descent data, along with S_R-linear morphisms that preserve Fil^{k-1} and commute with ϕ, N, and descent data.

Example 4.2. If $R = \mathcal{O}_E = \mathbb{Z}_p$ and $k = 2$, then $R - \mathrm{Mod}^{k-1}_{\mathrm{cris,dd}}$ is the category $\mathrm{Mod}^1_{\mathrm{cris,dd}}$.

Example 4.3. If $F = F' = \mathbb{Q}_p$, then our strongly divisible R-modules are precisely those strongly divisible R-modules of [BM02, Déf. 3.2.1.1] which satisfy the extra condition $N(\mathcal{M}) \subset J\mathcal{M}$; that is, our strongly divisible R-modules are all “crystalline”, whereas those of [BM02] may be “semistable”.

Definition 4.4. Let $I \subset R$ be an ideal containing $\mathfrak{m}_R^n$ for n sufficiently large. An *object of* $\underline{\mathrm{Mod}}^{k-1}_{\mathrm{dd}}$ *with an action of* R/I is an object $\mathcal{N}$ of $\underline{\mathrm{Mod}}^{k-1}_{\mathrm{dd}}$ together with an algebra map $R/I \to \mathrm{End}_{\underline{\mathrm{Mod}}^{k-1}_{\mathrm{dd}}}(\mathcal{N})$. Such an $\mathcal{N}$ is an S_R/IS_R-module.

Example 4.5. If $\mathcal{M}$ is a strongly divisible R-module and I is an arbitrary ideal of R, let $\mathrm{Fil}^{k-1}(\mathcal{M}/I\mathcal{M})$ be the image of $\mathrm{Fil}^{k-1}\mathcal{M}/I\mathrm{Fil}^{k-1}\mathcal{M} \hookrightarrow \mathcal{M}/I\mathcal{M}$. If R/I is flat, then $\mathcal{M}/I\mathcal{M}$ together with $\mathrm{Fil}^{k-1}(\mathcal{M}/I\mathcal{M})$ and the reductions modulo I of ϕ, N, $\widehat{g}$, is a strongly divisible R/I-module. If R/I is Artinian, then $\mathcal{M}/I\mathcal{M}$ together with $\mathrm{Fil}^{k-1}(\mathcal{M}/I\mathcal{M})$ and the reductions modulo I of ϕ, N, $\widehat{g}$, is an object of $\underline{\mathrm{Mod}}^{k-1}_{\mathrm{dd}}$ with an action of R/I.

We have the following weaker version of [BM02, Lem 3.2.1.3], adapted for the fact that given a morphism $f : \mathcal{N}^r \to \mathcal{N}$ in $\underline{\mathrm{Mod}}^{k-1}$, the identity $f(\mathrm{Fil}^{k-1}\mathcal{N}^r) = \mathrm{Fil}^{k-1}\mathcal{N} \cap f(\mathcal{N}^r)$ may not hold when the ramification index e is large.

Lemma 4.6. *Suppose I is an ideal of R containing $\mathfrak{m}_R^n$ for n sufficiently large, R' is a local Artinian $\mathcal{O}_E$-algebra with residue field a finite extension of $\mathbf{k}_E$, and $R/I \to R'$ is a local $\mathcal{O}_E$-algebra morphism. Suppose that $\mathcal{N}$ is an object of $\underline{\mathrm{Mod}}^{k-1}_{\mathrm{dd}}$ with R/I-action, and that either*

(1) $\mathcal{N} = \mathcal{M}/I\mathcal{M}$ *for some strongly divisible* R*-module* $\mathcal{M}$ *with tame descent data, or*
(2) R' *is isomorphic to* $(R/(p^r, I))^n$ *as an* R/I*-module.*

Then $\mathcal{N} \otimes_{R/I} R'$ *is an object of* $\underline{\mathrm{Mod}}_{\mathrm{dd}}^{k-1}$ *with* R'*-action, and* $\mathcal{N} \to \mathcal{N} \otimes_{R/I} R'$ *is a morphism in* $\underline{\mathrm{Mod}}_{\mathrm{dd}}^{k-1}$.

Proof. The result is clear if R' is a free R/I-module, so we may assume that $R \to R'$ is surjective. In case (2), the result follows as in [BM02] from the fact that $p^r\mathcal{N} \cap \mathrm{Fil}^{k-1}\mathcal{N}$ does equal $p^r\mathrm{Fil}^{k-1}\mathcal{N}$. In case (1), suppose that $R' = R/I'$ with $I' \supset I$. Then $N \otimes_{R/I} R' = \mathcal{M}/I'\mathcal{M}$. ☐

Corollary 4.7. *Suppose that* $R \to R'$ *is a finite local map of complete local noetherian flat* $\mathcal{O}_E$*-algebras, and suppose that either*

(1) *this map is surjective, or*
(2) *every non-zero ideal* I *of* R' *has* $I^m = (p^r)$ *for some positive integers* m *and* r.

If $\mathcal{M}$ *is a strongly divisible* R*-module with descent data, then* $\mathcal{M} \otimes_R R'$*, equipped with* $\phi \otimes 1$, $N \otimes 1$, $\widehat{g} \otimes 1$, *and the image of* $\mathrm{Fil}^{k-1}\mathcal{M} \otimes R'$*, is a strongly divisible* R'*-module with descent data.*

Proof. In the first case, use the proof of Lemma 4.6 and the fact that every ideal of R' is the image of an ideal of R, and pass to the appropriate inverse limit. Write $\mathcal{M}' = \mathcal{M} \otimes_R R'$. In the second case, we use the fact that we know $p^r\mathcal{M}' \cap \mathrm{Fil}^{k-1}\mathcal{M}'$ does equal $p^r\mathrm{Fil}^{k-1}\mathcal{M}'$. If $I\mathrm{Fil}^{k-1}\mathcal{M}' \subsetneq I\mathcal{M}' \cap \mathrm{Fil}^{k-1}\mathcal{M}'$, then inductively we would also have $I^m\mathrm{Fil}^{k-1}\mathcal{M}' \subsetneq I^m\mathcal{M}' \cap \mathrm{Fil}^{k-1}\mathcal{M}'$, a contradiction. ☐

Remark 4.8. In particular, the conclusions of Corollary 4.7 hold for any local $\mathcal{O}_E$-algebra map of the form $R \to \mathcal{O}_{E'}$ with E' a finite extension of E.

If $\mathcal{N}$ is an object of $\underline{\mathrm{Mod}}_{\mathrm{dd}}^{k-1}$, we may define, as in Section 3.5, $G_{F'}$-modules

$$V_{\mathrm{st,k}}^{F'}(\mathcal{N}) = \mathrm{Hom}_{'\underline{\mathrm{Mod}}^{k-1}}(\mathcal{N}, \widehat{A}_{\mathrm{st},\infty})$$

and

$$T_{\mathrm{st},k}^{F'}(\mathcal{N}) = V_{\mathrm{st,k}}^{F'}(\mathcal{N})\hat{}(k-1).$$

When $\mathcal{N}$ has an action of R/I, so does $T_{\mathrm{st},k}^{F'}(\mathcal{N})$, as in [BM02, Sec. 3.2.2].

Lemma 4.9. *Let* $I \subset I'$ *be ideals of* R *containing* $\mathfrak{m}_R^n$ *for* n *sufficiently large. Let* $\mathcal{M}$ *be a strongly divisible* R*-module of rank* d *with tame descent data. Then we have the following.*

(1) *The map* $T_{\mathrm{st},k}^{F'}(\mathcal{M}/I\mathcal{M}) \to T_{\mathrm{st},k}^{F'}(\mathcal{M}/I'\mathcal{M})$ *is surjective.*
(2) *The* R/I*-module* $T_{\mathrm{st},k}^{F'}(\mathcal{M}/I\mathcal{M})$ *is free of rank* d.

Proof. The proof is exactly the same as that of [BM02, Lem. 3.2.2.1], replacing [Bre98, Prop. 3.2.3.1, Cor. 3.2.3.2] with [Bre99, Lems. 2.3.1.1, 2.3.1.2, 2.3.1.3]. ☐

Lemma 4.10. *Let* I *be an ideal of* R *containing* $\mathfrak{m}_R^n$ *for* n *sufficiently large, let* R' *be an Artinian local* $\mathcal{O}_E$*-algebra with residue field a finite extension of* $\mathbf{k}_E$*, and let* $R/I \to R'$ *be a local morphism of* $\mathcal{O}_E$*-algebras. If* $\mathcal{M}$ *is a strongly divisible* R*-module with tame descent data, then*

$$T_{\mathrm{st},k}^{F'}(\mathcal{M}/I\mathcal{M}) \otimes_{R/I} R' \cong T_{\mathrm{st},k}^{F'}(\mathcal{M}/I\mathcal{M} \otimes_{R/I} R').$$

Proof. The proof is exactly the same as that of [BM02, Lem. 3.2.2.2], substituting Lemma 4.9 for [BM02, Lem. 3.2.2.1]. ☐

Definition 4.11. If $\mathcal{M}$ is a strongly divisible R-module, set

$$T^{F'}_{\mathrm{st},k}(\mathcal{M}) = \varprojlim_n T^{F'}_{\mathrm{st},k}(\mathcal{M}/\mathfrak{m}^n_R\mathcal{M}).$$

This is naturally an $R[G_{F'}]$-module.

Finally, using Lemmas 4.9 and 4.10 and passing to the limit, we have the following.

Corollary 4.12. *Let $\mathcal{M}$ be a strongly divisible R-module with descent data.*

(1) $T^{F'}_{\mathrm{st},k}(\mathcal{M})$ *is a free R-module of rank d with a continuous action of $G_{F'}$, and*

$$T^{F'}_{\mathrm{st},k}(\mathcal{M})/\mathfrak{m}^n_R \xrightarrow{\sim} T^{F'}_{\mathrm{st},k}(\mathcal{M}/\mathfrak{m}^n_R), .$$

(2) *If R' is another complete local noetherian flat $\mathcal{O}_E$-algebra with residue field a finite extension of $\mathbf{k}_E$, and if $R \to R'$ is a local map such that $\mathcal{M} \otimes_R R'$ is a strongly divisible R'-module with descent data, then*

$$T^{F'}_{\mathrm{st},k}(\mathcal{M}) \otimes_R R' \xrightarrow{\sim} T^{F'}_{\mathrm{st},k}(\mathcal{M} \otimes_R R').$$

Suppose $k = 2$. It remains to verify in this case that when $R = \mathcal{O}_E$, the category of strongly divisible R-modules with descent data corresponds to the category of lattices in potentially Barsotti-Tate E-representations of $G_{F'}$. Let $\mathcal{M}$ be a strongly divisible $\mathcal{O}_E$-module. Regarding $\mathcal{M}$ as a strongly divisible $\mathbb{Z}_p$-module, from Lemma 3.13 we obtain a filtered $(\varphi, N, F/F', \mathbb{Q}_p)$-module D such that

$$\mathcal{M} \otimes_{W(k)} F_0 \cong S_{\mathbb{Z}_p}[1/p] \otimes_{F_0} D,$$

and such that $D = \{x \in \mathcal{M} \otimes_{W(k)} F_0 \mid Nx = 0\}$. Since $Nx = 0$ implies $N(\alpha x) = 0$ for any $\alpha \in \mathcal{O}_E$, it follows that the action of $\mathcal{O}_E$ on $\mathcal{M}$ preserves D; in this manner D is a filtered $(\varphi, N, F/F', E)$-module, and

$$\mathcal{M} \otimes_{W(k)} F_0 \cong S_{\mathcal{O}_E}[1/p] \otimes_{F_0 \otimes E} D.$$

Suppose that the filtered $(\varphi, N, F/F', E)$-module D is $D^F_{\mathrm{st},2}(\rho)$ for the potentially Barsotti-Tate representation $\rho : G_{F'} \to \mathrm{GL}_d(E)$ becoming Barsotti-Tate over F. By the proof of [BM02, Lem. 3.2.3.1] (and noting that each map is now a $G_{F'}$-map, and not just a G_F-map) we conclude that $T^{F'}_{\mathrm{st},2}(\mathcal{M})$ is a $G_{F'}$-stable $\mathcal{O}_E$-lattice in ρ.

We now check the following.

Proposition 4.13. *Each $G_{F'}$-stable $\mathcal{O}_E$-lattice T in ρ is isomorphic to $T^{F'}_{\mathrm{st},k}(\mathcal{M})$ for some strongly divisible $\mathcal{O}_E$-module with descent data $\mathcal{M}$.*

Proof. We know T is $\mathbb{Z}_p$-isomorphic to $T^{F'}_{\mathrm{st},k}(\mathcal{M})$ for a strongly divisible $\mathbb{Z}_p$-module with descent data $\mathcal{M}$. We know from Corollary 3.6 that T gives rise to $\mathcal{M}$ via a p-divisible group Γ with descent data; since T is an $\mathcal{O}_E$-module, the p-divisible group Γ has an action of $\mathcal{O}_E$ by Tate's full faithfulness theorem in [Tat67], and so we obtain a map

$$\mathcal{O}_E \to \mathrm{End}_{\underline{\mathrm{Mod}}^1_{0,\mathrm{dd}}}(\mathcal{M}).$$

We must check that this makes $\mathcal{M}$ into a strongly divisible $\mathcal{O}_E$-module. To do this, we first note that $\mathcal{M}/(\mathrm{Fil}^1 S_{\mathbb{Z}_p})\mathcal{M}$ is a torsion-free $\mathcal{O}_E$-module: indeed, if $m \notin (\mathrm{Fil}^1 S_{\mathbb{Z}_p})\mathcal{M}$ but $am \in (\mathrm{Fil}^1 S_{\mathbb{Z}_p})\mathcal{M}$, then $p^r m \in (\mathrm{Fil}^1 S_{\mathbb{Z}_p})\mathcal{M}$ for sufficiently large r. Hence there would exist $m' \notin (\mathrm{Fil}^1 S_{\mathbb{Z}_p})\mathcal{M}$ such that $pm' \in (\mathrm{Fil}^1 S_{\mathbb{Z}_p})\mathcal{M}$, which is not the case. The proof now proceeds just as the proof of [BM02, Prop. 3.2.3.2].

Finally, recalling that the isomorphism $T_{\mathrm{st},2}^{F'}(\mathcal{M})[1/p] \cong \rho$ is compatible with $\mathcal{O}_E$-structures, we conclude that the strongly divisible $\mathcal{O}_E$-module $\mathcal{M}$ gives rise to the $\mathcal{O}_E$-lattice T.

□

Objects killed by p. As before, let $\mathbf{k}_F$ be the residue field of F, and let e be the absolute ramification index of F. Let BrMod be the category of *Breuil modules*, that is, the category of triples $(\mathcal{M}', \mathrm{Fil}^1\mathcal{M}', \phi_1)$ such that:

- $\mathcal{M}'$ is a finite rank free $\mathbf{k}_F[u]/u^{ep}$-module,
- $\mathrm{Fil}^1\mathcal{M}'$ is a submodule of $\mathcal{M}'$ containing $u^e\mathcal{M}'$, and
- $\phi_1 : \mathrm{Fil}^1\mathcal{M}' \to \mathcal{M}'$ is an additive map such that $\phi_1(hv) = h^p\phi_1(v)$ for any $h \in \mathbf{k}_F[u]/u^{ep}$ and $v \in \mathrm{Fil}^1\mathcal{M}'$, and $\phi_1(\mathrm{Fil}^1\mathcal{M}')$ generates $\mathcal{M}'$.

If $\mathcal{M}$ is an object of $\underline{\mathrm{Mod}}^1$ which is killed by p, then $\mathcal{M} \otimes_{S_1} \mathbf{k}_F[u]/u^{ep}$ is an object of BrMod; and in fact (see [Bre00, Prop. 2.1.2.2]) this induces an equivalence of categories T_0 between the subcategory of $\underline{\mathrm{Mod}}^1$ of objects that are killed by p and BrMod, with quasi-inverse T_0' given by $\mathcal{M}' \mapsto \mathcal{M}' \otimes_{\mathbf{k}_F[u]/u^{ep}} S_1$. Moreover (see Rem. 3.9) this extends to an equivalence between the subcategory of $\underline{\mathrm{Mod}}^1_{\mathrm{dd}}$ of objects that are killed by p, and the Breuil modules with descent data $\mathrm{BrMod}_{\mathrm{dd}}$ (described in [BCDT01, Th. 5.6.1] and [Sav04, Sec. 3.3]).

Here we note that if $\mathcal{M}$ is an object of $\underline{\mathrm{Mod}}^1_{\mathrm{dd}}$ with an action of R/I, then the corresponding Breuil module $T_0(\mathcal{M})$ also has an action of R/I. We thus obtain an equivalence of categories between the subcategory of $\underline{\mathrm{Mod}}^1_{\mathrm{dd}}$ of objects that are killed by p and have an action of R/I, and Breuil modules with descent data and an action of R/I. (See the proof of [Bre98, Prop. 2.2.2.1] to verify that the isomorphisms $T_0'(T_0(\mathcal{M})) \cong \mathcal{M}$ and $T_0(T_0'(\mathcal{M}')) \cong \mathcal{M}'$ are compatible with the actions of R/I.) Hence when we study objects of $\underline{\mathrm{Mod}}^1_{\mathrm{dd}}$ with an action of R/I which are killed by p, it suffices to consider the corresponding Breuil modules. By abuse of notation, if $\mathcal{M}'$ is a Breuil module with descent data, we write $T_{\mathrm{st},2}^{F'}(\mathcal{M}')$ for $T_{\mathrm{st},2}^{F'}(T_0'(\mathcal{M}'))$.

It is worth remarking that while it is certainly not the case that every Breuil module with an action of R/I is free as a $(\mathbf{k}_F \otimes R/I)[u]/u^{ep}$-module, this is true of Breuil modules arising as the reductions of strongly divisible modules.

We make the following observation.

Lemma 4.14. *Suppose that $\mathcal{M}'$ is a Breuil module with descent data satisfying $\mathrm{Fil}^1\mathcal{M}' = u^e\mathcal{M}'$. If $\mathcal{M}''$ is another Breuil module with descent data such that $T_{\mathrm{st},2}^{F'}(\mathcal{M}') = T_{\mathrm{st},2}^{F'}(\mathcal{M}'')$, then there is a nontrivial map $\mathcal{M}'' \to \mathcal{M}'$. In the terminology of* [Sav04, Def. 8.1], *$\mathcal{M}'$ is the maximal Breuil module of $\mathcal{M}''$.*

Proof. By the compatibility between Breuil modules and Dieudonné modules (see [BCDT01, Th. 5.1.3(3)]), we see that the group scheme $\mathcal{G}'$ corresponding to $\mathcal{M}'$ under the contravariant functor $\mathcal{G}_\pi$ (of [BCDT01, Th. 5.1.3(1)]) is étale. Let $(\mathcal{G}')^+$ be the maximal prolongation of the generic fibre of $\mathcal{G}'$ (see [Ray74]); by the universal property of the connected-étale sequence (see, e.g., [Tat97, 3.7(I)]), we find that $\mathcal{G}' = (\mathcal{G}')^+$. If $\mathcal{G}'' = \mathcal{G}_\pi(\mathcal{M}'')$, we conclude that there is a map $\mathcal{G}' \to \mathcal{G}''$ which induces an isomorphism on generic fibres, and therefore also a map $\mathcal{M}'' \to \mathcal{M}'$. □

Remark 4.15. Similarly, if $\mathrm{Fil}^1\mathcal{M}' = \mathcal{M}'$ we have the minimal Breuil module.

5. Strongly divisible modules for characters

In this section, we compute the strongly divisible $\mathcal{O}_E$-modules corresponding to lattices in the characters of Examples 2.13 and 2.14, in the case $k = 2$. The purpose is to list Breuil modules with descent data and an action of $\mathbf{k}_E = \mathcal{O}_E/\mathfrak{m}_E$ to which $T_{\mathrm{st},2}^{F'}$ associates the reduction mod $\mathfrak{m}_E$ of these characters.

These particularly simple strongly divisible $\mathcal{O}_E$-modules are given by the following propositions.

Proposition 5.1. *Let $F_1 = \mathbb{Q}_p(\zeta_p)$, fix $\pi = (-p)^{1/(p-1)}$ as the choice of uniformizer in $\mathcal{O}_{F_1}$, and consider the character $\epsilon\tilde{\omega}^j\lambda_a$ of $G_{\mathbb{Q}_p}$ as in Example 2.13. Then a strongly divisible $\mathcal{O}_E$-module in $S_{\mathcal{O}_E} \otimes D_{\mathrm{st},2}^{F_1}(\epsilon\tilde{\omega}^j\lambda_a)$ is given by*

$$\mathcal{M} = S_{\mathcal{O}_E} \cdot \mathbf{e}, \quad \mathrm{Fil}^1\mathcal{M} = \mathrm{Fil}^1 S_{\mathcal{O}_E} \cdot \mathbf{e},$$
$$\phi(\mathbf{e}) = a^{-1}\mathbf{e}, \qquad N\mathbf{e} = 0,$$
$$\widehat{g}(\mathbf{e}) = \tilde{\omega}^j(g)\mathbf{e} \text{ for } g \in \mathrm{Gal}(F_1/\mathbb{Q}_p).$$

Proof. The proof is clear. For example, since $\mathrm{Fil}^1 D_{\mathrm{st},2}^{F_1}(\epsilon\tilde{\omega}^j\lambda_a) = 0$, it follows that $\mathrm{Fil}^1\mathcal{M} = \{s(u)\mathbf{e} \mid s(\pi) = 0\}$; that is, $\mathrm{Fil}^1\mathcal{M} = \mathrm{Fil}^1 S_{\mathcal{O}_E} \cdot \mathbf{e}$. □

Similarly we have the following.

Proposition 5.2. *Let $F_2 = \mathbb{Q}_{p^2}(\varpi)$, as in Example 2.14, and fix ϖ as the choice of uniformizer in $\mathcal{O}_{F_2}$.*

(1) *A strongly divisible $\mathcal{O}_E$-module in $S_{\mathcal{O}_E} \otimes D_{\mathrm{st},2}^{F_2}(\epsilon\tilde{\omega}^j\lambda_a)$ is given by*

$$\mathcal{M} = S_{\mathcal{O}_E} \cdot \mathbf{e}, \quad \mathrm{Fil}^1\mathcal{M} = \mathrm{Fil}^1 S_{\mathcal{O}_E} \cdot \mathbf{e},$$
$$\phi(\mathbf{e}) = (1 \otimes a^{-1})\mathbf{e}, \quad N\mathbf{e} = 0,$$
$$\widehat{g}(\mathbf{e}) = (1 \otimes \tilde{\omega}^j(g))\mathbf{e} \text{ for } g \in \mathrm{Gal}(F_2/\mathbb{Q}_p).$$

(2) *Suppose that E is a finite extension of $\mathbb{Q}_{p^2}$. A strongly divisible $\mathcal{O}_E$-module in $S_{\mathcal{O}_E} \otimes D_{\mathrm{st},2}^{F_2}(\tilde{\omega}_2^m(\epsilon\lambda_a)|_{G_{\mathbb{Q}_{p^2}}})$ is given by:*

$$\mathcal{M} = S_{\mathcal{O}_E} \cdot \mathbf{e}, \quad \mathrm{Fil}^1\mathcal{M} = \mathrm{Fil}^1 S_{\mathcal{O}_E} \cdot \mathbf{e},$$
$$\phi(\mathbf{e}) = (1 \otimes a^{-1})\mathbf{e}, \quad N\mathbf{e} = 0,$$
$$\widehat{g}(\mathbf{e}) = (1 \otimes \tilde{\omega}_2^m(g))\mathbf{e} \text{ for } g \in \mathrm{Gal}(F_2/\mathbb{Q}_{p^2}).$$

Next, we have the following.

Proposition 5.3. *Let $F_1 = \mathbb{Q}_p(\zeta_p)$, fix $\pi = (-p)^{1/(p-1)}$ as the choice of uniformizer in $\mathcal{O}_{F_1}$, set $e_1 = p - 1$, and let $\mathbf{k}_E$ be the residue field of E. Let $\mathcal{M}'$ be the Breuil module with descent data and action of $\mathbf{k}_E$ given by*

$$\mathcal{M}' = (\mathbf{k}_E[u]/u^{e_1 p})\mathbf{e}, \quad \mathrm{Fil}^1\mathcal{M}' = u^{e_1}\mathcal{M}'$$
$$\phi_1(u^{e_1}\mathbf{e}) = \overline{a}^{-1}\mathbf{e}, \quad \widehat{g}(\mathbf{e}) = \omega^j(g)\mathbf{e} \text{ for } g \in \mathrm{Gal}(F_1/\mathbb{Q}_p).$$

Here $\overline{a}$ is the reduction of a modulo $\mathfrak{m}_E$. Then $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}') = \lambda_{\overline{a}} \cdot \omega^{j+1}$.

Proof. Note that $\mathbf{k}_E[u]/u^{e_1p} = \mathbf{k}_E \otimes \mathbb{F}_p[u]/u^{e_1p}$. The proposition follows directly from (1) of Corollary 4.12, once one checks that $\mathcal{M}'$ is the Breuil module corresponding to the reduction modulo $\mathfrak{m}_E$ of the strongly divisible $\mathcal{O}_E$-module $\mathcal{M}$ in Proposition 5.1. This is easy: for example, $(u^{e_1}+p)\mathbf{e} \in \mathrm{Fil}^1\mathcal{M}$ implies $u^{e_1}\mathbf{e} \in \mathrm{Fil}^1\mathcal{M}'$; and the equality $\phi_1((u^{e_1}+p)\mathbf{e}) = (\frac{u^{e_1p}}{p}+1)a^{-1}\mathbf{e}$ in $\mathcal{M}$ implies $\phi_1(u^{e_1}\mathbf{e}) = \overline{a}^{-1}\mathbf{e}$ in $\mathcal{M}'$. □

We denote the above Breuil modules by $\mathcal{M}_E(F_1/\mathbb{Q}_p, e_1, \overline{a}^{-1}, j)$. Similarly we have the following.

Proposition 5.4. *Let $F_2 = \mathbb{Q}_{p^2}(\varpi)$, fix ϖ as the choice of uniformizer in $\mathcal{O}_{F_2}$, set $e_2 = p^2-1$, suppose that E contains $\mathbb{Q}_{p^2}$, and let $\mathbf{k}_E$ be the residue field of E.*

(1) *Let $\mathcal{M}'$ be the Breuil module with descent data and action of $\mathbf{k}_E$ given by:*
$$\mathcal{M}' = (\mathbb{F}_{p^2}\otimes\mathbf{k}_E)[u]/u^{e_2p}\mathbf{e}, \quad \mathrm{Fil}^1\mathcal{M}' = u^{e_2}\mathcal{M}'$$
$$\phi_1(u^{e_2}\mathbf{e}) = (1\otimes\overline{a}^{-1})\mathbf{e}, \quad \widehat{g}(\mathbf{e}) = (1\otimes\omega^j(g))\mathbf{e} \text{ for } g\in\mathrm{Gal}(F_2/\mathbb{Q}_p).$$
Then $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}') = \lambda_{\overline{a}}\cdot\omega^{j+1}$.

(2) *Let $\mathcal{M}'$ be the Breuil module with descent data and action of $\mathbf{k}_E$ given by:*
$$\mathcal{M}' = (\mathbb{F}_{p^2}\otimes\mathbf{k}_E)[u]/u^{e_2p}\mathbf{e}, \quad \mathrm{Fil}^1\mathcal{M}' = u^{e_2}\mathcal{M}'$$
$$\phi_1(u^{e_2}\mathbf{e}) = (1\otimes\overline{a}^{-1})\mathbf{e}, \quad \widehat{g}(\mathbf{e}) = (1\otimes\omega_2^m(g))\mathbf{e} \text{ for } g\in\mathrm{Gal}(F_2/\mathbb{Q}_{p^2}).$$
Then $T_{\mathrm{st},2}^{\mathbb{Q}_{p^2}}(\mathcal{M}') = (\lambda_{\overline{a}})|_{G_{\mathbb{Q}_{p^2}}}\cdot\omega_2^{m+p+1}$.

Proof. The proof is the same as that of Proposition 5.3. □

The Breuil modules in parts (1) and (2) of the above Proposition are denoted by $\mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, \overline{a}^{-1}, j)$ and $\mathcal{M}_E(F_2/\mathbb{Q}_{p^2}, e_2, \overline{a}^{-1}, m)$, respectively.

Remark 5.5. When comparing Propositions 5.3 and 5.4 with [Sav04, Th. 6.3], one should remember that $T_{\mathrm{st},2}$ is a Tate twist of the dual of $V_{\mathrm{st},2}$. For example, when $E=\mathbb{Q}_p$, the Breuil modules in (1) of Proposition 5.4 are identified in [Sav04, Thm 6.3] with the character $\lambda_{\overline{a}^{-1}}\cdot\omega^{-j}$.

Remark 5.6. By Lemma 4.14, the Breuil modules of Propositions 5.3 and 5.4 are maximal.

6. Some strongly divisible modules

In this section, we list strongly divisible modules inside the weakly admissible filtered modules D_{x_1,x_2}, D'_{x_1,x_2}, and $D_{m,[a:b]}$ of Propositions 2.17, 2.20, and 2.18, and we use them to prove the main results of our paper.

6.1. **Elements of S.** We begin by constructing certain elements of the rings $S_{F_1,\mathcal{O}_E}$ and $S_{F_2,\mathcal{O}_E}$. Recall the notation of Propositions 2.17 and 2.20, and define $w\in\mathcal{O}_E^\times$ via $x_1x_2 = pw$. Set $e_1 = e(F_1/\mathbb{Q}_p) = p-1$ and $e_2 = e(F_2/\mathbb{Q}_p) = p^2-1$.

Lemma 6.1. *Let $x\in\mathcal{O}_E$. If $j=1$, suppose further that $x^2\not\equiv w \pmod{\mathfrak{m}_E}$. Then there exists a unique element $V_x\in S_{F_1,\mathcal{O}_E}$ satisfying*
$$V_x = 1 + \frac{x^2}{w}u^{p(p-1)(j-1)}\left(\frac{u^{e_1p}}{p}+1\right)\phi(V_x). \tag{6.2}$$

Proof. Suppose that $V_x = \sum_n v_n u^n$ solves 6.2. Then for $n > 0$, v_n satisfies

$$v_n = \frac{x^2}{w}\left(v_k + \frac{v_{k-e_1}}{p}\right) \tag{6.3}$$

where

$$kp + p(p-1)(j-1) = n$$

and v_k is taken to be zero if k is not a nonnegative integer. Since $n > 0$, both k and $k - e_1$ are strictly smaller than n, and so the existence and uniqueness of V (as a formal power series) follow inductively as soon as we know that the constant term in (6.2) can be satisfied.

If $j > 1$, the condition on v_0 is simply $v_0 = 1$. For $j = 1$, the constant term in (6.2) is

$$v_0 = 1 + \frac{x^2}{w} v_0.$$

This has a solution $v_0 \in \mathcal{O}_E$ exactly as long as $x^2 \not\equiv w \pmod{\mathfrak{m}_E}$.

It remains to check that V_x is actually an element of $S_{F_1,\mathcal{O}_E}$. Indeed, it follows inductively from (6.3) that if the denominator of v_n has p-adic valuation at least N, then $n \geq e_1 p(p^N - 1)/(p-1)$. In particular

$$v_n \in \frac{1}{p^{\lfloor n/e_1 p \rfloor}} \mathcal{O}_E.$$

Since $\frac{u^n}{p^{\lfloor n/e_1 p \rfloor}} \to 0$ in S_E as $n \to \infty$, the desired conclusion follows. ∎

Similarly, we define $U_x \in S_{F_1,\mathcal{O}_E}$ satisfying

$$U_x = 1 + \frac{x^2}{w} u^{p(p-1)(p-2-j)} \left(\frac{u^{e_1 p}}{p} + 1\right) \phi(U_x),$$

which exists provided that $x^2 \not\equiv w \pmod{\mathfrak{m}_E}$ in the case $j = p-2$, and is then unique.

We define analogous elements V'_x and U'_x in $S_{F_2,\mathcal{O}_E}$ by replacing u everywhere by u^{p+1} (e.g., replacing u^{e_1} by u^{e_2}). For example, V'_x satisfies

$$V'_x = 1 + (1 \otimes x^2 w^{-1}) u^{pe_2(j-1)} \left(\frac{u^{e_2 p}}{p} + 1\right) \phi(V'_x).$$

We remark that each coefficient of u in V'_x and U'_x is a power series in x. As a result, putting variables X_1, X_2 for x in V'_x and U'_x respectively, we obtain elements $V_{X_1}, U_{X_2} \in S_{F_2,\mathcal{O}_E[[X_1,X_2]]/(X_1X_2 - wp)}$ which specialize to V'_{x_1} and U'_{x_2} under the map $\mathcal{O}_E[[X_1,X_2]]/(X_1X_2 - wp) \to \mathcal{O}_E$ sending $X_1, X_2 \mapsto x_1, x_2$ when $0 < \mathrm{val}_p(x_1), \mathrm{val}_p(x_2) < 1$. Similarly, if $\mathrm{val}_p(x) = 0$ put $x = \tilde{x}(1+y)$ with $\tilde{x}$ the Teichmüller lift of the image of x in $\mathbf{k}_E$. Putting $\tilde{x}(1+Y)$ for x in V_x and U_x respectively, we obtain elements $V_Y, U_Y \in S_{F_1,\mathcal{O}_E[[Y]]}$ which specialize to V_x, U_x under the map $\mathcal{O}_E[[Y]] \to \mathcal{O}_E$ sending $Y \mapsto y$.

Next, recall the notation of Proposition 2.18, and define $w \in \mathcal{O}_E^\times$ via $x = pw$. Write $m = i + (p+1)j$ with $i \in \{0, \ldots, p\}$ and $j \in \mathbb{Z}/(p-1)\mathbb{Z}$. It is easy to see that $D_{m,[a:b]} \cong D_{pm,[bw:-a]}$, so without loss of generality we may assume that $a = 1$ and $\mathrm{val}_p(b) \geq 0$. Then we have the following.

Lemma 6.4. *If $i < p$, there is a unique $W \in S_{F_2,\mathcal{O}_E}$ satisfying*

$$W = -(1 \otimes w) + \left(1 + \frac{u^{pe_2}}{p}\right)(1 \otimes b^2) W \phi(W) u^{pe_2(p-i)}. \tag{6.5}$$

Proof. This follows inductively in the same manner as Lemma 6.1. For the base case, note that since $i < p$ the constant term w_0 is just $-(1 \otimes w)$. ☐

When $i = p$, we must solve the identity (6.5) somewhat more carefully. The constant term solves

$$w_0 = -(1 \otimes w) + (1 \otimes b^2)w_0^2. \tag{6.6}$$

Therefore, as long as $\mathcal{O}_E$ contains a root of the quadratic $b^2z^2 - z - w$ — that is, as long as $1 + 4wb^2$ is a square in E — the recursion can get started with $w_0 = 1 \otimes z$. If $\mathrm{val}_p(b) > 0$, by Hensel's lemma this is always possible; taking the square root of $1 + 4wb^2$ which is $1 \pmod{\mathfrak{m}_E}$, the corresponding root $z = (1 - \sqrt{1 + 4wb^2})/2b^2 \in \mathcal{O}_E$ can be expressed as a power series in b. If $\mathrm{val}_p(b) = 0$ and $1 + 4wb^2 \not\equiv 0 \pmod{\mathfrak{m}_E}$, write $b = \tilde{b}(1 + \beta)$ with $\mathrm{val}_p(\beta) > 0$ and $\tilde{b}$ the Teichmüller lift of the image of b in $\mathbf{k}_E$. Then either root z of the quadratic $b^2z^2 - z - w$ may be chosen and expressed as a power series in β; in this case we must assume that $1 + 4w\tilde{b}^2$ is a square in E. Finally, if $1 + 4wb^2 \equiv 0 \pmod{\mathfrak{m}_E}$, we must assume that $1 + 4wb^2$ is a square in E; in this case our root of $b^2z^2 - z - w$ may not be expressed as a power series in terms of b, but we shall see later that this does not matter. We obtain the following.

Lemma 6.7. (1) *If $i = p$, and if $1 + 4wb^2$ is a square in E when $\mathrm{val}_p(b) = 0$, then there is $W \in S_{F_2,\mathcal{O}_E}$ satisfying*

$$W = -(1 \otimes w) + \left(1 + \frac{u^{pe_2}}{p}\right)(1 \otimes b^2)W\phi(W)u^{pe_2(p-i)}.$$

(2) *If $i = 1$, $\mathrm{val}_p(b) > 0$, and w is a square in E, then there is $X \in S^{\times}_{F_2,\mathcal{O}_E}$ satisfying*

$$X(1 \otimes wb) = 1 \otimes w - \left(1 + \frac{u^{pe_2}}{p}\right)X\phi(X).$$

Proof. (1) The paragraph before the Lemma solves for the constant term of W. The recursion for the coefficient w_n of u^n is

$$w_n = (1 \otimes b^2)w_nw_0 + \text{lower terms}.$$

Since $w_0 = 1 \otimes z$ and $b^2z \not\equiv 1 \pmod{\mathfrak{m}_E}$, the recursion can be solved to obtain $W \in S_{F_2,\mathcal{O}_E}$.

(2) The constant term of X may be taken to be $1 \otimes x_0$ where x_0 is either root of $x_0^2 + wbx_0 - w$ in $\mathcal{O}_E^{\times}$. The recursion for the coefficient x_n of u^n is $x_n(x_0 + wb) =$ lower terms, and so the recursion can be solved to obtain $X \in S^{\times}_{F_2,\mathcal{O}_E}$. ☐

Moreover, if $\mathrm{val}_p(b) > 0$, then in all cases by putting the variable B for b we obtain an element W_B of $S_{F_2,\mathcal{O}_E[[B]]}$ which specializes to W under the map $\mathcal{O}_E[[B]] \to \mathcal{O}_E$ sending $B \mapsto b$. If $\mathrm{val}_p(b) = 0$ and we are away from the situation $i = p$ and $1 + 4w\tilde{b}^2 \equiv 0 \pmod{\mathfrak{m}_E}$, assume that $1 + 4w\tilde{b}^2$ is a square in E; then by putting $\tilde{b}(1+B)$ for b we obtain an element W'_B of $S_{F_2,\mathcal{O}_E[[B]]}$ which specializes to W under the map $\mathcal{O}_E[[B]] \to \mathcal{O}_E$ sending $B \mapsto \beta$. (In fact, when $\mathrm{val}_p(b) = 0$ and $i = p$, there are two such W'_B: one for each root of $b^2z^2 - z - w = 0$.)

Similarly, if $\mathrm{val}_p(b) > 0$ then by putting the variable B for b we obtain an element X_B of $S_{F_2,\mathcal{O}_E[[B]]}$ which specializes to X under the map $\mathcal{O}_E[[B]] \to \mathcal{O}_E$ sending $B \mapsto b$. Note that the image of X in $(\mathbb{F}_{p^2} \otimes \mathbf{k}_E)[u]/u^{e_2p}$ is $1 \otimes c$ with c a square root of $\overline{w}$.

6.2. **Strongly divisible modules.** With the special elements U, V, W in hand, we now present the strongly divisible modules that are contained inside the filtered modules of Propositions 2.17 and 2.18.

First, suppose we are in the situation of Proposition 2.17 or 2.20. Without loss of generality (twisting by an appropriate character) it suffices to consider the case $i = 0$. We begin by noting the following lemma.

Lemma 6.8. *In the two cases*

- $\mathrm{val}_p(x_1) = 0$*,* $j = 1$*, and* $x_1^2 \equiv w \pmod{\mathfrak{m}_E}$*;*
- $\mathrm{val}_p(x_2) = 0$*,* $j = p-2$*, and* $x_2^2 \equiv w \pmod{\mathfrak{m}_E}$*;*

the mod p *reduction of the representation corresponding to* D_{x_1,x_2} *does not have trivial centralizer.*

Proof. In the first case Example 2.13 tells us that the representation corresponding to D_{x_1,x_2} is an extension of $\epsilon\lambda_{x_1^{-1}}$ by $\tilde{\omega}\lambda_{x_1 w^{-1}}$, and the condition that $x_1^2 \equiv w \pmod{\mathfrak{m}_E}$ forces $x_1^{-1} \equiv x_1 w^{-1} \pmod{\mathfrak{m}_E}$. Therefore the two characters $\epsilon\lambda_{x_1^{-1}}$ and $\tilde{\omega}\lambda_{x_1 w^{-1}}$ have the same reduction modulo p. The second case is similar. □

In the remainder of this section, we therefore assume that we are not in either of the two cases of Lemma 6.8. Set $\mathcal{D}_{x_1,x_2} = S_{F_1,\mathcal{O}_E} \otimes D_{x_1,x_2}$ if $\mathrm{val}_p(x_1), \mathrm{val}_p(x_2)$ are integers and $\mathcal{D}_{x_1,x_2} = S_{F_2,\mathcal{O}_E} \otimes D'_{x_1,x_2}$ if $0 < \mathrm{val}_p(x_1), \mathrm{val}_p(x_2) < 1$. Then we have the following.

Proposition 6.9. *Put* $F = F_1$ *if* $\mathrm{val}_p(x_1), \mathrm{val}_p(x_2)$ *are integers and* $F = F_2$ *if* $0 < \mathrm{val}_p(x_1), \mathrm{val}_p(x_2) < 1$*. There exists a strongly divisible* $\mathcal{O}_E$*-module with descent data*

$$\mathcal{M}_{x_1,x_2} = S_{F,\mathcal{O}_E} \cdot g_1 + S_{F,\mathcal{O}_E} \cdot g_2$$

inside $\mathcal{D}_{x_1,x_2}$*, where:*

(1) *if* $\mathrm{val}_p(x_1) = 0$ *and* $\mathrm{val}_p(x_2) = 1$*, then*

$$\begin{aligned} g_1 &= -x_1 \mathbf{e}_1 \\ g_2 &= \mathbf{e}_2 + \frac{x_1^2}{w}\frac{u^{pj-e_1}}{p}(u^{e_1}+p)V_{x_1}\mathbf{e}_1 \ ; \end{aligned}$$

(2) *if* $\mathrm{val}_p(x_1) = 1$ *and* $\mathrm{val}_p(x_2) = 0$*, then*

$$\begin{aligned} g_1 &= -x_1 \mathbf{e}_1 + x_2 \frac{u^{p(e_1-j)-e_1}}{p}(u^{e_1}+p)U_{x_2}\mathbf{e}_2 \\ g_2 &= \mathbf{e}_2 \ ; \end{aligned}$$

(3) *if* $0 < \mathrm{val}_p(x_1), \mathrm{val}_p(x_2) < 1$*, then if* $k = (p+1)j$*,*

$$\begin{aligned} g_1 &= -x_1 \mathbf{e}_1 + x_2 \frac{u^{p(e_2-k)-e_2}}{p}(u^{e_2}+p)U'_{x_2}\mathbf{e}_2 \\ g_2 &= \mathbf{e}_2 + \frac{x_1^2}{w}\frac{u^{pk-e_2}}{p}(u^{e_2}+p)V'_{x_1}\mathbf{e}_1 \ . \end{aligned}$$

Proof. Abbreviate $\mathcal{M} = \mathcal{M}_{x_1,x_2}$. In each case, the only nontrivial steps are to compute $\mathrm{Fil}^1\mathcal{M}$, to verify that it satisfies $\mathrm{Fil}^1\mathcal{M} \cap I\mathcal{M} = I\mathrm{Fil}^1\mathcal{M}$, and to check that $\phi(\mathrm{Fil}^1\mathcal{M})$ lies inside $p\mathcal{M}$ and generates it over $S_{F,\mathcal{O}_E}$ or, equivalently, that $\phi_1(\mathrm{Fil}^1\mathcal{M})$ lies inside $\mathcal{M}$ and generates it over $S_{F,\mathcal{O}_E}$. Note that in each case, g_1 and g_2 are both eigenvectors for the action of $\mathrm{Gal}(F_1/\mathbb{Q}_p)$ (resp., $\mathrm{Gal}(F_2/\mathbb{Q}_p)$).

We begin with case (1), in which $\mathrm{val}_p(x_1) = 0$. It is easy to check that

$$\mathrm{Fil}^1\mathcal{M} = S_{F_1,\mathcal{O}_E} \cdot (-u^j g_1 + x_1 g_2) + (\mathrm{Fil}^1 S_{F_1,\mathcal{O}_E})\mathcal{M},$$

that $\phi(g_1) = x_1 g_1$, and using the defining equation for V_{x_1} from Lemma 6.1, that

$$\phi(g_2) = x_2 g_2 + u^{pj-e_1}(u^{e_1} + pV_{x_1})g_1.$$

From this it follows that

$$\phi(-u^j g_1 + x_1 g_2) = p(wg_2 + x_1 u^{pj-e_1} V_{x_1} g_1),$$

and we see easily from this that $\phi(\mathrm{Fil}^1\mathcal{M}) \subset p\mathcal{M}$ and generates it. The fact that $\mathrm{Fil}^1\mathcal{M} \cap I\mathcal{M} = I\mathrm{Fil}^1\mathcal{M}$ follows without difficulty from the analogous fact for $S_{F_1,\mathcal{O}_E}$.

Similarly, in case (2), in which $\mathrm{val}_p(x_2) = 0$, we have

$$\mathrm{Fil}^1\mathcal{M} = S_{F_1,\mathcal{O}_E} \cdot (x_2 g_1 + wu^{e_1-j} g_2) + (\mathrm{Fil}^1 S_{F_1,\mathcal{O}_E})\mathcal{M}.$$

We see that $\phi(g_2) = x_2 g_2$ and, by the defining equation for U_{x_2}, that

$$\phi(g_1) = x_1 g_1 - wu^{p(e_1-j)-e_1}(u^{e_1} + pU_{x_2})g_2.$$

It follows that

$$\phi(x_2 g_1 + wu^{e_1-j} g_2) = p(wg_1 - wx_2 u^{p(e_1-j)-e_1} U_{x_2} g_2),$$

and the other properties of $\mathcal{M}$ follow as above.

Finally, we turn to case (3), where $0 < \mathrm{val}_p(x_1), \mathrm{val}_p(x_2) < p$. We note that if polynomials $s(u), t(u)$ over $W(k) \otimes \mathcal{O}_E$ are such that $(1 \otimes x_1)s + u^k t$ is divisible by $u^{e_2} + p$, then (s,t) is a linear combination of $(-u^k, 1 \otimes x_1)$ and $(1 \otimes x_2, (1 \otimes w)u^{e_2-k})$. It follows that $\mathrm{Fil}^1\mathcal{M}$ is the submodule of $\mathcal{M}$ generated by $-u^k g_1 + (1 \otimes x_1)g_2$, $(1 \otimes x_2)g_1 + (1 \otimes w)u^{e_2-k} g_2$, and $(\mathrm{Fil}^1 S_{F_2,\mathcal{O}_E})\mathcal{M}$. Moreover, if $(s,t) = \alpha(-u^k, 1 \otimes x_1) + \beta(1 \otimes x_2, (1 \otimes w)u^{e_2-k})$ and the coefficients of s, t are in I, then so are the coefficients of α, β, and so $I\mathrm{Fil}^1\mathcal{M} = I\mathcal{M} \cap \mathrm{Fil}^1\mathcal{M}$. It remains to compute $\phi(g_1)$ and $\phi(g_2)$, and to verify that $\phi(-u^k g_1 + (1 \otimes x_1)g_2)$ and $\phi((1 \otimes x_2)g_1 + (1 \otimes w)u^{e_2-k} g_2)$ lie in $p\mathcal{M}$.

Set

$$D = \left(1 + U'_{x_2} V'_{x_1} \left(\frac{u^{e_2 p}}{p} + 2u^{(p-1)e_2} + pu^{(p-2)e_2}\right)\right),$$

an invertible element of $S_{F_2,\mathcal{O}_E}$. Inverting the matrix that yields g_1 and g_2 in terms of $(1 \otimes x_1)\mathbf{e}_1$ and $\mathbf{e}_2$ gives

$$\begin{aligned}
(1 \otimes x_1)\mathbf{e}_1 &= D^{-1}\left(-g_1 + (1 \otimes x_2)\frac{u^{p(e_2-k)-e_2}}{p}(u^{e_2} + p)U'_{x_2} g_2\right) \\
\mathbf{e}_2 &= D^{-1}\left((1 \otimes x_1 w^{-1})\frac{u^{pk-e_2}}{p}(u^{e_2} + p)V'_{x_1} g_1 + g_2\right)
\end{aligned}$$

Substituting into the expressions

$$\begin{aligned}
\phi(g_1) &= (1 \otimes x_1)g_1 - (1 \otimes w)u^{p(e_2-k)-e_2}(u^{e_2} + pU'_{x_2})\mathbf{e}_2, \\
\phi(g_2) &= (1 \otimes x_2)g_2 - u^{pk-e_2}(u^{e_2} + pV'_{x_1})((1 \otimes x_1)\mathbf{e}_1)
\end{aligned}$$

(which are obtained using the defining equations for V'_{x_1}, U'_{x_2}) and simplifying yields

$$\begin{aligned}\phi(g_1) &= (1\otimes x_1)D^{-1}\left(1+\left(\frac{u^{e_2 p}}{p}+u^{(p-1)e_2}\right)V'_{x_1}(U'_{x_2}-1)\right)g_1\\ &\quad -(1\otimes w)D^{-1}u^{p(e_2-k)-e_2}(u^{e_2}+pU'_{x_2})g_2,\\ \phi(g_2) &= D^{-1}u^{pk-e_2}(u^{e_2}+pV'_{x_1})g_1\\ &\quad +(1\otimes x_2)D^{-1}\left(1+\left(\frac{u^{e_2 p}}{p}+u^{(p-1)e_2}\right)U'_{x_2}(V'_{x_1}-1)\right)g_2.\end{aligned}$$

This confirms that $\phi(g_1), \phi(g_2) \in \mathcal{M}$. We then compute $\phi(-u^k g_1 + (1\otimes x_1)g_2)$ to be pD^{-1} times

$$\begin{aligned}& u^{pk-e_2}V'_{x_1}\left((1\otimes x_1)-(1\otimes x_2)(u^{e_2}+p)\left(\tfrac{u^{e_2 p}}{p}+1\right)\tfrac{u^{pe_2(p-1-j)}}{p}\phi(U'_{x_2})\right)g_1\\ + \quad & (1\otimes w)\left(1+\left(\tfrac{u^{e_2 p}}{p}+u^{(p-1)e_2}\right)U'_{x_2}V'_{x_1}+\tfrac{u^{e_2 p}}{p}\left(1-U'_{x_2}\right)\right)g_2\end{aligned}$$

and $\phi(x_2 g_1 + wu^{e_1-j}g_2)$ to be $p(1\otimes w)D^{-1}$ times

$$\begin{aligned}& \left(1+\left(\tfrac{u^{e_2 p}}{p}+u^{(p-1)e_2}\right)U'_{x_2}V'_{x_1}+\tfrac{u^{e_2 p}}{p}(1-V'_{x_1})\right)g_1\\ + \quad & u^{p(e_2-k)-e_2}U'_{x_2}\left(-(1\otimes x_2)+(1\otimes x_1)(u^{e_2}+p)\left(\tfrac{u^{e_2 p}}{p}+1\right)\tfrac{u^{pe_2 j}}{p}\phi(V'_{x_1})\right)g_2.\end{aligned}$$

In each case, the image lies inside $p\mathcal{M}$. Moreover, one checks without difficulty (by working modulo u) that $\phi_1(-u^k g_1 + x_1 g_2)$ and $\phi_1(x_2 g_1 + wu^{e_2-k}g_2)$ generate $\mathcal{M}$ over $S_{F_2,\mathcal{O}_E}$. This completes the proof. □

We turn next to the strongly divisible modules in the situation of Proposition 2.18. (Happily, this is actually simpler than the previous situation.) Extend E if necessary (i.e., when required by Lem. 6.7) to assume that $1+4wb^2$ or w is a square in E, and write $\mathcal{D}_{m,[1:b]} = S_{F_2,\mathcal{O}_E}\otimes D_{m,[1:b]}$. (Recall that we have without loss of generality assumed $a = 1$ and $\mathrm{val}_p(b) \geq 0$.) Set $k = (p-1)i$. We then have the following.

Proposition 6.10. *There exists a strongly divisible $\mathcal{O}_E$-module with descent data*

$$\mathcal{M}_{m,[1:b]} = S_{F_2,\mathcal{O}_E}\cdot g_1 + S_{F_2,\mathcal{O}_E}\cdot g_2$$

inside $\mathcal{D}_{m,[1:b]}$, where if $i > 1$ or $\mathrm{val}_p(b) = 0$ then

$$\begin{aligned}g_1 &= \mathbf{e}_1,\\ g_2 &= \frac{\mathbf{e}_2}{p}+(1\otimes b)W\frac{u^{p(e_2-k)}}{p}\mathbf{e}_1,\end{aligned}$$

while if $i = 1$ and $\mathrm{val}_p(b) > 0$ then we define instead

$$\begin{aligned}g_1 &= \mathbf{e}_1+\frac{X}{pw}u^{p(p-1)}\mathbf{e}_2\\ g_2 &= \mathbf{e}_2.\end{aligned}$$

We remark that the first set of formulas for g_1, g_2 will still define a strongly divisible module when $i = 1$ and $\mathrm{val}_p(b) > 0$; however, it is not the strongly divisible module that we wish to consider later on.

Proof. Put $\mathcal{M} = \mathcal{M}_{m,[1:b]}$. Suppose first that $i > 1$ or $\mathrm{val}_p(b) = 0$. We begin by noting that

$$\left(-\mathbf{e}_1 + \frac{1 \otimes b}{p} u^{e_2-k}\mathbf{e}_2\right) + \left(\frac{u^{e_2(p-i)}}{p}(1 \otimes b^2)W\right)(u^{e_2}+p)\,\mathbf{e}_1$$

is equal to

$$(1 \otimes b)u^{e_2-k}g_2 + (u^{e_2(p-i)}(1 \otimes b^2)W - 1)g_1,$$

and so this element of $\mathcal{M}$ lies in $\mathrm{Fil}^1\mathcal{M}$. We remark that $u^{e_2(p-i)}(1 \otimes b^2)W - 1$ is a unit in $S_{F_2,\mathcal{O}_E}$: this is clear when $i < p$; when $i = p$ use (6.6) to see that $b^2 w_0 - 1 \not\equiv 0 \pmod{\mathfrak{m}_E}$. Noting that g_2 is not an element of $\mathrm{Fil}^1\mathcal{M}$ (when $i = p$, this again uses the fact that $b^2 w_0 - 1 \not\equiv 0 \pmod{\mathfrak{m}_E}$) we find that

$$\mathrm{Fil}^1\mathcal{M} = S_{F_2,\mathcal{O}_E} \cdot ((1 \otimes b)u^{e_2-k}g_2 + (u^{e_2(p-i)}(1 \otimes b^2)W - 1)g_1) + (\mathrm{Fil}^1 S_{F_2,\mathcal{O}_E})\mathcal{M}.$$

From this, it is easy to check that $I\mathcal{M} \cap \mathrm{Fil}^1\mathcal{M} = I\mathrm{Fil}^1\mathcal{M}$. It remains to compute $\phi(g_1)$ and $\phi(g_2)$, and to verify that $\phi((1 \otimes b)u^{e_2-k}g_2 + (u^{e_2(p-i)}(1 \otimes b^2)W - 1)g_1)$ lies in $p\mathcal{M}$. Indeed

$$\phi(g_1) = \mathbf{e}_2 = pg_2 - bWu^{p(e_2-k)}g_1$$

and

$$\phi(g_2) = \left(w - (1 \otimes b^2)W\phi(W)\frac{u^{pe_2(p+1-i)}}{p}\right)g_1 + b\phi(W)u^{p^2(e_2-k)}g_2.$$

Then, after significant cancellation and using the defining equation for W from Lemma 6.4 (when $i < p$) or Lemma 6.7 (when $i = p$), we find

$$\phi((1 \otimes b)u^{e_2-k}g_2 + (u^{e_2(p-i)}(1 \otimes b^2)W - 1)g_1) = pwW^{-1}g_2.$$

For future reference, we record that $\phi((u^{e_2}+p)g_2)$ is equal to $p\left(\frac{u^{e_2 p}}{p} + 1\right)$ times

$$\left(\left((1 \otimes w) - (1 \otimes b^2)W\phi(W)\frac{u^{pe_2(p+1-i)}}{p}\right)g_1 + (1 \otimes b)\phi(W)u^{p^2(e_2-k)}g_2\right).$$

In particular, the coefficient of g_1 in this expression is a unit in $S_{F_2,\mathcal{O}_E}$, so $\phi_1(\mathrm{Fil}^1\mathcal{M})$ does generate $\mathcal{M}$ over $S_{F_2,\mathcal{O}_E}$.

Now suppose instead that $i = 1$ and $\mathrm{val}_p(b) > 0$. Observe that $h := u^{p-1}g_1 + \left(\frac{X}{w} + (1 \otimes b)\right)g_2$ lies in $\mathrm{Fil}^1\mathcal{M}$. Since $\frac{X}{w} + (1 \otimes b)$ is a unit in $S_{F_2,\mathcal{O}_E}$ and g_1 does not lie in $\mathrm{Fil}^1\mathcal{M}$, we deduce that $\mathrm{Fil}^1\mathcal{M} = S_{F_2,\mathcal{O}_E} \cdot h + (\mathrm{Fil}^1 S_{F_2,\mathcal{O}_E})\mathcal{M}$. From this it is easy to check that $I\mathcal{M} \cap \mathrm{Fil}^1\mathcal{M} = I\mathrm{Fil}^1\mathcal{M}$. Finally, we compute that

$$\begin{aligned}\phi(g_1) &= \phi(X)u^{p^2(p-1)}g_1 + \left(1 - X\phi(X)\frac{u^{pe_2}}{pw}\right)g_2 \\ \phi(g_2) &= pwg_1 - Xu^{p(p-1)}g_2\end{aligned}$$

both lie in $\mathcal{M}$; using the defining relation for X we find $\phi_1(h) = (1 \otimes w)X^{-1}g_1 \in \mathcal{M}$ and conclude that $\mathcal{M}$ is a strongly divisible module. □

6.3. **Reduction mod $\mathfrak{m}_E$.** For each of the strongly divisible modules $\mathcal{M}$ of Section 6.2, corresponding to a lattice in a Galois representation, we compute the reduction modulo $\mathfrak{m}_E$ of that lattice; that is, we compute $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)$.

Suppose first that we are in the situation of Propositions 2.17 and 2.20, excluding the cases of Lemma 6.8. We have the following.

Theorem 6.11. *Let $\mathcal{M} = \mathcal{M}_{x_1,x_2}$ be one of the strongly divisible modules of Proposition 6.9. Then we have the following.*

(1) *If $\mathrm{val}_p(x_1) = 0$, then $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)$ depends only on the reduction $\overline{x}_1$ of x_1 $\pmod{\mathfrak{m}_E}$, and in fact,*
$$T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E) \cong \begin{pmatrix} \lambda_{\overline{x}_1^{-1}}\omega & * \\ 0 & \lambda_{\overline{x}_1\overline{w}^{-1}}\omega^j \end{pmatrix}$$
with $* \neq 0$.

(2) *If $\mathrm{val}_p(x_2) = 0$, then $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)$ depends only on the reduction $\overline{x}_2$ of x_2 $\pmod{\mathfrak{m}_E}$, and in fact,*
$$T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E) \cong \begin{pmatrix} \lambda_{\overline{x}_2^{-1}}\omega^{1+j} & * \\ 0 & \lambda_{\overline{x}_2\overline{w}^{-1}} \end{pmatrix}$$
with $* \neq 0$.

(3) *If $0 < \mathrm{val}_p(x_1), \mathrm{val}_p(x_2) < 1$, then $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)$ is independent of x_1 and x_2 and satisfies*
$$T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_E)\,|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \cong \omega_2^{1+j} \oplus \omega_2^{p(1+j)}.$$

Proof. (1) By inspection, the Breuil module $\mathcal{M}' = T_0(\mathcal{M}/\mathfrak{m}_E)$ is generated by g_1 and g_2 over $\mathbf{k}_E[u]/u^{e_1 p}$ with
$$\mathrm{Fil}^1\mathcal{M}' = \mathbf{k}_E[u]/u^{e_1 p} \cdot (-u^j g_1 + \overline{x}_1 g_2) + \mathbf{k}_E[u]/u^{e_1 p} \cdot (u^{e_1} g_1),$$
$$\phi_1(-u^j g_1 + \overline{x}_1 g_2) = \overline{w} g_2 + \overline{x}_1 u^{pj-e_1}\overline{V}_{x_1} g_1,$$
and $\phi_1(u^{e_1}g_1) = \overline{x}_1 g_1$. Also, $\widehat{g}(g_1) = g_1$ and $\widehat{g}(g_2) = \tilde{\omega}^j(g)(g_2)$.

Let $\mathcal{M}_1 = \mathcal{M}_E(F_1/\mathbb{Q}_p, e_1, \overline{x}_1, 0)$. It follows from Proposition 5.3 that $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}_1) = \lambda_{\overline{x}_1^{-1}}\omega$. Let $\mathcal{M}_2 = \mathcal{M}_E(F_1/\mathbb{Q}_p, e_1, \overline{w}\overline{x}_1^{-1}, j-1)$. By Proposition 5.4, we have
$$T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}_2) = \lambda_{\overline{x}_1\overline{w}^{-1}}\omega^j.$$

But it is clear that $\mathcal{M}'$ has a submodule that is isomorphic to $\mathcal{M}_1$. Moreover, there is a map from $\mathcal{M}' \to \mathcal{M}_2$ sending $g_1 \mapsto 0$ and $g_2 \mapsto u^p\mathbf{e}$. It follows that $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}')$ has the desired form, and to see that $* \neq 0$, it suffices by Lemma 4.14 to check that there is no nontrivial map $\mathcal{M}' \to \mathcal{M}_1$. This is a standard calculation (that uses crucially the assumption that $\overline{w} \neq \overline{x}_1^2$ when $j = 1$).

(2) This is similar to (1).

(3) Extend E so that it contains $\mathbb{Q}_{p^2}$ and so that $\mathbf{k}_E$ contains a square root of $\overline{w}$. (We see the reason for the latter assumption towards the end of the argument.) Note that $\overline{U}'_{x_2} = \overline{V}'_{x_1} = 1$ in $\mathbb{F}_{p^2} \otimes \mathbf{k}_E[u]/u^{e_2 p}$, so that $\overline{D} = 1 + 2u^{(p-1)e_2}$. Therefore the Breuil module $T_0(\mathcal{M}/\mathfrak{m}_E)$ is
$$\mathcal{M}' = (\mathbb{F}_{p^2} \otimes \mathbf{k}_E)[u]/u^{e_2 p} \cdot g_1 \oplus (\mathbb{F}_{p^2} \otimes \mathbf{k}_E)[u]/u^{e_2 p} \cdot g_2$$
with $\mathrm{Fil}^1\mathcal{M}'$ generated over $(\mathbb{F}_{p^2} \otimes \mathbf{k}_E)[u]/u^{e_2 p}$ by $-u^k g_1$ and $(1 \otimes \overline{w})u^{e_2 - k} g_2$ with
$$\phi_1(-u^k g_1) = (1 \otimes \overline{w})\overline{D}^{-1}(1 + u^{(p-1)e_2}) g_2,$$

$$\phi_1((1\otimes\overline{w})u^{e_2-k}g_2) = (1\otimes\overline{w})\overline{D}^{-1}(1+u^{(p-1)e_2})g_1,$$
$$\widehat{g}(g_1) = g_1,\quad \widehat{g}(g_2) = (\tilde{\omega}_2(g)^k\otimes 1)g_2.$$

Replacing g_1 by $\overline{D}^{-1}(1+u^{(p-1)e_2})g_1$ and g_2 by $-\overline{D}^{-1}(1+u^{(p-1)e_2})g_2$ simplifies the form of the filtration and Frobenius to:

$$\mathrm{Fil}^1\mathcal{M}' = \mathbb{F}_{p^2}\otimes\mathbf{k}_E[u]/u^{e_2p}\cdot(u^kg_1) + \mathbb{F}_{p^2}\otimes\mathbf{k}_E[u]/u^{e_2p}\cdot(u^{e_2-k}g_2)$$

with

$$\phi_1(u^kg_1) = (1\otimes\overline{w})g_2,\quad \phi_1(u^{e_2-k}g_2) = -g_1.$$

Restrict the descent data on $\mathcal{M}'$ to $\mathrm{Gal}(F_2/\mathbb{Q}_{p^2})$, which amounts to restricting the representation $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}')$ to $G_{\mathbb{Q}_{p^2}}$. Denote this new Breuil module by $\mathcal{M}'_2$. Let $\mathcal{M}'' = \mathcal{M}_E(F_2/\mathbb{Q}_{p^2}, e_2, c, n)$. One checks that there is a nontrivial map from $\mathcal{M}'_2 \to \mathcal{M}''$ given by

$$g_1 \mapsto u^{p(p-j)}\alpha\mathbf{e},$$
$$g_2 \mapsto u^{p(1+j)}\beta\mathbf{e},$$

provided that

- $\phi(\beta)(1\otimes c) = -\alpha$,
- $\phi(\alpha)(1\otimes c) = (1\otimes\overline{w})\beta$,
- $(\tilde{\omega}_2^{p(j-p)}\otimes 1)\alpha = (1\otimes\tilde{\omega}_2^n)\alpha$,
- $(\tilde{\omega}_2^{j-p}\otimes 1)\beta = (1\otimes\tilde{\omega}_2^n)\beta$.

Then it is possible to satisfy the above conditions with $c = \sqrt{-\overline{w}}$ and either $n = p(j-p)$ or $n = j-p$: in the former case, take $\alpha\in\mathbb{F}_{p^2}\otimes\mathbf{k}_E$ which is annihilated by $(\tilde{\omega}_2^{p(j-p)}\otimes 1) - (1\otimes\tilde{\omega}_2^{p(j-p)})$, and in the latter case, take α which is annihilated by $(\tilde{\omega}_2^{p(j-p)}\otimes 1) - (1\otimes\tilde{\omega}_2^{j-p})$. By Proposition 5.4, it follows that

$$T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}')\,|_{G_{\mathbb{Q}_{p^2}}} \cong \lambda_{\sqrt{-\overline{w}}^{-1}}\,|_{G_{\mathbb{Q}_{p^2}}}\otimes(\tilde{\omega}_2^{j+1}\oplus\tilde{\omega}_2^{p(1+j)}).$$

The result follows. □

In the situation of Proposition 2.18, we have the following.

Theorem 6.12. *Let $\mathcal{M} = \mathcal{M}_{m,[1:b]}$ be one of the strongly divisible modules of Proposition 6.10. Then we have the following.*

(1) *If $\mathrm{val}_p(b) = 0$ and $1 < i < p$, then $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)$ depends only on the reduction $\overline{b}$ of b (mod $\mathfrak{m}_E$), and*

$$T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E) \cong \begin{pmatrix}\lambda_{\overline{bw}^{-1}}\omega^{i+j} & * \\ 0 & \lambda_{-\overline{b}}\omega^{1+j}\end{pmatrix}$$

with $*\neq 0$ and peu ramifié if $i = 2$.

(2) *If $\mathrm{val}_p(b) = 0$ and $i = 1$, then $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)$ depends only on the reduction $\overline{b}$ of b (mod $\mathfrak{m}_E$), and*

$$T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)\otimes_{\mathbf{k}_E}\overline{\mathbb{F}}_p \cong \begin{pmatrix}\lambda_{r_+}\omega^{1+j} & * \\ 0 & \lambda_{r_-}\omega^{1+j}\end{pmatrix}$$

where $r_\pm = -\frac{1}{2}(\overline{b}\pm\sqrt{\overline{b}^2+4\overline{w}^{-1}})$ and $* = 0$ if $r_+\neq r_-$. In any case $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)$ does not have trivial endomorphisms.

(3) *If* $\mathrm{val}_p(b) = 0$ *and* $i = p$*, then* $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)$ *depends only on the reduction* $\overline{b}$ *of* $b \pmod{\mathfrak{m}_E}$*, and*

$$T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E) \cong \begin{pmatrix} \lambda_{\overline{bw_-/w}}\omega^{1+j} & * \\ 0 & \lambda_{\overline{bw_+/w}}\omega^{1+j} \end{pmatrix}$$

where w_+ *is the root of* $b^2z^2 - z - w = 0$ *such that the constant term of* W *is* $-(1 \otimes w_+)$*, and* w_- *is the other root. If* $w_+ \not\equiv w_- \pmod{\mathfrak{m}_E}$ *(i.e., if* $1 + 4b^2w \not\equiv 0 \pmod{\mathfrak{m}_E}$*)), then* $* \neq 0$*; the two choices for* W *give lattices with different reductions. If* $1 + 4b^2w \equiv 0 \pmod{\mathfrak{m}_E}$*, then* $* = 0$*.*

(4) *If* $i > 1$ *and* $\mathrm{val}_p(b) > 0$*, then* $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)$ *is independent of* b *and*

$$T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)\,|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \cong \omega_2^{m+p} \oplus \omega_2^{pm+1}.$$

(5) *If* $i = 1$ *and* $\mathrm{val}_p(b) > 0$*, then* $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E)$ *is independent of* b *and*

$$T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/\mathfrak{m}_E) \cong \begin{pmatrix} \lambda_{-c^{-1}}\omega^{1+j} & * \\ 0 & \lambda_{c^{-1}}\omega^{1+j} \end{pmatrix}$$

with $* \neq 0$*.*

Proof. The Breuil module $\mathcal{M}' = T_0(\mathcal{M}/\mathfrak{m}_E)$ in all cases satisfies

$$\mathcal{M}' = (\mathbb{F}_{p^2} \otimes \mathbf{k}_E)[u]/u^{e_2p} \cdot g_1 \oplus (\mathbb{F}_{p^2} \otimes \mathbf{k}_E)[u]/u^{e_2p} \cdot g_2$$

with $\mathrm{Fil}^1\mathcal{M}'$ generated by $(1 \otimes \overline{b})u^{e_2-k}g_2 + (u^{e_2(p-i)}(1 \otimes \overline{b}^2)\overline{W} - 1)g_1$ and $u^{e_2}g_2$, with

$$\phi_1((1 \otimes \overline{b})u^{e_2-k}g_2 + (u^{e_2(p-i)}(1 \otimes \overline{b}^2)\overline{W} - 1)g_1) = (1 \otimes \overline{w})\overline{W}^{-1}g_2$$

and

$$\phi_1(u^{e_2}g_2) = (1 \otimes \overline{w})g_1 + (1 \otimes \overline{b})\phi(\overline{W})u^{p^2(e_2-k)}g_2$$

and $\widehat{g}(g_1) = (\tilde{\omega}_2^m(g) \otimes 1)g_1$, $\widehat{g}(g_2) = (\tilde{\omega}_2^{pm}(g) \otimes 1)g_2$.

Suppose first that $\mathrm{val}_p(b) = 0$. If $i < p$, then $\overline{W} = -1 \otimes \overline{w}$ and $p^2(e_2 - k) > pe_2$. Set $X = 1 + u^{e_2(p-i)}(1 \otimes \overline{b}^2)\overline{w}$, and observe that $\phi(X) = 1$, so that ϕ_1 simplifies to:

$$\phi_1(g_1 - (1 \otimes \overline{b})X^{-1}u^{e_2-k}g_2) = g_2$$

$$\phi_1(u^{e_2}g_2) = (1 \otimes \overline{w})g_1.$$

Write $g_1' = g_1 + Cu^{kp}g_2$. Observing that

$$u^kg_1' = u^k(g_1 - (1 \otimes \overline{b})X^{-1}u^{e_2-k}g_2) + (Cu^{e_2(i-1)} + (1 \otimes \overline{b})X^{-1})u^{e_2}g_2,$$

we obtain $\phi_1(u^kg_1') = (1 \otimes \overline{w})(\phi(C)u^{e_2p(i-1)} + (1 \otimes \overline{b}))g_1'$, provided that

$$(1 \otimes \overline{w})(\phi(C)u^{e_2p(i-1)} + (1 \otimes \overline{b}))C = 1.$$

If $1 < i < p$, this is satisfied with $C = (1 \otimes \overline{bw})^{-1}$. If $i = 1$, this is satisfied with C equal to either root of $c^2 + \overline{b}c - \overline{w}^{-1} = 0$, extending E if necessary to ensure that this equation has roots in $\mathbf{k}_E$.

If $1 < i < p$, this shows that $\mathcal{M}'$ has a sub-Breuil module $\mathcal{M}''$ generated by g_1' with $\mathrm{Fil}^1\mathcal{M}'' = u^k\mathcal{M}''$ satisfying $\phi_1(u^kg_1') = \overline{bw}g_1'$ and $\widehat{g}(g_1') = (\tilde{\omega}_2^m(g) \otimes 1)g_1'$. Since there is a map

$$\mathcal{M}'' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, \overline{bw}, i + j - 1)$$

obtained by sending $g_1' \mapsto u^{p(p+1-i)}\mathbf{e}$, we see that $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}')$ has a subcharacter equal to $\lambda_{\overline{bw}^{-1}}\omega^{i+j}$. By considering the determinant, the quotient character must be $\lambda_{-\overline{b}}\omega^{1+j}$; alternately, one may check that there is a nontrivial map from $\mathcal{M}' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, -\overline{b}^{-1}, j)$ (sending $g_2 \mapsto u^{pi}\mathbf{e}$ and $g_1 \mapsto -\overline{bw}^{-1}u^{p^2 i}\mathbf{e}$). Finally, to see that $* \neq 0$, by Lemma 4.14 one checks that there is no map $\mathcal{M}' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, \overline{bw}, i+j-1)$ (assume such a map exists, and use the commutativity with ϕ_1 and $\widehat{g}$ to see that this implies $i = p$). The peu ramifié claim follows from Lemma 6.13.

On the other hand, if $i = 1$ and the roots of $c^2 + \overline{b}c - \overline{w}^{-1} = 0$ are distinct, this shows that $\mathcal{M}'$ has *two* sub-Breuil modules, hence two distinct subcharacters equal to $\lambda_{r_\pm}\omega^{1+j}$, where $r_\pm = -\frac{1}{2}(\overline{b} \pm \sqrt{\overline{b}^2 + 4\overline{w}^{-1}})$. It follows that the representation is split. If the roots of $c^2 + \overline{b}c - \overline{w}^{-1} = 0$ are equal (i.e., if $4 + \overline{w}\overline{b}^2 = 0$), then we only obtain one subcharacter, equal to $\lambda_{-\overline{b}/2}\omega^{1+j}$. But then, by considering the determinant, we see that the quotient character is the same as the subcharacter (since $(-\overline{b}/2)^2 = -\overline{w}^{-1}$) and so $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}')$ does not have trivial endomorphisms.

Now consider $\mathrm{val}_p(b) = 0$ and $i = p$. Here $\overline{W} = 1 \otimes \overline{w}_+$, where w_+ is a chosen root of $b^2w_+^2 - w_+ - w = 0$. Let w_- be the other root. Since $b^2w_+ - 1 = w/w_+$, setting $\beta = 1 \otimes bw_+/w \pmod{\mathfrak{m}_E}$ we have $\phi_1(g_1 + \beta u^{p-1}g_2) = g_2$. Set $g_1' = g_1 + \beta u^{p^2(p-1)}g_2$. Since $p^2(p-1) \geq 2e_2$, we see that $\mathrm{Fil}^1\mathcal{M}'$ is generated by $u^{e_2}g_2$ and $g_1' + \beta u^{p-1}g_2$ with

$$\phi_1(u^{e_2}g_2) = (1 \otimes \overline{w})g_1', \quad \phi_1(g_1' + \beta u^{p-1}g_2) = g_2.$$

Setting $g_1'' = g_1' - (1 \otimes \overline{bw_0}^{-1})u^{p^2(p-1)}g_2$, one computes that $\phi_1(u^{p(p-1)}g_1'') = -(1 \otimes \overline{bw_+})g_1''$. Therefore $\mathcal{M}'$ has a sub-Breuil module generated by g_1'' with $\mathrm{Fil}^1\mathcal{M}'' = u^{p(p-1)}g_1''$. There is a map $\mathcal{M}'' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, -\overline{bw}_+, j)$ sending $g_1'' \mapsto u^p\mathbf{e}$, so the subcharacter is $\lambda_{\overline{bw_-/w}}\omega^{1+j}$. Considering the determinant, the quotient character is $\lambda_{\overline{bw_+/w}}\omega^{1+j}$. Finally, one checks when there exists a map $\mathcal{M}' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, -\overline{bw}_+, j)$: one sees that such a map must be of the form $g_2 \mapsto u^{p^2}\mathbf{e}$ and $g_1 \mapsto 0$. This commutes with ϕ_1 on $g_1 + \beta u^{p-1}g_2$ if and only if $-\beta^2\overline{w} = 1$, which occurs if and only if $1 + 4b^2w \equiv 0 \pmod{\mathfrak{m}_E}$. In particular, $* \neq 0$ if $1 + 4b^2w \not\equiv 0 \pmod{\mathfrak{m}_E}$. This settles part (3).

In part (4), the hypothesis that $\mathrm{val}_p(b) > 0$ simplifies $\mathcal{M}' = T_0(\mathcal{M}/\mathfrak{m}_E)$ dramatically: namely, $\mathrm{Fil}^1\mathcal{M}'$ is generated by g_1 and $u^{e_2}g_2$ with $\phi_1(g_1) = g_2$ and $\phi_1(u^{e_2}g_2) = (1 \otimes \overline{w})g_1$. The identification of $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}')$ proceeds as in case (3) of Theorem 6.11. In particular, let $\mathcal{M}_2'$ denote $\mathcal{M}'$ with the descent data restricted to $\mathrm{Gal}(F_2/\mathbb{Q}_{p^2})$. Then a map $\mathcal{M}_2' \to \mathcal{M}_E(F_2/\mathbb{Q}_{p^2}, e_2, c, n)$ must be of the form $g_1 \mapsto \alpha u^{p^2}\mathbf{e}$ and $g_2 \mapsto \phi(\alpha)(1 \otimes c)u^p\mathbf{e}$, and such a map exists if and only if c is a square root of $\overline{w}$ and α is annihilated by $(\tilde{\omega}_2^{m-1} \otimes 1) - (1 \otimes \tilde{\omega}_2^n)$. Extending E if necessary so that $\overline{w}$ has a square root in $\mathbf{k}_E$, such a map then exists for $n = m-1$ and for $n = p(m-1)$. In the former case we get the character $(\lambda_{c^{-1}})\,|_{G_{\mathbb{Q}_{p^2}}}\omega_2^{m+p}$, and in the latter case we get the character $(\lambda_{c^{-1}})\,|_{G_{\mathbb{Q}_{p^2}}}\omega_2^{pm+1}$. The result follows.

For part (5), write $\mathcal{M}' = T_0(\mathcal{M}/\mathfrak{m}_E)$. Then $\mathrm{Fil}^1\mathcal{M}'$ is generated by $u^{p-1}g_1 + c^{-1}g_2$ and $u^{e_2}g_1$, with $\phi_1(u^{p-1}g_1 + c^{-1}g_2) = cg_1$ and $\phi_1(u^{e_2}g_1) = u^{p^2(p-1)}cg_1 + g_2$. Note that $\phi_1(u^{p(p-1)}g_2) = -cg_2$. There is evidently a nontrivial map $\mathcal{M}' \to \mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, c, j)$ sending $g_2 \mapsto 0$ and $g_1 \mapsto u^{p^2}\mathbf{e}$. On the other hand if $f : \mathcal{M}' \to$

$\mathcal{M}_E(F_2/\mathbb{Q}_p, e_2, d, n)$ is a nontrivial map sending $g_1 \mapsto \alpha\mathbf{e}$ and $g_2 \mapsto \beta\mathbf{e}$, then α, β must both be polynomials in u^p since g_1, g_2 are in the image of ϕ_1. On the other hand if $\beta \neq 0$ then the relation $f \circ \phi_1 = \phi_1 \circ f$ on $u^{p(p-1)} g_2$ implies that β is a unit times u^p; but then $f(u^{p-1} g_1 + c^{-1} g_2) \in \langle u^{e_2}\mathbf{e}\rangle$ implies that α has a linear term, a contradiction. Therefore $\beta = 0$, and then it is easy to check that $c = d$ and $j = n$. It follows that $* \neq 0$. □

Lemma 6.13. *Let $\mathbf{k}$ be a finite field of characteristic p, and suppose that $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbf{k})$ is très ramifié. If $\overline{\rho}\,|_{G_F}$ extends to a finite flat $\mathbf{k}$-vector space scheme over the ring of integers $\mathcal{O}_F$, then $p\,|e(F)$.*

Proof. This lemma follows from the proof of [Edi92, Lem. 8.2] and the discussion that follows it. Namely, let $r = [\mathbf{k} : \mathbb{F}_p]$, so that $\overline{\rho}$ corresponds to an element $\sigma = (x_1, \ldots, x_r)$ in $(\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^p)^r$; the assumption that $\overline{\rho}$ is très ramifié implies that σ does not lie in $(\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^p)^r$, that is, that some $\mathrm{val}_p(x_i) \not\equiv 0 \pmod p$. If the image σ_F of σ in $(F^\times/(F^\times)^p)^r$ then lies in $(\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^p)^r$, it is evident that $p\,|e(F)$. □

Remark 6.14. The behavior in the cases $i = 1$ and $i = p$, $\mathrm{val}_p(b) = 0$, is the same as that observed in [Sav04, Prop. 8.4]. We also note that this provides examples of a Galois representation containing both a lattice whose reduction is split and a lattice whose reduction is reducible and nonsplit.

Corollary 6.15. *Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ be a potentially crystalline representation with Hodge-Tate weights $\{0, 1\}$, and T a Galois-stable lattice inside ρ such that the reduction $T/\mathfrak{m}_E$ has trivial endomorphisms.*

(1) *If $\tau(\rho) = \tilde{\omega}^i \oplus \tilde{\omega}^j$ with $i \not\equiv j \pmod{p-1}$, then $(T/\mathfrak{m}_E)\,|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p$ has one of the three forms*
   - $\begin{pmatrix} \omega^{1+i} & * \\ 0 & \omega^j \end{pmatrix}$,
   - $\begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^i \end{pmatrix}$,
   - $\omega_2^{1+\{j-i\}+(p+1)i} \oplus \omega_2^{p-\{j-i\}+(p+1)j}$ *where $\{a\}$ denotes the unique integer in $\{0, \ldots, p-2\}$ which is congruent to $a \pmod{p-1}$.*

(2) *If $\tau(\rho) = \tilde{\omega}_2^m \oplus \tilde{\omega}_2^{pm}$ with $p+1 \nmid m$, then $(T/\mathfrak{m}_E)\,|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p$ has one of the four forms*
   - $\begin{pmatrix} \omega^{i+j} & * \\ 0 & \omega^{1+j} \end{pmatrix}$ *with $*$ peu ramifié when $i = 2$,*
   - $\begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^{i+j} \end{pmatrix}$ *with $*$ peu ramifié when $i = p-1$,*
   - $\omega_2^{p+m} \oplus \omega_2^{1+pm}$,
   - $\omega_2^{1+m} \oplus \omega_2^{p(1+m)}$.

Proof. Part (1) follows, twisting by $\tilde{\omega}^i$, from the corresponding result for type $1 \oplus \tilde{\omega}^{j-i}$. We know that $D_{\mathrm{st},2}^{F_1}(\rho)$ is described by Proposition 2.17. If $\mathrm{val}_p(x_1) = 0$ or $\mathrm{val}_p(x_2) = 0$, then ρ is actually reducible and the only possible possible reduction of ρ with trivial endomorphisms is given by part (1) or (2) of Theorem 6.11. If $0 < \mathrm{val}_p(x_1), \mathrm{val}_p(x_2) < 1$, then the reduction is given by part (3) of Theorem 6.11, and it is unique (hence irreducible).

For part (2), recall the isomorphism $D_{m,[a:b]} \cong D_{pm,[bw:-a]}$. Since $p+1 \nmid m$, we know that $D^{F_2}_{\mathrm{st},2}(\rho)$ is $D_{m,[a:b]}$ for some $[a:b]$. Suppose first that $\mathrm{val}_p(a) = \mathrm{val}_p(b)$. If $i \neq 1, p$, then applying part (1) of Theorem 6.12 to $D_{m,[a:b]}$ yields a lattice with a reduction of the first kind in the list, and applying the same result to $D_{pm,[bw:-a]}$ yields a lattice with a reduction of the second kind. These are distinct, and so are the two nontrivial reductions of ρ with trivial endomorphisms (see, e.g., Lem. 9.1.1 of Breuil's Barcelona notes [Bre] for the proof that there are at most two). If $i = 1, p$, then part (3) of Theorem 6.12 gives two distinct reductions (since we have assumed that $T/\mathfrak{m}_E$ has trivial endomorphisms).

Suppose next that $\mathrm{val}_p(b) > \mathrm{val}_p(a)$. If $i > 1$ then part (4) of Theorem 6.12 yields a reduction of the third kind on the above list, necessarily unique since it is irreducible; if $i = 1$, then part (5) of Theorem 6.12 yields two reductions (one for each choice of x_0 in (2) of Lemma 6.7) that are both of the first and second kind.

Finally, if $\mathrm{val}_p(b) < \mathrm{val}_p(a)$, then the previous paragraph applied to $D_{pm,[bw:-a]}$ gives a unique reduction of the fourth kind when $i < p$, and two reductions of the first/second kind when $i = p$. □

Remark 6.16. Observe that the reductions in Corollary 6.15 are the same as those in [CDT99, Conjs. 1.2.2, 1.2.3]. Note also that it follows from the proof of Corollary 6.15 that, up to isomorphism, we have actually listed in Propositions 6.9 and 6.10 *all* lattices (in such ρ) whose reductions have trivial endomorphisms.

Remark 6.17. We elaborate on the need for the trivial endomorphisms hypothesis in Corollary 6.15. Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ be a potentially crystalline representation with Hodge-Tate weights $\{0,1\}$ and $\tau(\rho)$ tame and nonscalar. If ρ is decomposable, then its reduction is easy to compute, so we assume that ρ is indecomposable. If ρ is not one of the reducible representations considered in Lemma 6.8, then in either Theorem 6.11 or Theorem 6.12 we have computed the reduction of at least one lattice contained in ρ. Therefore, in all cases we know the semisimplification of the reduction of ρ. However, when the semisimplification is split, we do not claim to have found all of the lattices T contained in ρ such that $T/\mathfrak{m}_E$ has nontrivial endomorphisms.

6.4. **Application to modular forms.** We now apply the results of Section 6.3 to give a new computation of the reduction mod p of the local (at p) respresentation attached to a modular form of weight 2 for $\Gamma_1(pN)$ (a result due variously to Deligne, Serre, Fontaine, Gross [Gro90], Edixhoven [Edi92],...).

Proposition 6.18. *Let N be a positive integer relatively prime to p, let χ_p be the Teichmüller character modulo p, and let χ_N be a Dirichlet character modulo N. Suppose that $f \in S_2(\Gamma_1(pN), \chi_p^j \chi_N)$ is a normalized cuspidal newform with $j \in \{1, \dots, p-2\}$. Let $\rho_{f,p}$ be the restriction to $G_{\mathbb{Q}_p}$ of the mod p Galois representation attached to f. Then we have the following.*

- *If f has slope 0, then* $\overline{\rho}_{f,p} \cong \begin{pmatrix} \lambda_{\chi_N(p)/a_p}\omega^{j+1} & * \\ 0 & \lambda_{a_p} \end{pmatrix}$.
- *If f has slope 1, then* $\overline{\rho}_{f,p} \cong \begin{pmatrix} \lambda_{a_p/p}\omega & * \\ 0 & \lambda_{\chi_N(p)(p/a_p)}\omega^j \end{pmatrix}$.
- *If f has slope in the interval $(0,1)$, then* $\overline{\rho}_{f,p}|_{I_p} \cong \omega_2^{1+j} \oplus \omega_2^{p(1+j)}$.

Proof. Let $\rho_{f,p}$ be the restriction to $G_{\mathbb{Q}_p}$ of the p-adic Galois representation attached to f, so that $\overline{\rho}_{f,p}$ is a reduction of $\rho_{f,p}$ mod p. We briefly summarize the (well-known) computation of $D^{F_1}_{\mathrm{st},2}(\rho_{f,p})$ (see, e.g., [Bre, Sec. 3.4] for more details (of a dual version)). Faltings [Fal87, Fal97] shows that $\rho_{f,p}$ is potentially crystalline, becoming crystalline over F_1 with Hodge-Tate weights $(0, 1)$. By theorems of Saito [Sai97] and Deligne, Langlands, and Carayol [Car86], we find that $\tau(\rho_{f,p}) = 1 \oplus \tilde{\omega}^j$ and, if $\rho_{f,p}$ is indecomposable, $D^{F_1}_{\mathrm{st},2}(\rho_{f,p}) = D_{pa_p^{-1}, a_p\chi_N(p)^{-1}}$. The result now follows from Theorem 6.11. (Note that we do not really need the strong input of Theorem 6.11 in the case where f has integer slope, because $\rho_{f,p}$ is reducible, but we do require it when the slope is is not an integer.) $\square$

Remark 6.19. Techniques of Coleman and Iovita [CI] may be used to prove that the representation $\rho_{f,p}$ attached to a weight 2 newform for $\Gamma_0(p^2N)$ with $(p, N) = 1$ becomes crystalline over F_2. Thus Theorem 6.12 reduces the problem of computing $\overline{\rho}_{f,p}$ to the problem of computing $D_{\mathrm{st}}(\rho_{f,p})$ for such forms.

6.5. **Families of Galois lattices.** We now describe explicitly how to arrange our Galois lattices into families. Recall the elements $V_Y, U_Y \in S_{F_1,\mathcal{O}_E[[Y]]}$, $V_{X_1}, U_{X_2} \in S_{F_2,\mathcal{O}_E[[X_1,X_2]]/(X_1X_2-wp)}$, and $W_B, W'_B \in S_{F_2,\mathcal{O}_E[[B]]}$ which we described in section 6.1.

Remark 6.20. For brevity, we omit the description of N in the strongly divisible modules below. In each case, the desired description is clear from the corresponding strongly divisible $\mathcal{O}_E$-modules we have already constructed (and well-defined using, e.g., the fact that $x_1^2 g_1, x_2 g_2 \in \mathcal{M}_{x_1,x_2}$ in the case when $0 < \mathrm{val}_p(x_1), \mathrm{val}_p(x_2) < 1$, and that p divides $N(W)$).

Proposition 6.21. *There exist strongly divisible modules with descent data as follows.*

(1) *Denoting* $x_1 = \tilde{x}_1(1+Y)$, $x_2 = pw\tilde{x}_1^{-1}(1+Y)^{-1}$ *and assuming* $\tilde{x}_1^2 \not\equiv w \pmod{\mathfrak{m}_E}$ *when* $j = 1$,

$$\mathcal{M}_{Y_1} = S_{F_1,\mathcal{O}_E[[Y]]} \cdot g_1 + S_{F_1,\mathcal{O}_E[[Y]]} \cdot g_2,$$
$$\mathrm{Fil}^1\mathcal{M}_{Y_1} = S_{F_1,\mathcal{O}_E[[Y]]} \cdot (-u^j g_1 + x_1 g_2) + (\mathrm{Fil}^1 S_{F_1,\mathcal{O}_E[[Y]]})\mathcal{M},$$
$$\phi(g_1) = x_1 g_1, \quad \phi(g_2) = x_2 g_2 + u^{pj-e_1}(u^{e_1} + pV_Y)g_1,$$
$$\widehat{g}(g_1) = g_1, \quad \widehat{g}(g_2) = \tilde{\omega}^j(g)g_2.$$

(2) *Denoting* $x_2 = \tilde{x}_2(1+Y)$, $x_1 = pw\tilde{x}_2^{-1}(1+Y)^{-1}$ *and assuming* $\tilde{x}_2^2 \not\equiv w \pmod{\mathfrak{m}_E}$ *when* $j = p-2$,

$$\mathcal{M}_{Y_2} = S_{F_1,\mathcal{O}_E[[Y]]} \cdot g_1 + S_{F_1,\mathcal{O}_E[[Y]]} \cdot g_2,$$
$$\mathrm{Fil}^1\mathcal{M}_{Y_2} = S_{F_1,\mathcal{O}_E[[Y]]} \cdot (x_2 g_1 + wu^{e_1-j} g_2) + (\mathrm{Fil}^1 S_{F_1,\mathcal{O}_E[[Y]]})\mathcal{M},$$
$$\phi(g_1) = x_1 g_1 - wu^{p(e_1-j)-e_1}(u^{e_1} + pU_Y)g_2, \quad \phi(g_2) = x_2 g_2,$$
$$\widehat{g}(g_1) = g_1, \quad \widehat{g}(g_2) = \tilde{\omega}^j(g)g_2.$$

(3) *Denoting*

$$D = \left(1 + U_{X_2}V_{X_1}\left(\frac{u^{e_2 p}}{p} + 2u^{(p-1)e_2} + pu^{(p-2)e_2}\right)\right),$$

$$\mathcal{M}_{X_1,X_2} = S_{F_2,\mathcal{O}_E[[X_1,X_2]]/(X_1X_2-pw)} \cdot g_1 + S_{F_2,\mathcal{O}_E[[X_1,X_2]]/(X_1X_2-pw)} \cdot g_2,$$

$$\begin{aligned}\mathrm{Fil}^1\mathcal{M}_{X_1,X_2} &= S_{F_2,\mathcal{O}_E[[X_1,X_2]]/(X_1X_2-pw)}\cdot(-u^kg_1+(1\otimes X_1)g_2)\\ &\quad +S_{F_2,\mathcal{O}_E[[X_1,X_2]]/(X_1X_2-pw)}\cdot((1\otimes X_2)g_1+(1\otimes w)u^{e_2-k}g_2)\\ &\quad +(\mathrm{Fil}^1S_{F_2,\mathcal{O}_E[[X_1,X_2]]/(X_1X_2-pw)})\mathcal{M},\end{aligned}$$

$$\begin{aligned}\phi(g_1) &= (1\otimes X_1)D^{-1}\left(1+\left(\frac{u^{e_2p}}{p}+u^{(p-1)e_2}\right)V_{X_1}(U_{X_2}-1)\right)g_1\\ &\quad -(1\otimes w)D^{-1}u^{p(e_2-k)-e_2}(u^{e_2}+pU_{X_2})g_2,\\ \phi(g_2) &= D^{-1}u^{pk-e_2}(u^{e_2}+pV_{X_1})g_1\\ &\quad +(1\otimes X_2)D^{-1}\left(1+\left(\frac{u^{e_2p}}{p}+u^{(p-1)e_2}\right)U_{X_2}(V_{X_1}-1)\right)g_2,\end{aligned}$$

$$\widehat{g}(g_1)=g_1,\quad \widehat{g}(g_2)=(\tilde{\omega}^j(g)\otimes 1)g_2.$$

(4) *Denoting* $b=\tilde{b}(1+B)$, *letting* $2\le i\le p$, *and if* $i=p$ *assuming that* $1+4w^2\tilde{b}\not\equiv 0 \pmod{\mathfrak{m}_E}$ *and is a square in* E,

$$\mathcal{M}'_B=(S_{F_2,\mathcal{O}_E[[B]]})\cdot g_1+(S_{F_2,\mathcal{O}_E[[B]]})\cdot g_2,$$

$$\begin{aligned}\mathrm{Fil}^1\mathcal{M}'_B = S_{F_2,\mathcal{O}_E[[B]]}\cdot((1\otimes b)u^{e_2-k}g_2+(u^{e_2(p-i)}(1\otimes b^2)W'_B-1)g_1)\\ +(\mathrm{Fil}^1S_{F_2,\mathcal{O}_E[[B]]})\mathcal{M},\end{aligned}$$

$$\begin{aligned}\phi(g_1) &= pg_2-bW'_Bu^{p(e_2-k)}g_1,\\ \phi(g_2) &= \left(w-(1\otimes b^2)W'_B\phi(W'_B)\frac{u^{pe_2(p+1-i)}}{p}\right)g_1+b\phi(W'_B)u^{p^2(e_2-k)}g_2,\end{aligned}$$

$$\widehat{g}(g_1)=(\tilde{\omega}_2^m(g)\otimes 1)g_1,\quad \widehat{g}(g_2)=(\tilde{\omega}_2^{pm}(g)\otimes 1)g_2.$$

(5) *If* $i>1$,

$$\mathcal{M}_B=(S_{F_2,\mathcal{O}_E[[B]]})\cdot g_1+(S_{F_2,\mathcal{O}_E[[B]]})\cdot g_2,$$

$$\begin{aligned}\mathrm{Fil}^1\mathcal{M}_B = S_{F_2,\mathcal{O}_E[[B]]}\cdot((1\otimes B)u^{e_2-k}g_2+(u^{e_2(p-i)}(1\otimes B^2)W_B-1)g_1)\\ +(\mathrm{Fil}^1S_{F_2,\mathcal{O}_E[[B]]})\mathcal{M},\end{aligned}$$

$$\begin{aligned}\phi(g_1) &= pg_2-BW_Bu^{p(e_2-k)}g_1,\\ \phi(g_2) &= \left(w-(1\otimes B^2)W_B\phi(W_B)\frac{u^{pe_2(p+1-i)}}{p}\right)g_1+B\phi(W_B)u^{p^2(e_2-k)}g_2,\end{aligned}$$

$$\widehat{g}(g_1)=(\tilde{\omega}_2^m(g)\otimes 1)g_1,\quad \widehat{g}(g_2)=(\tilde{\omega}_2^{pm}(g)\otimes 1)g_2.$$

(6) *If* $i=1$ *and assuming that* w *is a square in* E,

$$\mathcal{M}_X=(S_{F_2,\mathcal{O}_E[[B]]})\cdot g_1\oplus(S_{F_2,\mathcal{O}_E[[B]]})\cdot g_2,$$

$$\mathrm{Fil}^1\mathcal{M}_X=S_{F_2,\mathcal{O}_E[[B]]}\cdot(u^{p-1}g_1+(w^{-1}X_B+(1\otimes B))g_2)+(\mathrm{Fil}^1S_{F_2,\mathcal{O}_E[[B]]})\mathcal{M}_X,$$

$$\begin{aligned}\phi(g_1) &= \phi(X_B)u^{p^2(p-1)}g_1+\left(1-X_B\phi(X_B)\frac{u^{pe_2}}{pw}\right)g_2,\\ \phi(g_2) &= pwg_1-X_Bu^{p(p-1)}g_2,\end{aligned}$$

$$\widehat{g}(g_1)=(\tilde{\omega}_2^m\otimes 1)g_1,\qquad \widehat{g}(g_2)=(\tilde{\omega}_2^{pm}\otimes 1)g_2.$$

Proof. In each case, the proof that these formulas define a strongly divisible module is identical to the proof that the corresponding strongly divisible $\mathcal{O}_E$-modules with descent data of Proposition 6.9 or 6.10 are indeed strongly divisible $\mathcal{O}_E$-modules. □

We adopt the following notation. For each strongly divisible R-module $\mathcal{M}$ in Proposition 6.21, set $R(\mathcal{M}) = R$ (so that, for example, $R(\mathcal{M}_{Y_1}) = \mathcal{O}_E[[Y_1]]$). Set $\tau(\mathcal{M}) = 1 \oplus \tilde{\omega}^j$ in the first three cases, and $\tau(\mathcal{M}) = \tilde{\omega}_2^m \oplus \tilde{\omega}_2^{pm}$ in the final two cases. Finally, set $\overline{\rho}(\mathcal{M}) = T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M}/\mathfrak{m}_{R(\mathcal{M})})$.

6.6. **Deformation rings.** We now come to our main results.

Theorem 6.22. *Conjecture 1.2.2 of* [CDT99] *holds; that is, suppose that* $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbf{k}_E)$ *has trivial endomorphisms. Suppose that* $\tau \cong \tilde{\omega}^i \oplus \tilde{\omega}^j$ *with* $i \not\equiv j \pmod{p-1}$. *Then we have the following:*

(1) $R(2, \tau, \overline{\rho})_{\mathcal{O}_E} = 0$ *if* $\overline{\rho}\,|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \notin \left\{ \begin{pmatrix} \omega^{1+i} & * \\ 0 & \omega^j \end{pmatrix}, \begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^i \end{pmatrix}, \omega_2^k \oplus \omega_2^{pk} \right\}$ *with* $k = 1 + \{j - i\} + (p+1)i$;

(2) $R(2, \tau, \overline{\rho})_{\mathcal{O}_E} = \mathcal{O}_E[[Y]]$ *if* $\overline{\rho}\,|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \in \left\{ \begin{pmatrix} \omega^{1+i} & * \\ 0 & \omega^j \end{pmatrix}, \begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^i \end{pmatrix} \right\}$;

(3) $R(2, \tau, \overline{\rho})_{\mathcal{O}_E} = \mathcal{O}_E[[X_1, X_2]]/(X_1X_2 - pw)$ *if* $\overline{\rho}\,|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \cong \omega_2^k \oplus \omega_2^{pk}$ *with* $k = 1 + \{j - i\} + (p+1)i$, *assuming that* E *contains* $\mathbb{Q}_{p^2}$ *and that* $\mathbf{k}_E$ *contains a square root of* $\det(\overline{\rho}(\mathrm{Frob}_p))$.

Theorem 6.23. *Conjecture 1.2.3 of* [CDT99] *holds; that is, suppose that* $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbf{k}_E)$ *has trivial endomorphisms. Suppose that* $\tau \cong \tilde{\omega}_2^m \oplus \tilde{\omega}_2^{pm}$ *with* $p + 1 \nmid m$.

(1) $R(2, \tau, \overline{\rho})_{\mathcal{O}_E} = \mathcal{O}_E[[B]]$ *if* $\overline{\rho}\,|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \in \left\{ \begin{pmatrix} \omega^{i+j} & * \\ 0 & \omega^{1+j} \end{pmatrix}, \begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^{i+j} \end{pmatrix} \right\}$, *the first* $*$ *peu ramifié when* $i = 2$ *and the second when* $i = p - 1$;

(2) $R(2, \tau, \overline{\rho})_{\mathcal{O}_E} = \mathcal{O}_E[[B]]$ *if* $\overline{\rho}|_{I_p} \otimes_{\mathbf{k}_E} \overline{\mathbb{F}}_p \in \left\{ \omega_2^{p+m} \oplus \omega_2^{1+pm}, \omega_2^{1+m} \oplus \omega_2^{p(1+m)} \right\}$;

(3) $R(2, \tau, \overline{\rho})_{\mathcal{O}_E} = 0$ *otherwise.*

Theorem 6.24. *Conjecture 2.2.2.4 of* [BM02] *(and so, in particular,* [CDT99, Conj. 1.2.1]*) holds for* $k = 2$ *and* τ *tame.*

Proof. We remark that it suffices to prove Theorem 6.23 and part (2) of Theorem 6.22 after extending E in a manner dependent only on $\overline{\rho}$: indeed, once this result (and the corresponding case of [BM02, Conj. 2.2.2.4]) has been established, [BM02, Lems. 5.1.8, 2.2.2.5] yield the result for our original E.

Part (1) of Theorem 6.22 and part (3) of Theorem 6.23 follow immediately from Corollary 6.15. In the cases concerning type $\tilde{\omega}^i \oplus \tilde{\omega}^j$, we may suppose without loss of generality that $i = 0$. We claim that for each strongly divisible module $\mathcal{M}$ of Proposition 6.21, the $R(\mathcal{M})$-representation $T_{\mathrm{st},2}^{\mathbb{Q}_p}(\mathcal{M})$ is actually the universal deformation of $\overline{\rho}$ to $R(2, \tau(\mathcal{M}), \overline{\rho}(\mathcal{M}))_{\mathcal{O}_E}$. As in the proof of [BM02, Th. 5.3.1], after all of the work that we have done (the fact that we have found every lattice in a deformation of $\overline{\rho}$ of type $(2, \tau(\mathcal{M}))$; cf. Prop. 2.21 and Rems. 4.8, 6.16), it is essentially formal that there is a canonical injection

$$R(2, \tau(\mathcal{M}), \overline{\rho}(\mathcal{M}))_{\mathcal{O}_E} \to R(\mathcal{M}).$$

Abbreviate $R = R(\mathcal{M})$. It remains to show that this map is a surjection; once this is done, the rest of Theorems 6.22, 6.23, and 6.24 follows as in [BM02, Sec. 5.3].

For this surjectivity, it suffices to see that $T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}/(\mathfrak{m}_R^2,\mathfrak{m}_E))$ cannot be defined over a $\mathbf{k}_E$-subalgebra of $R/(\mathfrak{m}_R^2,\mathfrak{m}_E)$. The method used in [BM02] is unavailable, as $T_{\mathrm{st},2}$ is not fully faithful, so we must resort to another (somewhat more unpleasant) method. We outline the proof, after which we give the proof in detail in the most daunting case (part (3) of Th. 6.22).

In most of our cases, $R/(\mathfrak{m}_R^2,\mathfrak{m}_E)=\mathbf{k}_E[X]/(X^2)$ for a variable X. Consider the Breuil module $\mathcal{M}'_X=T_0(\mathcal{M}/(\mathfrak{m}_R^2,\mathfrak{m}_E))$. If the representation $\overline{\rho}_X=T^{\mathbb{Q}_p}_{\mathrm{st},2}(\mathcal{M}'_X)$ is defined over a $\mathbf{k}_E$-subalgebra, that subalgebra can only be $\mathbf{k}_E$, and in particular $\overline{\rho}_X$ (regarded simply as a representation over $\mathbf{k}_E$) has a subrepresentation $\overline{\rho}'$ such that the composition $\overline{\rho}'\to\overline{\rho}_X\to\overline{\rho}(\mathcal{M})$ is an isomorphism, where the rightmost map is reduction modulo X. By a scheme-theoretic closure argument, $\mathcal{M}'_X$ has a sub-Breuil module $\mathcal{M}'$ (with action of $\mathbf{k}_E$) so that $\mathcal{M}'\to\mathcal{M}'_X\to\mathcal{M}'_X/X\mathcal{M}'_X$ corresponds to a map on group schemes which is an isomorphism on generic fibres. (Recall that since $\mathcal{M}$ is a strongly divisible module, reduction modulo X actually corresponds to the map $\mathcal{M}'_X\to\mathcal{M}'_X/X\mathcal{M}'_X$ on Breuil modules.) In practice, it is too complicated to show directly that such $\mathcal{M}'$ does not exist. Fortunately, we know that in every case (possibly restricting $\overline{\rho}$ to $G_{\mathbb{Q}_{p^2}}$ or extending E if necessary), $\overline{\rho}(\mathcal{M})$ has a subcharacter χ. From Remark 4.15 and the results of section 5, we can compute the minimal Breuil module $\mathcal{M}''$ corresponding to χ. What one proves is that the image of every map $\mathcal{M}''\to\mathcal{M}'_X$ falls inside $X\mathcal{M}'_X$, and so the map $\mathcal{M}''\to\mathcal{M}'_X/X\mathcal{M}'_X$ is zero and the sought-for $\mathcal{M}'$ cannot exist.

We demonstrate how this argument can be applied to part (3) of Theorem 6.22. In this case $R/(\mathfrak{m}_R^2,\mathfrak{m}_E)=\overline{R}=\mathbf{k}_E[X_1,X_2]/(X_1^2,X_1X_2,X_2^2)$, so let L be a linear form in X_1 and X_2 and suppose that $\overline{\rho}_X$ is defined over the subalgebra $\mathbf{k}_E[L]$. Let the corresponding subrepresentation of $\overline{\rho}_X$ be $\overline{\rho}_L$. Then the representation $\overline{\rho}_X/(L)$ is actually defined over $\mathbf{k}_E$, and we may apply the argument of the previous paragraph.

We now do this explicitly. Suppose E is sufficiently large that $-\overline{w}$ is a square in $\mathbf{k}_E$. Since $X_1^2=X_2^2=0$, we see that $V_{X_1}=U_{X_2}=1$ in $\mathbb{F}_{p^2}\otimes\overline{R}[u]/u^{e_2p}$. We compute $\mathcal{M}'_{X_1,X_2}=T_0(\mathcal{M}/(\mathfrak{m}_R^2,\mathfrak{m}_E))$ explicitly from Proposition 6.21 and the calculations in the proof of Proposition 6.9 and obtain, after a simplifying change of basis, that $\mathcal{M}'_{X_1,X_2}$ may be generated by g_1,g_2 in such a way that $\mathrm{Fil}^1\mathcal{M}'_{X_1,X_2}$ is generated by $h_1=-u^kg_1+(X_1-X_2u^{e_2(p-1-j)})g_2$ and $h_2=(1\otimes\overline{w})u^{e_2-k}g_2+(X_2-X_1u^{e_2j})g_1$ satisfying

$$\phi_1(-u^kg_1+(X_1-X_2u^{e_2(p-1-j)})g_2)=(1\otimes\overline{w})g_2,$$
$$\phi_1((1\otimes\overline{w})u^{e_2-k}g_2+(X_2-X_1u^{e_2j})g_1)=(1\otimes\overline{w})g_1.$$

The minimal Breuil module $\mathcal{M}''$ of the desired subrepresentation χ of $\overline{\rho}(\mathcal{M})$ restricted to $G_{\mathbb{Q}_{p^2}}$ is such that $\mathrm{Fil}^1\mathcal{M}''=\mathcal{M}''$, and for some generator $\mathbf{e}$, we have $\phi_1(\mathbf{e})=(1\otimes c)\mathbf{e}$ with $c^2=-\overline{w}$. Suppose that we have a nonzero map $f:\mathcal{M}''\to\mathcal{M}'_{X_1,X_2}/(L)$, let $\overline{X}_1,\overline{X}_2$ denote the images of X_1 and X_2 in $\overline{R}/(L)$, and fix L' a non-zero nilpotent in $\mathbf{k}_E[X_1,X_2](L,X_1^2,X_1X_2,X_2^2)$. Our map f must send

$$\mathbf{e}\mapsto\alpha h_1+\beta h_2.$$

Write $\alpha=\alpha_0u^r+\alpha_Lu^tL'$ and $\beta=\beta_0u^s+\beta_Lu^vL'$ with $\alpha_0,\alpha_L,\beta_0,\beta_L$ polynomials in u^e over $\mathbb{F}_{p^2}\otimes\mathbf{k}_E$ which either are zero or have nonzero constant term. We wish to show that $\alpha_0=\beta_0=0$. We consider the relation $f\phi_1(\mathbf{e})=\phi_1f(\mathbf{e})$, first paying attention only to the terms not involving nilpotents:

$$\phi(\beta_0)u^{ps}c = -\alpha_0 u^{r+k},$$
$$\phi(\alpha_0)u^{pr} = \beta_0 u^{e_2-k+s}c.$$

If α_0, β_0 are nonzero, we must therefore have $r = p - j$ and $s = 1 + j$. We turn next to the terms involving nilpotents. The g_1-term in this relation is:

$$w\phi(\beta_L)u^{pv}L' = -c\alpha_L u^{t+k}L' + c\beta_0 u^{1+j}(\overline{X}_2 - \overline{X}_1 u^{e_2 j}).$$

But if $\beta_0\overline{X}_2 \neq 0$, equality could not hold here, because there can be no other terms of degree $1 + j$ in u! If $\beta_0 \neq 0$ it follows that $\overline{X}_2 = 0$. But similar consideration of the g_2-term yields $\overline{X}_1 = 0$. Since $\overline{X}_1$ and $\overline{X}_2$ cannot both be zero, it follows that $\alpha_0 = \beta_0 = 0$, and we are done.

We note very briefly some of the features of this calculation for the other parts of Theorems 6.22 and 6.23. In part (2) of Theorem 6.22, the case $\tilde{x}_1^2 \equiv w \pmod{\mathfrak{m}_E}$ requires slightly more work (in most cases an α_0 is forced to be zero on its own, but in the more complicated case, one needs to use $j \neq 1$ and consider β_0 as well to see that $\alpha_0 = 0$). There is a similar feature in part (1) of Theorem 6.23 when $\tilde{b}^2 w \equiv \pm 1 \pmod{\mathfrak{m}_E}$; in this case, there is a β_0 which satisfies $\phi(\beta_0) = \mp\beta_0$, and then an equation of the form $\pm\beta_B = \phi(\beta_0) - \phi(\beta_B)$ implies $\beta_0 = 0$. (Apply ϕ to this equation again.) ☐

Corollary 6.25. *The Breuil-Mézard conjecture* [BM02, Conj. 2.3.1.1] *holds for* $k = 2$ *and* τ *tame.*

Proof. This is immediate from the computation of $\mu_{\mathrm{aut}}(2, \overline{\rho}, \tau)$ with τ tame. ☐

Corollary 6.26. *Theorem 1.6 holds.*

Acknowledgements. The author is especially indebted to Christophe Breuil for numerous discussions; without Breuil's generosity this work would never have been completed. The author is grateful to Richard Taylor for suggesting this problem, and to the anonymous referee for his or her helpful comments. The author thanks the Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France, for its warm hospitality during a year-long stay in 2002-2003.

References

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$, J.A.M.S. **14** (2001), 843–939.

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart.

[Bre] Christophe Breuil, *p-adic Hodge theory, deformations, and local Langlands*, Notes from a course given at Centre de Recerca Matematica, July 18 to 28, 2001, available at http://www.crm.es/Publications/ps-pdf-preprints/Quadern20-1.pdf.

[Bre98] ———, *Construction de représentations p-adiques semi-stables*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 3, 281–327.

[Bre99] ———, *Représentations semi-stables et modules fortement divisibles*, Invent. math. **136** (1999), no. 1, 89–122.

[Bre00] ———, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549.

[Car86] Henri Carayol, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 409–468.

[CDT99] Brian Conrad, Fred Diamond, and Richard Taylor, *Modularity of certain potentially Barsotti-Tate Galois representations*, J.A.M.S. **12** (1999), 521–567.

[CF00] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations p-adiques semi-stables*, Invent. math. **140** (2000), no. 1, 1–43.

[CI] Robert Coleman and Adrian Iovita, *Hidden structures on semi-stable curves*, Preprint.

[Edi92] Bas Edixhoven, *The weight in Serre's conjectures on modular forms*, Invent. math. **109** (1992), no. 3, 563–594.

[Fal87] Gerd Faltings, *Hodge-Tate structures and modular forms*, Math. Ann. **278** (1987), no. 1-4, 133–149.

[Fal97] ———, *Crystalline cohomology of semistable curve—the* $\mathbf{Q}_p$*-theory*, J. Algebraic Geom. **6** (1997), no. 1, 1–18.

[FM95] Jean-Marc Fontaine and Barry Mazur, *Geometric Galois Representations*, Elliptic curves, modular forms, & Fermat's last theorem (John Coates and Shing-Tung Yau, eds.), International Press, 1995, pp. 41–78.

[Fon94] Jean-Marc Fontaine, *Representations p-adiques semi-stables*, Astérisque **223** (1994), 113–184.

[Gro90] Benedict H. Gross, *A tameness criterion for Galois representations associated to modular forms (mod p)*, Duke Math. J. **61** (1990), no. 2, 445–517.

[Kis] Mark Kisin, *Moduli of finite flat group schemes and modularity*, Preprint.

[Ray74] Michel Raynaud, *Schémas en groupes de type* $(p, p, \ldots, p)$, Bull. Soc. Math. France **102** (1974), 241–280.

[Sai97] Takeshi Saito, *Modular forms and p-adic Hodge theory*, Invent. Math. **129** (1997), no. 3, 607–620.

[Sav04] David Savitt, *Modularity of some potentially Barsotti-Tate Galois representations*, Compos. Math. **140** (2004), no. 1, 31–63.

[Tat67] John Tate, *p-divisible groups*, Proc. Conf. Local Fields (Driebergen, 1966), Springer-Verlag, 1967, pp. 158–183.

[Tat97] ———, *Finite flat group schemes*, Modular forms and Fermat's last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 121–154.

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