1. Representations of adelic groups

Let $G$ be a linear algebraic group over a global field $k$, with ring of adeles $\mathbb{A}$. For this chapter, a representation $\pi$ of $G(\mathbb{A})$ will always be a smooth representation of the finite adeles $G(\mathbb{A}_f)$, and topologized as the strict direct limit

$$\lim_{\rightarrow} \pi_{K_f}$$

over a basis of open compact neighborhoods of the identity in $K_f$. Any property of representations of groups over local fields will be applied to representations of $G(\mathbb{A})$, by restricting it to the spaces of $K^S$-invariants, considered as a $G(k_S)$-representation, where $S$ is a finite set of places including the Archimedean ones, and $K^S$ is any compact open subgroup of the adeles away from $S$. For example, $\pi$ is an SF-representation, or smooth representation of moderate growth, if $\pi_{K^S}$ is such a representation of $G(k_S)$ (Definition 1.2), for any such $K^S$.

If $k$ has Archimedean places, we will denote by $\mathfrak{g}$ the complexified Lie algebra $\mathfrak{g}(k_\infty) \otimes_{\mathbb{R}} \mathbb{C}$, by $U(\mathfrak{g})$ its universal enveloping algebra, and by $Z(\mathfrak{g})$ the (Harish-Chandra) center of $U(\mathfrak{g})$.

2. The space of automorphic forms

**Definition 2.1.** A continuous function $f$ on the automorphic space $[G]$ is of moderate growth if on one, equivalently any, Siegel fundamental set $\Omega A_e K$ (in the language and notation of Definitions 5.4, 5.5), and a norm on the space $a = \text{Hom}(\mathbb{G}_m, A) \otimes \mathbb{R}$, the function satisfies a bound of the form

$$|f(\omega a k)| \ll e^{s \log(a)}$$

for some $s > 0$, where $\log : A(\mathbb{A}) \to a$ is the logarithmic map of (2.6.1).

For fixed $s$ and norm, the functions above form a Banach space with norm sup$(|f(\omega a k)|e^{-s \log(a)})$. The space $C_{mg}([G])$ of moderate growth functions is the direct limit of these spaces.
A function is of uniform moderate growth if it belongs to the space $C_{mg}(G)^{\infty}$ of smooth vectors in this space. Equivalently, if it is fixed under an open compact subgroup $K$ of the finite adeles $G(k_f)$, and for every $D \in U(g(k_\infty))$, the function $Df$ satisfies a bound as above, for the same $s$.

**Remark 2.2.** Equivalent definitions of moderate growth are the following:

First, fix a closed embedding of $G$ into an affine space with coordinates $(x_1, \ldots, x_n)$, and define, for every place $v$, $\|g\| = \max\{1, |x_1(g)|, \ldots, |x_n(g)|\}$ on $G(k_v)$. Define $\|g\| = \prod_v \|g_v\|$ on $G(\mathbb{A})$. (This makes sense, because almost all factors are 1.) Then, a continuous function $f$ on $\bar{G}$ is of moderate growth iff $|f(g)| \ll \|g\|^s$ for some $s > 0$. See [MgW95, I.2.2] for the equivalence.

Another, more geometric, equivalent definition is the following: Consider any equivariant toroidal (full) compactification $\overline{G}$ (Definition 4.3). Recall that the complement of $\overline{G}$ is the union of $P$-cusps, as $P$ ranges over all conjugacy classes of parabolics, and that the $P$-cusp has a neighborhood which maps, with compact fibers (Definition 2.3b) such as $\mathbb{A}_P(k_\infty) \times \mathbb{A}_P(k_\infty)[G]_P$. Now, fix a compact subset $U$ of the $P$-cusp; it has a neighborhood of the form $U \times V$, where $V$ is a neighborhood of the closed $\mathbb{A}_P(k_\infty)$-orbit in $\mathbb{A}_P(k_\infty)$; use $\tilde{V}$ to denote its intersection with the open $\mathbb{A}_P(k_\infty)$-orbit. Then, “moderate growth” means that for any cover of the boundary of $\bar{G}$ by such compact sets, the function is bounded on $U \times \tilde{V}$ by a multiple of $\epsilon^{-s}$, where $\epsilon$ is an “algebraic distance function” from the closed orbit in $\tilde{V}$, that is, if the closed orbit is given by the vanishing of algebraic coordinates $x_1, \ldots, x_n$, then $\epsilon \sim \max_i |x_i|$.

**Proposition 2.3.** Let $\pi$ be a Fréchet representation of moderate growth of $G(\mathbb{A})$. (See Section 4.) Any morphism $l: \pi \to C(\bar{G})$ factors through a continuous map to $C_{mg}(G)$. If it is a smooth Fréchet representation of moderate growth, it factors through a continuous map to $C_{mg}(\bar{G})^{\infty}$.

**Proof.** We will use the first equivalent characterization of moderate growth of Remark 2.2. Since the map $l: \pi \to C(\bar{G})$ is continuous (the space on the right considered as a Fréchet space), for every $K^S$: compact open subgroup of $G(k_S)$ there is a continuous seminorm $q$ on $\pi^{K^S}$ such that $|l(v)(1)| \leq q(v)$. With $\pi$ being an $F$-representation, we may assume that $q$ is $G(k_S)$-continuous, and then by (6.2.1) we have that $|l(v)(g)| = |l(\pi(g)v)(1)| \leq q(\pi(g)v) \leq ||g||^s q(v)$ for some $s > 0$. Thus, the map factors continuously through $C_{mg}(\bar{G})$. Passing to smooth vectors, we get a continuous map $\pi^\infty \to C_{mg}(\bar{G})^{\infty}$. Passing to smooth vectors, we get

If, in addition, $\pi$ is admissible, elements in its image have the following properties:

**Proposition 2.4.** Let $\pi$ be an admissible smooth Fréchet representation of moderate growth of $G(\mathbb{A})$, and $l : \pi \to C^{\infty}(\bar{G})$ a morphism. Fix a maximal compact subgroup $K_\infty$ of $G(k_\infty)$. Elements $f$ in the image of $\pi^{K_\infty}$ have the following properties:

1. $f$ is of uniform moderate growth;
2. $f$ is $K_f$-finite, for every compact open subgroup of $G(k_f)$;
3. $f$ is $K_\infty$-finite;
4. $f$ is $Z(g)$-finite, if $k$ has Archimedean places. In the function field case, $f$ is finite under the Bernstein center of $G(k_\infty)$, for some chosen place $\infty$. 

Conversely, every such function $f$ generates an admissible SF-subrepresentation of $C_{mg}(\mathbb{G})^{\infty}$.

**Proof.** The first property is contained in Proposition 2.3, the second and third are obvious, and the fourth follows from the fact that $Z(g)$ preserves the finite-dimensional $K_fK_\infty$-isotypic space of $f$.

Vice versa, if $f$ satisfies these properties, the statement to prove is admissibility. This follows from Harish-Chandra’s theorem, which says that the space of functions as above with fixed $K_fK_\infty$-type, and annihilated by a fixed ideal of finite codimension in $Z(g)$, is finite-dimensional. [Not included yet in the notes.] □

**Definition 2.5.** The space $\mathcal{A}(\mathbb{G})$ of automorphic forms on $\mathbb{G}$ is the sum of all admissible subrepresentations of $C_{mg}(\mathbb{G})^{\infty}$ that are generated (in the sense of closure of the $G(\mathbb{A})$-translates) by their $K_\infty$-finite vectors. An automorphic representation is any irreducible subquotient of the space of automorphic forms.

**Remark 2.6.** The definition of automorphic forms given above is not standard. Usually, all the conditions of Proposition 2.4 are imposed on automorphic forms, while we omitted $K_\infty$-finiteness. The problem with the standard definition is that it depends on the choice of $K_\infty$ (only up to translation by $G(k_\infty)$-though, since all $K_\infty$ are conjugate), and it doesn’t produce a representation of $G(k_\infty)$, but a $(g, K_\infty)$-module. On the other hand, the definition that we gave contains the clumsy requirement that the representation is generated by its $K_\infty$-finite vectors (a condition, though, that clearly does not depend on the choice of $K_\infty$). This condition is equivalent to a finite length condition (for the $K_S$-invariants of the representation, where $K_S$ is an open subgroup away from a finite number of places $S$), i.e., we do not allow for “automorphic forms” to be approximable by vectors belonging to an increasing sum of finite-length representations, without belonging to a finite-length subsum.

At this point, it is not clear from the definition that the subrepresentation of $C_{mg}(\mathbb{G})^{\infty}$ generated by an automorphic form has bounded growth. However, it is true, and follows from the existence of exponents [also behind Harish-Chandra’s finiteness theorem, not included yet].

### 3. Modular and cusp forms (analytic theory)

Let $G = \text{GL}_2(\mathbb{R})^0$, acting on the complex upper half plane $\mathcal{H}$ on the left by Möbius transformations. Let $\Gamma$ be a discrete subgroup, so that $\Gamma \backslash \mathcal{H}$ has finite volume. We may assume $-I \in \Gamma$ and $\Gamma \subset \text{SL}_2(\mathbb{R})$. Let $k \in \mathbb{Z}_{>0}$ and let $\chi : \Gamma \to S^1$ be a character satisfying $\chi(-I) = (-1)^k$.

**Definition 3.1.** Define $\mathcal{M}_k(\Gamma, \chi)$ to be the set of all holomorphic functions $f$ on $\mathcal{H}$ satisfying the following two conditions: $f(\gamma z) = \chi(\gamma)(cz+d)^k f(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$; and $f$ is holomorphic at the cusps of $\Gamma$. Let $\mathcal{S}_k(\Gamma, \chi) := \{ f \in \mathcal{M}_k(\Gamma, \chi) \mid f$ vanishes at the cusps $\}$. Elements of $\mathcal{M}_k(\Gamma, \chi)$ are called modular forms, while those of $\mathcal{S}_k(\Gamma, \chi)$ are called cusp forms.

Now, we present modular and cusp forms as sections of some line bundles, following Deligne [Del73]. Towards this purpose, we construct a line bundle on $\mathcal{H}$ by presenting it as a space equipped with a universal elliptic curve and pushing forward the sheaf of differentials. We will demonstrate an action of $G$ on the sections.
of the sheaf, and then explain the relationship of modular forms and cusp forms to this line bundle.

Let Isom(R², C) be the set of isomorphisms of R² and C, as R-vector spaces, and Hom⁻(R², C) the subset of orientation-reversing ones. The structure of a complex vector space on C endows it with a natural structure of a two-dimensional complex submanifold of C², with a free action of C×. Moreover, it has a left action of GL₂(C), induced from its right action on R², whose elements we think of as row vectors: g·T := T(vg). The quotient Hom⁻(R², C)/C× is thus a one-dimensional complex manifold with a GL₂(C)-action, which parametrizes orientation-reversing complex structures on R². We identify this quotient with the complex upper half plane H, by sending the class of a homomorphism T to T(1,0)/(0,1) ∈ H, and then GL₂(C) acts on the left by M¨obius transformations γ · z = \( \frac{az+b}{cz+d} \) for γ = \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

By construction, the space H comes equipped with a complex line bundle \( \omega⁻¹ \), whose pullback to Hom⁻(R², C) is the structure sheaf \( \mathcal{O} \) of holomorphic functions. Equivalently, sections of \( \omega⁻¹ \) are real analytic functions \( \sigma : H \rightarrow \mathbb{R} \), such that for one, equivalently any, complex analytic lift \( H ∋ z \mapsto T_z \in \text{Hom}⁻(\mathbb{R}², C) \), the composition \( z \mapsto T_z(σ(z)) \) is holomorphic. (Note that such lifts T exist, for example by fixing that \( T(0,1) = 1 \), otherwise we would need to make the same statements locally.) The action of GL₂(C) induces a GL₂(C)-equivariant structure on \( \omega⁻¹ \); we define this as a right action on sections, by

\[ σ|γ(z) = σ(γz)γ, \]

where the exponent denotes the right action of γ on R² (and the number “1” stands for the first power of \( ω⁻¹ \)).

Notice that, by construction we have an isomorphism of the associated real analytic vector bundle \( \omega⁻¹ R \) with the constant real analytic vector bundle with fiber \( R² \), such that constant sections of \( R² \) correspond to holomorphic sections. (Such a structure is called a variation of complex structure on \( R² \) over H.) This gives rise to a GL₂(C)-equivariant surjection

\[ \mathcal{O} \otimes \mathbb{C}² \rightarrow \omega⁻¹ \]

of complex vector bundles on H (where \( \mathbb{C}² \) denotes the constant sheaf).

Note that \( \omega⁻¹ \) can be trivialized as a complex vector bundle (after all, H is simply connected), but not GL₂(C)-equivariantly so. In fact:

**Lemma 3.2.** The section \( H ∋ z \mapsto T_z ∈ \text{Hom}⁻(\mathbb{R}², C) \) determined by \( T_z(0,1) = 1 \) (hence \( T_z(1,0) = z \)) induces a trivialization of \( \omega⁻¹ \), such that the (right) action of \( γ = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ∈ \text{GL}_2(\mathbb{R}) \) on sections of \( ω^k \) is given by

\[ f|kγ(z) = (cz+d)^{-k}f(γz). \]

Moreover, there is an equivariant isomorphism of complex line bundles, \( ω² = Ω^1 \) (the bundle of holomorphic 1-forms) on H.

**Proof.** For the first statement, if \( σ = (σ₁, σ₂) \) is a section of \( ω⁻¹ \), and \( f \) is the corresponding section of the trivial line bundle induced by this trivialization, then \( f \) is given by \( f(z) = T_z ∘ σ(z) = zσ₁(z) + σ₂(z) \).

\[ \text{That’s unfortunate, and contrary to the convention of Deligne, but reproduces the usual action on } \mathcal{H}. \]
The action of an element \( \gamma \) as above is hence given by \( f|_1 \gamma (z) = T_2 \circ (\sigma_1 \gamma)(z) = T_2(\sigma(z) \gamma) = (az + b) \sigma_1(\gamma z) + (cz + d) \sigma_2(\gamma z) = (cz + d) f(\gamma z) \), and the case of a general power of \( \omega \) is immediate.

The sheaf \( \Omega^1 \) of differential one-forms can be trivialized by the global section \( dz \) on \( \mathcal{H} \), and then it is immediate to check that the right action of \( \gamma \) sends the form \( f(z)dz \to f|_2 \gamma(z)dz \), identifying the trivializations of \( \Omega^1 \) and \( \omega^2 \) equivariantly. \( \square \)

Now let \( \Gamma \) be as above, and assume additionally that \( \Gamma \) is torsion free. The latter assumption ensures that it acts properly discontinuously on \( \mathcal{H} \) and \( \Gamma \setminus \mathcal{H} =: Y_\Gamma \) has a unique complex manifold structure making \( \mathcal{H} \to \Gamma \setminus \mathcal{H} \) a local analytic isomorphism. If \( Y_\Gamma \) is not compact, we let \( X_\Gamma \) be the compactification that is obtained by adding cusps. [Cusps and their comparison to adelic reduction theory need to be added.]

The sheaves \( \omega^{-1}, \Omega^1 \) being equivariant, they extend to the quotient \( Y_\Gamma \), and they admit a natural extension to \( X_\Gamma \), to be denoted by the same symbols: the sheaf \( \Omega^1 \) as the sheaf of one-forms; for the sheaf \( \omega^{-1} \), we may assume (by applying a Möbius transformation) that the cusp of interest is the one-point compactification of \( \Gamma_\infty \setminus \mathcal{H} \) at \( \infty \), where \( \Gamma_\infty \) is a discrete subgroup of upper triangular unipotent matrices; then we declare the \( \Gamma_\infty \)-invariant section \( \sigma: \mathcal{H} \to \mathbb{R}^2 \), \( \sigma(z) = (0,1) \) of \( \omega^{-1} \) to extend to a non-zero section at the cusp. The comparison of Lemma 3.2 extends to the cusps as follows:

**Lemma 3.3.** In a neighborhood of a cusp \( \infty \), we have

\[
\Omega = \omega^2(\infty).
\]

**Proof.** Identifying a neighborhood of the cusp with a neighborhood of \( i\infty \) in \( \Gamma_\infty \setminus \mathcal{H} \), as above, and using the trivialization of Lemma 3.2 the non-zero section \( \sigma(z) = (0,1) \) of \( \omega^{-1} \) corresponds to the constant function \( f^{-1}(z) = 1 \), and therefore the function \( f_2 = f^{-2} = 1 \) corresponds to a non-zero section of \( \omega^2 \) in a neighborhood of the cusp. If \( \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \) is a generator for \( \Gamma_\infty \), a holomorphic coordinate at the cusp is given by \( q_\alpha = e^{2\pi i z_\alpha} \), so \( dz = \frac{\alpha}{2\pi i} dq_\alpha \), and we see that the corresponding differential form \( f_2 dz = dz \) has a pole of order 1 at \( q_\alpha = 0 \). \( \square \)

Now we are ready to express modular forms and cusp forms in terms of these line bundles; the proposition below could also have been taken as the definition.

**Proposition 3.4.** Under the trivializations of Lemma 3.2 \( \mathcal{M}_k(\Gamma, 1) \) is the image of \( H^0(X_\Gamma, \omega^{\otimes k}) \to H^0(Y_\Gamma, \omega^{\otimes k}) \), and \( \mathcal{S}_k(\Gamma, 1) \) is the image of \( H^0(X_\Gamma, \omega^{\otimes k}(-D)) \to H^0(Y_\Gamma, \omega^{\otimes k}) \), where \( D \) denotes the divisor corresponding to the cusps of \( X_\Gamma \).

**Proof.** This is immediate from Lemma 3.2 except for the behavior at the cusps. We have seen that, by definition, the constant function corresponds under the above trivialization to a section of \( \omega^k \) over \( \Gamma_\infty \setminus \mathcal{H} \), hence modular forms of weight \( k \) extend to sections of \( \omega^k \) at the cusps, and cusp forms extend to sections vanishing at the cusps. \( \square \)

Now let \( \mathcal{F}_\mathbb{R} \) be the dual of the constant sheaf \( \mathbb{R}^2 \) over \( \mathcal{H} \). By using the standard symplectic form on \( \mathbb{R}^2 \), we can and will identify it with \( \mathbb{R}^2 \otimes \det^{-1} \), equivariantly under the \( \text{GL}_2(\mathbb{R}) \)-action. Let \( \mathcal{F} = \mathcal{F}_\mathbb{R} \otimes \mathbb{C} \). The surjection (3.1.1) induces, dually, an injection \( \omega \hookrightarrow \mathcal{O} \otimes \mathcal{F} \otimes \det^{-1} \) which, for every non-negative integer \( k \),
gives rise to an injection
\[(3.4.1) \quad \omega^k \to O \otimes \text{Sym}^k \mathcal{F} \otimes \text{det}^{-k}.\]

In particular, the restriction to SL_2(\mathbb{R}) is an equivariant injection to \(O \otimes \text{Sym}^k \mathcal{F}\), and both sheaves descend to \(Y_{\Gamma}\).

From these maps, (3.3.1), and de Rham cohomology, we obtain:
\[(3.4.2) \quad H^0(Y_{\Gamma}, \omega^k) \to H^0(Y_{\Gamma}, \omega^{k-2} \otimes \Omega^1_{Y_{\Gamma}}) \to H^1(Y_{\Gamma}, \text{Sym}^{k-2} \mathcal{F})\]

**Theorem 3.5** (Shimura isomorphism). The map (3.4.2) carries \(S_k(\Gamma, 1)\) into \(H^1(Y_{\Gamma}, \text{Sym}^{k-2} \mathcal{F})\) and induces an isomorphism
\[(3.5.1) \quad S_k(\Gamma, 1) \oplus S_k(\Gamma, 1) \cong H^1(Y_{\Gamma}, \text{Sym}^{k-2} \mathcal{F}),\]
where \(H^\bullet\) denotes the image of \(H^\bullet\) (cohomology with compact supports to cohomology without supports).

**Proof.** See [Del73] for references and further discussion.

Now we describe the above sheaves and constructions in terms of moduli of elliptic curves. The benefit of doing so is that it allows to endow the spaces \(Y_{\Gamma}, X_{\Gamma}\), and the above sheaves, with algebro-geometric structure over the rational numbers or appropriate rings of integers.

**Definition 3.6.** An elliptic curve in the category of complex manifolds is a pair \((E, e)\) consisting of a compact Riemann surface of genus one, and a point on it. More generally, an elliptic curve over a complex manifold \(S\) is a smooth (submersive) morphism of complex manifolds \(E \to S\), whose fibers are elliptic curves, equipped with a section \(e : S \to E\).

Equivalent definitions mention the abelian group structure on the elliptic curve, which arises by identifying it with its Jacobian, by sending a point \(x\) to the divisor \((x) - (e)\).

The space \(S = \text{Isom}(\mathbb{R}^2, \mathbb{C})\) comes equipped with an elliptic curve \(E_0\), defined by the following short exact sequence of sheaves:
\[0 \to \mathbb{Z}^2 \to \omega^1 \to E_0 \to 0.\]

Moreover, this elliptic curve comes equipped with the following structure:
- an identification of its fundamental group with \(\mathbb{Z}^2\); equivalently, an identification of the local system of homology groups \((R^1 f_* \mathbb{Z})^\vee\) (where \(R^1 f_*\) denotes the first derived functor of pushforward, i.e., fiberwise cohomology), or cohomology groups \(R^1 f_* \mathbb{Z}\), with the constant sheaf \(\mathbb{Z}^2\);
- an identification of the analytic sheaf \(e^* \Omega^1_{E_0/S} = f_* \Omega^1_{E_0/S}\) with the sheaf \(\omega\).

**Proposition 3.7.**
1. The functor which associates to each complex manifold \(S\) the set of isomorphism classes of elliptic curves \((f : E \to S, e : S \to E)\), equipped with isomorphisms \(e^* \Omega^1_{E} \simeq O\) and \(R^1 f_* \mathbb{Z} \simeq \mathbb{Z}^2\) is representable by the complex manifold \(\text{Isom}(\mathbb{R}^2, \mathbb{C})\), equipped with a universal elliptic curve \(E_0\).
2. The functor which associates to each analytic space \(S\) the set of isomorphism classes of elliptic curves over \(S\), equipped with an isomorphism \(R^1 f_* \mathbb{Z} \simeq \mathbb{Z}^2\) is represented by the complex manifold \(\text{Isom}(\mathbb{R}^2, \mathbb{C})/\mathbb{C}^\times\).

**Proof.** Omitted.
The exterior square $\wedge^2 R^1 f_* \mathbb{Z}$ is canonically trivialized by the fundamental class corresponding to the complex orientation, and $\mathcal{H} = \text{Hom}^{-}(\mathbb{R}^2, \mathbb{C})/\mathbb{C}^\times$ represents those isomorphisms for which the induced isomorphism $\wedge^2 R^1 f_* \mathbb{Z} \cong \wedge^2 \mathbb{Z}$ sends this class to $e_2 \wedge e_1$, in the standard basis.

The map (3.1.1), now, and its dual (3.4.1) corresponds to the Hodge filtration (3.7.1)

$$0 \rightarrow \omega = R^0 f_* \Omega^1_{E_0/S} \rightarrow O \otimes R^1 f_* \mathbb{R} \rightarrow \omega^{-1} = (R^0 f_* \Omega^1_{E_0/S})^\vee = R^1 f_* \mathcal{O}_{E_0} \rightarrow 0.$$  

(Note that in this discussion we have been ignoring determinant factors when identifying the constant sheaf with its dual, since the factors above do not carry an action of $GL_2(\mathbb{R})^0$, but only of its subgroup $SL_2(\mathbb{Z})$.)

### 4. Maaß forms

**Definition 4.1.** For fixed $k$ (“weight”) $\in \mathbb{Z}$, define Maaß differential operators on $C^\infty(\mathcal{H})$ as follows:

$$R_k := i y \frac{\partial}{\partial z} + y \frac{\partial}{\partial \bar{z}} + \frac{k}{2},$$

$$L_k := -i y \frac{\partial}{\partial z} + y \frac{\partial}{\partial \bar{z}} - \frac{k}{2},$$

$$\Delta_k := -g^2 (\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{z}^2}) + i k y \frac{\partial}{\partial z} = -R_{k-2} L_k + \frac{k}{2} (1 - \frac{k}{2}).$$

Let $G = GL_2(\mathbb{R})^0$ act on $C^\infty(\mathcal{H})$ by $(f|_k g)(z) = \left( \frac{az + d}{cz + d} \right)^k f(gz)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Let $\Gamma$ be a discontinuous subgroup of $SL_2(\mathbb{R}) \subset G$ containing $-I$ with $\Gamma \backslash \mathcal{H}$ of finite volume, and let $\chi$ be a unitary character of $\Gamma$. We take $C^\infty(\Gamma \backslash \mathcal{H}, \chi, k)$ to denote the set of functions, $f$, $\in C^\infty(\mathcal{H})$ satisfying $\chi(\gamma)(z) = (f|_k \gamma)(z)$, for $\gamma \in \Gamma$. (Note that this forces $\chi(-I) = (-1)^k$.) A short calculation gives us the following lemma:

**Lemma 4.2.** $R_k$ and $L_k$ act as weight raising and lowering operators respectively, namely: $R_k : C^\infty(\Gamma \backslash \mathcal{H}, \chi, k) \rightarrow C^\infty(\Gamma \backslash \mathcal{H}, \chi, k + 2)$; $L_k : C^\infty(\Gamma \backslash \mathcal{H}, \chi, k) \rightarrow C^\infty(\Gamma \backslash \mathcal{H}, \chi, k - 2)$; $\Delta_k : C^\infty(\Gamma \backslash \mathcal{H}, \chi, k) \rightarrow C^\infty(\Gamma \backslash \mathcal{H}, \chi, k)$.

**Proof.** Left to the reader. $\square$

**Definition 4.3.** A Maaß form of weight $k$ for $\Gamma$ is a smooth complex valued function $f$ on $\mathcal{H}$ that satisfies:

1. $f \in C^\infty(\Gamma \backslash \mathcal{H}, \chi, k)$, for some $\chi$;
2. $\Delta_k f = \lambda f$, some $\lambda \in \mathbb{C}$;
3. $f$ has moderate growth at cusps of $\Gamma$.

Here, moderate growth at infinity means that $f(x + iy)$ is bounded by a polynomial in $y$ as $y \rightarrow \infty$. For a general cusp of $\Gamma$, $a \in \mathbb{R} \cup \infty$, let $\xi \in SL_2(\mathbb{R})$ be such that $\xi(\infty) = a$. Then $f$ is said to be of moderate growth at $a$ if $f|_k \xi \in C^\infty(\xi^{-1} \Gamma \xi \backslash \mathcal{H}, \xi \chi \xi^{-1}, k)$ is of moderate growth at $\infty$.

**Remark 4.4.** Observe that the vanishing of a Maaß form $f$ under the operator $L_k$ is equivalent to $y^{-k/2} f$ satisfying the Cauchy–Riemann equations. This defines an embedding

$$\mathcal{M}_k(\Gamma, \chi) \hookrightarrow C^\infty(\Gamma \backslash \mathcal{H}, \chi, k).$$

As $y^{k/2} f' \in \text{ker}(L_k)$, it is $\Delta_k$-eigenfunction with eigenvalue $\frac{k}{2} (1 - \frac{k}{2})$. Moderate growth is automatic, hence the embedding above identifies holomorphic modular
forms with a subspace of the Maaß forms, determined by the vanishing of the weight-lowering operators.

When we pass to $G = \text{GL}_2(\mathbb{R})^0$-representations in the next section, it will turn out that the vanishing of the weight-lowering operator places the image of holomorphic modular forms in the discrete series representation with a certain eigenvalue for the Casimir operator corresponding to $k$.

5. Classical automorphic forms

With notation as in the previous section, namely $G = \text{GL}_2(\mathbb{R})^0$ and $\chi$ a character of the lattice $\Gamma \subset \text{SL}_2(\mathbb{R})$, let $C^\infty(\Gamma \backslash G, \chi)$ denote the space of complex-valued smooth functions $F$ on $G$ satisfying $F(\gamma g) = \chi(\gamma)F(g)$ for $\gamma \in \Gamma$, $g \in G$. $G$ acts on this space by right translation. Let $C^\infty(\Gamma \backslash G, \chi, k) \subset C^\infty(\Gamma \backslash G, \chi)$ be the set of functions, $F$, additionally satisfying $F(\kappa g_\theta) = e^{ik\theta}F(g)$, for $\kappa \in SO_2(\mathbb{R})$ that gives clockwise rotation by $\theta$.

**Proposition 5.1.** There exists an inclusion $\sigma_k : C^\infty(\Gamma \backslash \mathcal{H}, \chi, k) \to C^\infty(\Gamma \backslash G, \chi, k)$ given by $f \mapsto \big( F : g \mapsto (f|_K)(g) \big)$. Furthermore, there exist elements $R, L$ and $\Delta$ of $U(\mathfrak{g}_\mathbb{C})$ acting on $C^\infty(\Gamma \backslash G, \chi, k)$, that commute with the action of $R_k, L_k$ and $\Delta_k$ respectively. $\Delta$ is (up to a scalar) the Casimir element of $U(\mathfrak{g})$.

**Proof.** That the image $F$ of $f$ belongs to $C^\infty(\Gamma \backslash G, \chi, k)$ is quickly checked. For the rest of the statements, notice first that every element $g \in G$ can be uniquely written as $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \kappa_\theta$. Here, $x, y$ and $u$ are uniquely determined, while $\theta$ is uniquely determined mod $2\pi$. Define the following elements of $U(\mathfrak{g}_\mathbb{C})$: $R := \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $L := \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$.

Computing the action of $R, L$ and the Casimir element $\Delta$ in terms of $x, y, u$ and $\theta$, we get that $dR = e^{i\theta}(iy \frac{\partial}{\partial y} + y \frac{\partial}{\partial y} + \frac{1}{2i} \frac{\partial}{\partial u})$, $dL = e^{i\theta}(-i\frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{1}{2i} \frac{\partial}{\partial u})$ and $d\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y^2 \frac{\partial^2}{\partial x \partial y}$. One checks that these operators commute with the action of $R_k, L_k$ and $\Delta_k$ in the desired fashion.

For details of the calculation, refer to [Bum97 Theorem 2.2.5].

Let $\omega$ be a (unitary) character of the center $Z(G)$ of $G$, agreeing with $\chi$ on $-I$. Consider $C^\infty(\Gamma \backslash G, \chi, \omega) \subset C^\infty(\Gamma \backslash G, \chi)$ denoting functions, $F$, that additionally satisfy $F(zg) = \omega(z)F(g)$.

**Definition 5.2.** $\mathcal{A}(\Gamma \backslash G, \chi, \omega)$ is the subspace of those elements of $C^\infty(\Gamma \backslash G, \chi, \omega)$ that are $Z(U(\mathfrak{g}_\mathbb{C}))$-finite, $K$-finite, and satisfy the condition of moderate growth below. Elements of $\mathcal{A}(\Gamma \backslash G, \chi, \omega)$ are called automorphic forms.

Moderate growth here means that $\exists k > 0$ such that $\forall D \in U(\mathfrak{g}_\mathbb{C})$, $|Df(g)|$ has order of growth less than $\|g\|^k$ where $\|g\|$ can be defined to be a height function obtained by pulling back the maximum function along the embedding $G = \text{GL}_2(\mathbb{R})^0 \hookrightarrow \mathbb{R}^5$. This is the embedding which sends a matrix to its 4 coordinates and the determinant. Observe that since $\omega$ is a unitary character, $|Df(g)|$ is in fact a well defined function on $\Gamma \backslash \text{SL}_2(\mathbb{R})$.

Below, we explain the relationship between Maaß forms and Classical automorphic forms.
Remark 5.3. Notice that this will also cover the relationship between modular forms and classical automorphic forms since modular forms give rise to certain Maass forms (kernels of $L_k$ operators) (Remark 4.4).

Let $f$ be a Maass form of weight $k$, for character $\chi$. Let $F(g) := (f|_k g)(i) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(gi), g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

Proposition 5.4. The above map $f \mapsto F$ identifies Maass forms of weight $k$ for character $\chi$ with $\mathcal{A}(\Gamma \setminus G, \chi, \omega) \cap C^\infty(\Gamma \setminus G, \chi, k)$, where $\omega$ is the character which is trivial on the identity component of $\mathbb{R}^\times$ and equal to $(-1)^k$ on $-1$.

Proof. Let $z = \begin{pmatrix} r \\ r \end{pmatrix} \in Z(G), F(zg) = ((f|_k z)|_k g)(i) = \begin{pmatrix} -c & d \\ c & -d \end{pmatrix} k f(gz)(gi) = (\begin{pmatrix} r \\ r \end{pmatrix}) k \begin{pmatrix} -c & d \\ c & -d \end{pmatrix} k f(gi) = \omega(r) F(g)$

Now, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. $F(\gamma g) = ((f|_k \gamma)|_k g)(i) = \chi(\gamma)(f|_k g)(i) = \chi(\gamma) F(g)$.

Therefore, $F \in \mathcal{A}(\Gamma \setminus G, \chi, \omega)$.

By Proposition 5.1 as $f$ is an eigenvector of $\Delta_k$, so is $F = \sigma_k(f)$ of $\Delta$. Therefore, $F$ is $Z(U(\mathbb{G}))$-finite, as the latter is generated by $\Delta$ (by Theorem 5.7, Theorem 4.2 and Definition 4.4). Additionally, being an element of $C^\infty(\Gamma \setminus G, \chi, k)$, $F$ is also $K = SO_2(\mathbb{R})$-finite. Finally, the moderate growth conditions for $f$ and $F$ turn out to be equivalent, as in Remark 2.2. □

6. Classical and adelic automorphic forms

Let $K_0(N) = \prod_p \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p)|c \equiv 0 \mod N$, and $K_1(N) = \prod_p \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p)|c \equiv 0, d \equiv 1 \mod N$. $\mathbb{A}$ refers to the ring of adeles of $\mathbb{Q}$.

Our objective now is to present a classical automorphic form (which is, in particular, a function on $\Gamma_0(N) \setminus GL_2(\mathbb{R})^0$) as an automorphic form on $GL_2(A)$. We will first construct an isomorphism between $\Gamma_0(N) \setminus GL_2(\mathbb{R})^0$ and a quotient of $GL_2(A)$, and then demonstrate a way to pull back elements of $\mathcal{A}(\Gamma \setminus G, \chi, \omega)$ to automorphic forms on $GL_2(A)$ with suitably defined characters of $K_0(N)$ and $Z(GL_2(A))$, where $Z$ denotes the center of $GL_2$.

Lemma 6.1. (1) For $i = 0, 1$, $\Gamma_i(N) \setminus GL_2(\mathbb{R})^0$ and $GL_2(\mathbb{Q}) \setminus GL_2(A)/K_0(N)$ are isomorphic as $GL_2(\mathbb{R})^0$-spaces, under the map induced by $GL_2(\mathbb{R})^0 \hookrightarrow GL_2(A)$.

(2) The above induces an isomorphism of $SL_2(\mathbb{R})$-spaces $\Gamma_0(N) \setminus SL_2(\mathbb{R})$ and $Z(A)GL_2(\mathbb{Q}) \setminus GL_2(A)/K_0(N)$

Proof. Consider the determinant map $GL_2(\mathbb{Q}) \setminus GL_2(A) \to \mathbb{Q}^\times \mathbb{A}^\times$. Every fiber is represented by an element of $GL_2(\mathbb{R})$, and by the strong approximation Theorem 6.5 applied to the group $SL_2$, each fiber is acted upon transitively by $SL_2(\mathbb{R})S$, for any open subgroup $S$ of $SL_2(A_f)$. Therefore, the map of double coset spaces $GL_2(\mathbb{Q}) \setminus GL_2(A)/K_i(N)GL_2(\mathbb{R})^0 \to \mathbb{Q}^\times \mathbb{A}^\times/\text{det}(K_i(N))\mathbb{R}_+^*$ is a bijection. The right hand side represents the narrow class group of $\mathbb{Q}$, hence has only one element. Therefore, $GL_2(A) = GL_2(\mathbb{Q})GL_2(\mathbb{R})^0K_i(N)$.

We obtain a surjection $f : GL_2(\mathbb{R})^0 \to GL_2(\mathbb{Q}) \setminus GL_2(A)/K_i(N)$. Suppose $f(g) = f(g')$. Then, $g = \gamma g'k$ for some $\gamma \in GL_2(\mathbb{Q}), k \in K_i(N)$. Writing $\gamma = \gamma_f \gamma_{\infty}$ (where
\( \gamma_f \) corresponds to the part in finite adeles and \( \gamma_\infty \) is the part corresponding to archimedean places. \( g = \gamma_f k_\infty g' = \gamma_\infty g' \). Therefore, \( \gamma_f = k^{-1} \in K_i(N) \) and \( \gamma_\infty = g' q^{-1} \) has positive determinant. So, \( \gamma = \gamma_f \gamma_\infty \in GL_2(\mathbb{Q}) \cap GL_2(\mathbb{R})^0 K_i(N) = \Gamma_i(N) \Rightarrow g \in \Gamma_i(N) g' \). We get the first isomorphism in the statement of the proposition.

Taking quotients by \( Z(\mathbb{R}^+) \) on both sides, and using the fact that \( Z(\mathbb{A}) = Z(\mathbb{R}^+) Z(\mathbb{Q}) (Z(\mathbb{A}) \cap K_0(N)) \) (again by the triviality of the narrow class group), we get the second statement of the proposition.

Next, we explain how to produce from \( \chi \) and \( \omega \) characters \( \lambda \) and \( \tilde{\omega} \) of \( K_0(N) \) and \( Z(GL_2(\mathbb{A})) \) respectively. For this, we use again the fact that

\[
\mathbb{A}^\times / \mathbb{Q}^\times \cong \mathbb{R}^*_+ \prod_{p<\infty} \mathbb{Z}^\times_p,
\]

as in the proof of Lemma 6.1.

To construct a character \( \lambda : K_0(N) \to \mathbb{C}^\times \), we proceed as follows. Projection to the places dividing \( N \) and the chinese remainder theorem gives us \( \mathbb{A}^\times / \mathbb{Q}^\times \cong \mathbb{R}^*_+ \prod_{p<\infty} \mathbb{Z}^\times_p \to \prod_{p\mid N} \mathbb{Z}^\times_p \to (\mathbb{Z}/N\mathbb{Z})^\times \). Composing this projection with \( \chi \), we get a character \( \tilde{\chi} : \mathbb{A}^\times / \mathbb{Q}^\times \to \mathbb{C}^\times \). Let \( \rho : K_0(N) \to \mathbb{A}^\times / \mathbb{Q}^\times \) be the map given by sending \( \left( \left( \begin{array}{cc} a_p & b_p \\ c_p & d_p \end{array} \right) \right)_p \to (d_p)_p \). Define the character \( \lambda \) to be \( \tilde{\chi}^{-1} \circ \rho \). Observe that

if \( l \) is a prime not dividing \( N \), \( l = (l)_p \leq 1 \in \mathbb{R}^*_+ \prod_{p} \mathbb{Z}^\times_p \), where for \( p \neq l \), \( a_p = l^{-1} \) and \( a_l = 1 \). Therefore, \( l \) projects to \( 1^{-1} \in (\mathbb{Z}/N\mathbb{Z})^\times \). By multiplicativity, any \( d \) coprime to \( N \) projects to \( \bar{d}^{-1} \in (\mathbb{Z}/N\mathbb{Z})^\times \). Composing this map with \( \chi \), we observe that \( \tilde{\chi}(d) = \chi^{-1}(\bar{d}) \). If \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K_0(N) \) such that \( a, b, c, d \in \mathbb{Z} \),

then the above discussion shows that \( \lambda(\gamma^{-1}) = \tilde{\chi}^{-1}(\rho(\gamma^{-1})) = \chi(\bar{d}) \).

The central character \( \tilde{\omega} \) is given as follows: \( Z(GL_2(\mathbb{A})) \cong \mathbb{A}^\times \to \mathbb{A}^\times / \mathbb{Q}^\times \cong \mathbb{R}^*_+ \prod_{p<\infty} \mathbb{Z}^\times_p \to \mathbb{C}^\times \). Here \( \mu \) is the map sending \( (a_p)_p \leq 1 \to \omega(a_\infty) \chi^{-1}(\{a_\infty\}) \).

Proposition 6.2. Let \( \tilde{\omega} \) be as defined above. The isomorphisms of Lemma 6.1 give rise to an inclusion of \( \mathcal{A}(\Gamma \backslash G, \chi, \omega) \) into the set of automorphic forms on \( GL_2(\mathbb{A}) \) with central quasi-character \( \tilde{\omega} \).

Proof. Let \( \lambda : K_0(N) \to \mathbb{C}^\times \) be as defined above. Let \( F \in \mathcal{A}(\Gamma \backslash G, \chi, \omega) \). Let \( g \in GL_2(\mathbb{A}) \). We have then that \( g = \gamma g_\infty k \) for some \( \gamma \in GL_2(\mathbb{Q}) \), and \( k \in K_0(N) \).

Consider the function \( \phi : g \mapsto F(g_\infty) \lambda(k) \). To show this is well-defined, we need to show that if \( g'_\infty = \gamma g_\infty k \) then \( F(g'_\infty) = F(g_\infty) \lambda(k) \).

Notice that \( g'_\infty = \gamma g_\infty k \Rightarrow g'_\infty = \gamma f_\infty g_\infty k \) (writing \( \gamma \) as a product of the finite part and the archimedean part). Let \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \). Therefore, \( g'_\infty = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) g_\infty \) and \( k = \gamma^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1}_p \). This implies that \( F(g'_\infty) = F(g_\infty) \chi(d) \), and now all we have to show is that \( \chi(d) = \lambda((\gamma^{-1})_p)_p \chi^{-1} \). But this follows from the construction \( \lambda \).

\( Z(GL_2(\mathbb{A})) \cong \mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}^*_+ \prod_p \mathbb{Z}^\times_p \) can be verified to be acting via the quasi-character \( \tilde{\omega} \) by computing separately the action of \( \mathbb{Q}^\times, \mathbb{R}^*_+ \) and \( \prod_p \mathbb{Z}^\times_p \). Let \( g \) again be equal to \( g = \gamma g_\infty k \) for some \( \gamma \in GL_2(\mathbb{Q}) \), and \( k \in K_0(N) \). For
\[ z = (z)_p \leq \infty \in \mathbb{Q}^\times, \phi(zg) = F(g_\infty)\lambda(k) = \tilde{\omega}(z)\phi(g). \] For \( z_\infty \in \mathbb{R}_+, \phi(z_\infty g) = F(z_\infty g_\infty)\lambda(k) = \omega(z_\infty)F(g_\infty)\lambda(k) = \omega(z)\phi(g). \] For \( z = (z_p)_p \in \prod_p \mathbb{Z}_p^\times, \phi(zg) = F(g_\infty)\lambda(k)\lambda((z_p)_{p<\infty}) = \phi(g)\tilde{\chi}^{-1}(z_p) = \tilde{\omega}(z)\phi(g). \]

**Proposition 6.3.** Suppose \( p \nmid N \). If \( F \in \mathcal{A}(\Gamma\backslash G, \chi, \omega) \) is an eigenfunction of the classical Hecke operator \( T_p \), then it is an eigenfunction of the measure given by the characteristic function of \( GL_2(\mathbb{Z}_p)\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) \), with the same eigenvalue. Furthermore, \( F \) is an eigenfunction of (the characteristic function of) \( GL_2(\mathbb{Z}_p)\begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p) \) with eigenvalue \( \chi(p) \).

**Proof.** [Discussion of classical Hecke operators to be added.]

7. Other chapters

(1) Introduction
(2) Basic Representation Theory
(3) Representations of compact groups
(4) Lie groups and Lie algebras; general properties
(5) Structure of finite-dimensional Lie algebras
(6) Verma modules
(7) Linear algebraic groups
(8) Forms and covers of reductive groups, and the \( L \)-group
(9) Galois cohomology of linear algebraic groups
(10) Representations of reductive groups over local fields
(11) Plancherel formula: reduction to discrete spectra
(12) Construction of discrete series
(13) The automorphic space
(14) Automorphic forms
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