Hamiltonian spaces and periods of automorphic forms

Yiannis Sakellaridis, Johns Hopkins University.
Joint with David Ben-Zvi (UT Austin) and Akshay Venkatesh (IAS).

45th KAST International Symposium, February 26, 2021.

Outline

1. Periods

2. Hamiltonian spaces

3. Examples

4. Summary and trailer

5. Geometrization/categorification

Riemann: $\int_0^\infty y^{\frac{s}{2}} \sum_{n=1}^\infty e^{-n^2\pi y} dx = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, and proof of functional equation based on symmetry of the theta series:

$$\theta(y) = \sum_{n=1}^{\infty} e^{-\pi n^2 y} = y^{-\frac{1}{2}}\theta(y^{-1}).$$

I am going to use the language of adeles.

$$\text{For } \mathbb{Q}: \quad \mathbb{A} = (\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}) \times \mathbb{R} = \prod_p{}' \mathbb{Q}_p \times \mathbb{R}.$$

$$\Phi = \bigotimes_{p<\infty} 1_{\mathbb{Z}_p} \otimes \Phi_\infty, \ \Phi_\infty \in \mathcal{S}(\mathbb{R}), \text{ then}$$

$$\sum_{q\in\mathbb{Q}} \Phi(q) = \sum_{\substack{q\in\mathbb{Q} \\ \forall p<\infty, q_p\in\mathbb{Z}_p}} \Phi_\infty(q) = \sum_{n\in\mathbb{Z}} \Phi_\infty(n)$$

but $\Phi_r = 1_{r^i\mathbb{Z}_r}$ for a prime r would correspond to $\sum_{n\in r^i\mathbb{Z}} \Phi_\infty(n)$.

$$\theta(y) = \sum_{n=1}^{\infty} e^{-\pi n^2 y} = y^{-\frac{1}{2}}\theta(y^{-1}).$$

Iwasawa–Tate reformulation as ($x \leftrightarrow \sqrt{y}$, $F = \mathbb{Q}$)

$$\int_{F^\times \backslash \mathbb{A}^\times} \chi(x) \sum_{\gamma \in F^\times} \Phi(\gamma x) d^\times x = L(\chi, s) \text{ for suitable choice of } \Phi,$$

where the critical local calculation is that, for $\Phi_v = 1_{\mathfrak{o}_v}$,
$\chi_v : F_v^\times / \mathfrak{o}_v^\times \to \mathbb{C}$,

$$\int_{k_v^\times} \chi_v(x)\Phi_v(x) d^\times x = \sum_{i=0}^{\infty} \chi_v(\varpi^i) = \frac{1}{1 - \chi_v(\varpi)} = L_v(\chi_v, 0).$$

Very soon we will switch to function fields.

$$\text{For a function field } F = \mathbb{F}(C) : \quad \mathbb{A} = \prod_{v \in |C|}{}' F_v,$$

$$F^\times \backslash \mathbb{A}^\times / \prod_v \mathfrak{o}_v^\times = \mathrm{Pic}(C)(\mathbb{F}).$$

Notation: For an algebraic group G over F, $[G] = G(F)\backslash G(\mathbb{A})$, and $K = \prod_v G(\mathfrak{o}_v)$.

E.g., for $G = \mathbb{G}_m = \mathrm{GL}_1$, $[G]/K = \mathrm{Pic}(C)$.

In this case, with $\Phi = \prod_v 1_{\mathfrak{o}_v}$, the function $\Theta_\Phi(L) = \sum_{\gamma \in F^\times} \Phi(\gamma L) \in C^\infty([\mathbb{G}_m]/K)$ has the meaning (thinking of L as a line bundle on the curve C)

$$\Theta_\Phi(L) = \#H^0(C, L)$$

Indeed, representing L by a divisor in $\mathbb{A}^\times/K$, its sections are those meromorphic functions $\gamma \in F^\times$ such that $\mathrm{val}_v(\gamma L) \geq 0$ at all v.

There is a universe of examples of "theta series" that pair with automorphic forms to produce L-functions. Before we proceed, let us remember that an L-function for an automorphic representation π with (conjectural, in the number field case) Langlands parameter

$$\phi : \mathcal{W}_F \to {}^L G$$

is determined by a finite-dimensional representation ρ of its Langlands dual group

$$\rho : {}^L G \to \mathrm{GL}(V)$$

gives

$$L(\pi, \rho, s) = \prod_v L_v(\pi_v, \rho, s) = \prod_v L_v(\rho \circ \phi_v, s).$$

Moreover, the parameter s is a red herring: we can write $L(\pi, \rho, s) = L(\pi, \rho', 0)$ by setting $\rho' =$ the product of ρ by

$${}^L G = \check{G} \rtimes \mathcal{W}_F \to \mathcal{W}_F \xrightarrow{|\bullet|^s} \mathbb{C}^\times \hookrightarrow \mathrm{GL}(V).$$

Therefore, we will often just write $L(\pi, \rho)$ for $L(\pi, \rho, 0)$, but depending on ρ, this may actually correspond what one usually considers as an L-function at a different s.

Outline

1. Periods

2. Hamiltonian spaces

3. Examples

4. Summary and trailer

5. Geometrization/categorification

Introducing a "philosophy" of where to find integrals representing automorphic L-functions.

Quantization of suitable Hamiltonian G-spaces M.

"Suitable":

- M is *smooth* and *affine*. One should be able to relax both assumptions, but not the object of this work.
- M has symplectic structure, and a G-action by symplectomorphisms.
- M is Hamiltonian, i.e., the $\mathfrak{g}$-action is by Hamiltonian vector fields induced from a moment map $M \xrightarrow{\mu} \mathfrak{g}^*$.
  Basic examples: $M = T^*X$ for a G-space X, or $M =$ a symplectic representation of G.
- M has a *multiplicity-free property*, "coisotropic": the Poisson algebra $F[M]^G$ is commutative.
  Example: If $M = T^*X$, then M is coisotropic $\iff$ X is *spherical*, i.e., $F[X]$ is a multiplicity-free sum of (highest weight) G-modules.
- Some technical properties, to ensure that our space is sufficiently

What is “quantization”?

- Locally, produce a unitary representation ω_v of $G(F_v)$ out of M. Functions in $F[M]$ deform to operators on ω_v, and their Poisson bracket deforms to commutator of operators.
- A basic vector $\Phi_v^0 \in \omega_v$ for almost all v.
- Globally, an automorphic realization $\omega := \bigotimes'_v \omega_v^\infty \to C^\infty([G])$.

Basic examples of quantization:

- $M =$ a symplectic representation, then $\omega_v =$ the Weil/oscillator representation. Schrödinger model (ignoring half-density twists throughout):

$$X \subset M \text{ a Lagrangian}, \omega_v = L^2(X(F_v)),$$

with right translations by $G(F_v)$, if X is G-stable, otherwise involving Fourier transforms, and may need to switch from G to a metaplectic cover $\tilde{G}$ (will ignore, mostly).
It is known that this comes with an automorphic realization

$$\omega \to C^\infty([\widetilde{Sp}]) \xrightarrow{\text{restr.}} C^\infty([G]),$$

which on the Schrödinger model is just the theta series

$$\mathcal{S}(X(\mathbb{A})) \ni \Phi \mapsto \Theta_\Phi(g) = \sum_{\gamma \in X(F)} (\omega(g)\Phi)(\gamma) \in C^\infty([G]).$$

- $M = T^*X$ for some G-space X, then $\omega_v = L^2(X(F_v))$, $\Phi_v^0 = 1_{X(\mathfrak{o}_v)}$, and $\Theta_\Phi = \sum_{\gamma \in X(F)} \Phi(\gamma g) \in C^\infty([G])$.

When $\Phi = \bigotimes_v \Phi_v^0$, where $\Phi_v^0 = 1_{X(\mathfrak{o}_v)}$, for all v (function field case), we'll be writing Θ_M or Θ_X for Θ_Φ.

Questions one can ask about theta series:

1. Evaluate the integral above against a suitably chosen automorphic form $f \in \pi$:

$$\langle \Theta_\Phi, f \rangle = \int_{[G]} f(g) \Theta_\Phi(g) dg.$$

2. Evaluate the L^2-norm of the projection of Θ_Φ to π

$$\left\langle \Theta_\Phi, \overline{\Theta_\Phi} \right\rangle_\pi .$$

   This is the same as summing $| \langle \Theta_\Phi, f \rangle |^2$ over all f in an ON basis of π.
   The pairing $\left\langle \Theta_\Phi, \overline{\Theta_\Phi} \right\rangle$ and its spectral decomposition are called the *relative trace formula* (Jacquet).

We expect to see L-values!

Themes to be explored:

- Examples of quantization. Which familiar constructions in the theory of automorphic forms does it encode?
- The Euler factorization principle: Relations between global periods and the local unitary structure (Plancherel formula).
- Relations with L-functions. (Mostly in the talk by A. Venkatesh.)
- Geometrization/categorification of periods/theta series.

Outline

1. Periods

2. Hamiltonian spaces

3. Examples

4. Summary and trailer

5. Geometrization/categorification

The linear paradigm: $M =$ a symplectic G-space.

- Generalizing Tate, Godement–Jacquet used $X = \mathrm{Mat}_n$ under the action of $G = (\mathrm{GL}_n \times \mathrm{GL}_n)/\mathbb{G}_m$ (set $[G] = G(F)\backslash G(\mathbb{A})$), $f_1 \otimes f_2 \in \tilde{\tau} \otimes \tau$ a cusp form,

$$\int_{[G]} f_1(g_1) f_2(g_2) \sum_{\gamma \in \mathrm{Mat}_n(F)} \Phi(g_1^{-1}\gamma g_2) \left| \frac{\det g_2}{\det g_1} \right|^{\frac{n}{2}} d(g_1, g_2) =$$

$$\int_{\mathrm{GL}_n(\mathbb{A})} \langle f_1, \tau(x) f_2 \rangle \, \Phi(x) |\det x|^{\frac{n}{2}} dx = L(\tau, \mathrm{Std}, \frac{1-n}{2} \frac{1}{2})$$

(for suitable Φ and f).

- The theta correspondence (Howe duality):
  $G = G_1 \times G_2 \hookrightarrow \mathrm{Sp}(M)$ a dual pair (ignore metaplectic covers).
  Weil representation $\omega = \otimes'_v \omega_v$ of $\mathrm{Sp}(M)(\mathbb{A})$,
  $\Phi \in \omega$, theta series Θ_Φ.
  Rallis inner product formula, say for $G_1 = \mathrm{SO}_{2n}$, $G_2 = \mathrm{Sp}_{2n}$:
  $\pi = \tau \otimes \theta(\tau)$, $\Phi \in \omega$, then

  $$\langle \Theta_\Phi, \Theta_\Phi \rangle_\pi = \prod_v \langle \Phi_v, \Phi_v \rangle_{\pi_v},$$

  where

  $$\langle \Phi_v, \Phi_v \rangle_{\omega_v} = \int \langle \Phi_v, \Phi_v \rangle_{\pi_v} \mu(\tau_v)$$

  is the Plancherel formula for ω_v (with respect to the Plancherel measure $\mu(\pi_v)$ on $\widehat{SO_{2n}(F_v)}$).
  Moreover

  $$\langle \Phi_v, \Phi_v \rangle_{\pi_v} = L_v(\tau_v, \mathrm{Std}, \frac{1}{2})$$

  when $\Phi_v = \Phi_v^0$, the basic vector. (Omitting factors that don't depend on the representation!)

The period paradigm

•Hecke: $X = \mathrm{GL}_2$, $G = \mathrm{GL}_1 \times \mathrm{GL}_2$, $\chi \otimes \pi \ni \chi \otimes f$,

$$\int_{[\mathbb{G}_m]} \chi(a) f \begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a = L(\chi \otimes \pi, \mathrm{Std}, \frac{1}{2}) \text{ for suitable choice of } f.$$

This can also be written as an integral against a theta series. E.g., over function fields, f everywhere unramified, there is a canonical choice $\Phi = \prod_v 1_{X(\mathfrak{o}_v)}$. The associated theta series will be denoted by Θ_X,

$$\Theta_X(g) = \sum_{\gamma \in X(F)} \Phi(\gamma g) \in C^\infty([G]),$$

where $[G] = G(F) \backslash G(\mathbb{A})$, and we have

$$\int_{[\mathbb{G}_m]} \chi(a) f \begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a = \int_{[G]} \chi(a) f(g) \Theta_X(a, g) d(a, g).$$

$$\int_{[\mathbb{G}_m]} \chi(a) f \begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a = \int_{[G]} \chi(a) f(g) \Theta_X(a,g) d(a,g) = L(\chi \otimes \pi, \mathrm{Std}, \frac{1}{2}).$$

In this case, $[G]/K = \mathrm{Pic}(C) \times \mathrm{Bun}_2(C) =$ pairs (line bundle, rank 2 vector bundle) on C,
and $\Theta_X(L, V)$ counts the number of morphisms $L \to V$.

Remark: For the formula above to hold, we need a normalization of f with L^2-norm $\langle f, \bar{f} \rangle_{[G]} = L(\pi, \mathrm{Ad}, 1)$. When $\|f\|^2 = 1$, we get

$$\int_{[\mathbb{G}_m]} \chi(a) f \begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a = \frac{L(\chi \otimes \pi, \mathrm{Std}, \frac{1}{2})}{\sqrt{L(\pi, \mathrm{Ad}, 1)}}$$

instead.

•Generalizing this, Waldspurger proved (for $X = T\backslash(T \times \mathrm{GL}_2) = \mathrm{SO}_2\backslash\mathrm{SO}_2 \times \mathrm{GSO}_3$, $T \hookrightarrow \mathrm{GL}_2$ a non-split 1-dimensional torus, and much later Gross–Prasad conjectured (for $X = \mathrm{SO}_n\backslash(\mathrm{SO}_n \times \mathrm{SO}_{n+1}) = H\backslash G$), followed by much more general conjectures by Gan–Gross–Prasad, that

$$\left|\int_{[H]} f(h)dh\right|^2 = \frac{1}{|S_\phi|}\frac{L(\pi, \otimes, \frac{1}{2})}{L(\pi, \mathrm{Ad}, 1)},$$

again for suitable $f \in \pi$.

The exact conjecture is actually due to Ichino–Ikeda: for arbitrary $f \in \pi = \otimes'_v \pi_v$, writing $f = \otimes_v f_v$, the RHS is

$$\frac{1}{|S_\phi|}\prod_v^* \int_{H(k_v)} \langle \pi(h) f_v, f_v \rangle \, dh_v.$$

Translation to theta series: Again, we can replace the above integrals by pairings against theta series

$$\Theta_\Phi(g) = \sum_{\gamma \in X(F)} \Phi(\gamma g) \in C^\infty([G]),$$

where $\Phi \in \mathcal{S}(X(\mathbb{A}))$. Over function fields, $\Phi = \prod_v 1_{X(\mathfrak{o}_v)}$ (write $\Theta_\Phi = \Theta_X$ for this specific Φ), f unramified,

$$\int_{[H]} f(h)dh = \int_{[G]} f(g)\Theta_X(g)dg.$$

The II conjecture states that

$$\langle \Theta_X, \overline{\Theta_X} \rangle_\pi = \frac{1}{|S_\phi|} \times \prod_v^* \text{Plancherel density } J_{\pi_v}(\Phi_v),$$

where

$$\langle \Phi_v, \overline{\Phi_v} \rangle_{L^2(X(F_v))} = \int J_{\pi_v}(\Phi_v)\mu(\pi_v),$$

with $\mu(\pi_v)$ the Plancherel measure of G.

Combining the "period" and the "linear" paradigms:

$M = T^*X$, with X a G-variety or a (not necessarily G-stable) Lagrangian in a symplectic vector space.

$\rightsquigarrow$ unitary representations $\omega_v = L^2(X(F_v))$ at every place v.

The Plancherel decomposition

$$\langle \Phi_v, \overline{\Phi_v} \rangle_{\omega_v} = \int J_{\pi_v}(\Phi_v)$$

involves the unitary dual of G or a different group. In the Langlands philosophy, we should be thinking of subgroups of the Langlands dual group of G, e.g.:

- $G = \mathrm{GL}_n \times \mathrm{GL}_n$, $X = \mathrm{Mat}_n$, then $\check{G} = \mathrm{GL}_n \times \mathrm{GL}_n$, but only $\pi = \tilde{\tau} \otimes \tau$ appearing, corresponds to

$$\check{G}_M := \mathrm{GL}_n \hookrightarrow \check{G},$$

embedded as $g \mapsto (g^C, g)$, with g^C the Chevalley involution (up to conjugacy, ${}^t g^{-1}$).

- $G = \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}$, $M =$ a symplectic vector space, $\omega_v =$ the Weil representation, then $\check{G} = \mathrm{SO}_{2n} \times \mathrm{SO}_{2n+1}$, but only $\pi = \tau \otimes \theta(\tau)$ appears, so $\check{G}_M = \mathrm{SO}_{2n} \hookrightarrow \check{G}$, embedded "diagonally".
- $G = \mathrm{SO}_{2n} \times \mathrm{SO}_{2n+1}$, $X = \mathrm{SO}_{2n} \backslash G$, then $\check{G} = \mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}$, and all tempered representations of G appear (rather: all tempered L-packets — local GGP conjecture proven by Waldspurger, Moeglin). Hence, $\check{G}_M = \check{G}$.

The relative trace formula and Ichino–Ikeda conjecture, in all cases: For a (suitable) automorphic representation π,

$$\left\langle \Theta_\Phi, \overline{\Theta_\Phi} \right\rangle_\pi = ? \cdot \prod_v \left\langle \Phi_v, \overline{\Phi_v} \right\rangle_{\pi_v}.$$

Other types of "periods":

- The Whittaker case:
  Let $N \subset G$ the unipotent radical of a Borel subgroup, $f : N \to \mathbb{G}_a$ a generic additive character (e.g., for GL_n, sum of entries just above the diagonal), use the same letter for its differential, $f \in n^*$ or for a nilpotent lift $f \in \mathfrak{g}^*$, and let ψ be a composition $[N] \to [\mathbb{G}_a] \to \mathbb{C}^\times$.

$$M = T^*G /\!/_f N = (f + \mathfrak{n}^\perp) \times^N G$$

quantizes to

$$\omega_v = L^2(N(F_v)\backslash G(F_v), \psi_v), \text{ the Whittaker model.}$$

Its theta series $\Theta_\Phi = \sum_{\gamma \in N\backslash G(F)} \Phi(\gamma g)$ is the well-known *Poincaré series*.
For $\Phi =$ the basic function at every place,

$$\langle \Theta_\Phi, \Theta_\Phi \rangle_\pi = \frac{1}{|S_\phi|} \frac{1}{L(\pi, \mathrm{Ad}, 1)}$$

(conjecturally, known for GL_n and some other groups.)

The Whittaker case is a case where the minimal nilpotent orbit in the image of the moment map $M \to \mathfrak{g}^*$ is not $\{0\}$, but the orbit of $f \in \mathfrak{g}^*$. In general, our assumptions imply that $\mathbb{G}_m \times G$ has a unique closed orbit $M_0 \subset M$ with nilpotent image $\mathcal{O} = fG \subset \mathfrak{g}^*$, and that (at least up to equality of formal neighborhoods of M_0),

$$M = S \times^{HU}_{(\mathfrak{h}\oplus\mathfrak{u}_+)^*} T^*G,$$

where:

- H is a reductive subgroup (the stabilizer of a point on M_0, with nilpotent image $f \in \mathfrak{g}^*$);
- S is a symplectic H-representation;
- U is the positive unipotent radical (= h acts with weights ≥ 1) of a parabolic attached to an $\mathfrak{sl}_2$-triple (h, e, f) containing f.
- $U_+ \subset U$ is the subgroup where h acts with weights ≥ 2, so f defines a character $U_+ \to \mathbb{G}_a$, and U/U_+ is a symplectic vector space; thus $U \times^{U_+} \mathbb{G}_a$ is a Heisenberg group.
- $S \to \mathfrak{h}^*$ is the moment map, and $S \to \mathfrak{u}_+^*$ has image f.

$$M = S \times^{HU}_{(\mathfrak{h}\oplus\mathfrak{u}_+)^*} T^*G,$$

Quantizations of such M comprise a mixture of the cases that we've seen:

- the linear case $H = G$, $M = S$;
- the period/homogeneous case $M = T^*(H\backslash G)$;
- the Whittaker case, $f \neq 0$ and $U = U_+$;

plus one more: when $U_+ \neq U$, we need to include a Weil representation of the Heisenberg group, e.g.

Fourier–Jacobi models (studied by Gan–Gross–Prasad):

$$\mathrm{Ind}^{\widetilde{\mathrm{Sp}}(W')}_{\widetilde{\mathrm{Sp}}(W)\ltimes U}\omega_\psi,$$

where W is a symplectic space, $W' = W \oplus l \oplus l'$ is its sum with a 2-dimensional symplectic space, $U =$ the unipotent radical of the parabolic stabilizing the isotropic subspace (line) l, and ω_ψ the oscillator representation associated to $l^\vee \otimes W$.

Outline

1. Periods

2. Hamiltonian spaces

3. Examples

4. Summary and trailer

5. Geometrization/categorification

What we have seen:

- General multiplicity-free, smooth affine Hamiltonian spaces

$$M = S \times^{HU}_{(\mathfrak{h}\oplus\mathfrak{u}_+)^*} T^*G,$$

admit quantizations covering a broad range of "periods" in the theory of automorphic forms.
- Quantization consists of local unitary representations ω_v and global theta series $\Theta_M \in C^\infty([G])$.
- Conjecturally, the Plancherel decomposition of ω_v entails Langlands parameters from a different dual group $\check{G}_X \subset \check{G}$:

$$\left\langle \Phi_v, \overline{\Phi_v} \right\rangle = \int_{\widehat{G_X}} J_{\pi_v}(\Phi_v)\mu(\pi_v).$$

- Conjecturally, the components of the spectral decomposition of

$$\left\langle \Theta_M, \overline{\Theta_M} \right\rangle \ , \quad \text{(relative trace formula)}$$

admit Euler products

$$\left\langle \Theta_M, \overline{\Theta_M} \right\rangle =? \cdot \prod_v J_{\pi_v}(\Phi_v^0).$$

- Somehow, the local factors $J_{\pi_v}(\Phi_v^0)$ turn out to be local L-factors.

What we will see:

- On the "spectral side" (Langlands parameters) there is a similar Hamiltonian space $\check{M}$ "explaining" these phenomena.
- The role of H is played by the dual group $\check{G}_X \subset \check{G}$, so

$$\check{M} = \check{S} \times^{\check{G}_X \check{U}}_{\check{\mathfrak{g}}_X \oplus \check{\mathfrak{u}}_+} T^*\check{G}.$$

- The $\mathfrak{sl}_2$-triple $(\check{h}, \check{e}, \check{f})$ giving rise to $\check{U}, \check{U}_+$ is one canonically associated to the space M, as follows (up to now, swept under the rug):

- It is not *Langlands* parameters conjecturally entering the Plancherel decomposition of ω_v, but *Arthur* parameters, associated to a unique conjugacy class of $\mathfrak{sl}_2 \subset \check{\mathfrak{g}}$, e.g.:
  $M = T^*(\mathrm{pt}) = \mathrm{pt} \Rightarrow \omega_v =$ the trivial representation, $\check{G}_X = 1$, but the trivial representation has non-trivial Langlands parameter.
  However, it has *Arthur* parameter with $\mathfrak{sl}_2 \hookrightarrow \check{\mathfrak{g}}$ principal.
  For a principal $\mathfrak{sl}_2$-triple $(\check{h}, \check{e}, \check{f})$, we obtain $\check{U} = \check{U}_+ = \check{N}$ (= maximal unipotent in $\check{G}$), and

$$\check{M} = (\check{f} + \check{\mathfrak{n}}^\perp) \times^N \check{G}$$

  the cotangent space of the Whittaker model of $\check{G}$!
  Hence, the dual of (trivial period for G) is (Whittaker period for $\check{G}$).
- We'll see that this duality is involutive, i.e.,

  the dual of (trivial period for G) is (Whittaker period for $\check{G}$).

Outline

1. Periods

2. Hamiltonian spaces

3. Examples

4. Summary and trailer

5. Geometrization/categorification

To formulate conjectures about this duality, we need to switch to the categorical setting of geometric Langlands.

Local story: Derived endomorphisms and the Plancherel formula

Now let $\mathbb{F} = \overline{\mathbb{F}_q}$, $F = \mathbb{F}((t)) \supset \mathfrak{o} = \mathbb{F}[[t]]$, and for an affine variety X think of $X(F) = LX(\mathbb{F})$, $X(\mathfrak{o}) = L^+(\mathbb{F})$, points of the loop and the arc space.

Set $\mathrm{Shv}(LX/L^+G)$ denote an appropriate — bounded DG – category of L^+G-equivariant "sheaves" on LX — should be l-adic for translation to functions, but once we abstract from functions one can also take $\mathbb{F} = \mathbb{C}$ and work with D-modules. Let k: coefficient field, characteristic 0.

Sheaf–function dictionary: In an l-adic setting, the inner product of functions should be obtained as Frobenius trace of derived homomorphisms (Ext) of sheaves:

Given two l-adic sheaves $\mathcal{F}, \mathcal{G}$ on an $\mathbb{F}_q$-variety Y, let f and $g^\vee$ be the trace functions associated to respectively $\mathcal{F}$ and $D\mathcal{G}$, with D the Verdier dual. Then

$$\sum_{Y(\mathbb{F}_q)} f(y) g^\vee(y) = \operatorname{tr}(\operatorname{Fr}_q, \operatorname{Hom}(\mathcal{F}, \mathcal{G})^\vee).$$

The basic function corresponds to the (Verdier self-dual, for appropriate normalization) constant sheaf $\underline{k}_{L^+G}$.

Therefore, our goal will be to analyze $\operatorname{End}(\underline{k}_{L^+G})$, and relate it to L-functions (in lecture of David Ben-Zvi).

Global story: The period sheaf

(Ignoring half-twists.)

$F = \mathbb{F}(C)$.

Recall, Iwasawa–Tate case: $X = \mathbb{A}^1$, $G = \mathbb{G}_m$.

$\Phi = \prod_v 1_{\mathfrak{o}_v} \rightsquigarrow \Theta_X \in C^\infty(\mathrm{Pic}(C)(\mathbb{F}))$,

with $\Theta_X(L) = \#H^0(C, L)$.

Geometrically, $\mathrm{Pic}(C) = \mathrm{Maps}(C, \mathrm{pt}/G)$.

Let $\mathrm{Pic}^X(C) = \mathrm{Maps}(C, \mathbb{A}^1/G)$, the stack classifying $L \in \mathrm{Pic}(C)$ together with section $C \to L$.

$\pi : \mathrm{Pic}^X(C) \to \mathrm{Pic}(C)$ forgetful.

Then Θ_X is the trace of Frobenius on $\mathcal{P}_X = \pi_! \underline{k}$.

This is the *period sheaf*.

Similarly, for a G-space X, we have

$$\pi : \mathrm{Bun}_G^X = \mathrm{Maps}(C, X/G) \to \mathrm{Bun}_G = \mathrm{Maps}(C, \mathrm{pt}/G).$$

And Θ_X is the trace of Frobenius on $\mathcal{P}_X = \pi_! \underline{k}$.

(If X is not G-stable, but a Lagrangian in the symplectic G-space M, the generalization of this construction has been explained in papers of Lysenko.)

Questions on the period sheaf (in lecture of Akshay Venkatesh):

1. Evaluate its pairing with Hecke eigensheaves $\mathcal{F}$:

$$\mathrm{Hom}(\mathcal{P}_X, \mathcal{F}) \quad \text{(derived homomorphisms, i.e., Ext).}$$

   The Frobenius trace on this corresponds to

$$\left\langle \Theta_X, f^\vee \right\rangle, \quad f^\vee :$$

   the automorphic form associated to the Verdier dual of $\mathcal{F}$.

2. Evaluate the (derived) endomorphisms of $\mathcal{P}_X$:

$$\mathrm{End}(\mathcal{P}_X)$$

   (corresponding, roughly, to the "relative trace formula" inner product $\langle \Theta_X, \Theta_X \rangle$).

In the lectures by Venkatesh and Ben-Zvi, you will see how to conjecturaly answer these questions on the spectral side (using the Hamiltonian $\check{G}$-space $\check{M}$), in a way that recovers the relevant L-functions.

Thank you! 감사합니다!