COMPATIBLE COMPLEX STRUCTURES ON SYMPLECTIC RATIONAL RULED SURFACES

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In fond memory of Raoul Bott

ABSTRACT. In this paper we study the topology of the space $\mathcal{I}_\omega$ of complex structures compatible with a fixed symplectic form ω, using the framework of Donaldson. By comparing our analysis of the space $\mathcal{I}_\omega$ with results of McDuff on the space $\mathcal{J}_\omega$ of compatible almost complex structures on rational ruled surfaces, we find that $\mathcal{I}_\omega$ is contractible in this case.

We then apply this result to study the topology of the symplectomorphism group of a rational ruled surface, extending results of Abreu and McDuff.

1. INTRODUCTION

The work of Gromov on J-holomorphic curves [Gr] has provided tools for understanding the topology of the group of symplectomorphisms of certain four-dimensional symplectic manifolds. Gromov began the study of the group of symplectomorphisms on $S^2 \times S^2$ with the standard symplectic form $\sigma \oplus \sigma$. He showed that this group is homotopy equivalent to a semidirect product of $\mathbb{Z}/2$ with $SO(3) \times SO(3)$. Later Abreu [Ab] continued this study by analyzing the structure of the group of symplectomorphism on $S^2 \times S^2$ with symplectic form $\omega = \lambda\sigma \oplus \sigma,\ 1 < \lambda \leq 2$. This work was extended by Abreu-McDuff [AM] to arbitrary λ, leading to a complete calculation of the rational cohomology ring of the classifying space of the symplectomorphism group (modulo a mistake which is corrected below in Theorem 1.3). There are corresponding results for the structure of the symplectomorphism group of the non trivial S^2-bundle over S^2 (for an arbitrary symplectic form). The basic idea in the work mentioned above is to analyze the action of the symplectomorphism group on the contractible space of compatible almost complex structures $\mathcal{J}_\omega$.

In [Do] Donaldson showed that the action of the symplectomorphism group on $\mathcal{J}_\omega$ is Hamiltonian, and the moment map for this action is the Hermitian scalar curvature of the corresponding almost Kähler metric. He also showed that, restricted to the space $\mathcal{I}_\omega$ of compatible integrable structures, this action fits into the general framework of infinite dimensional geometric invariant theory, going back to Atiyah and Bott [AB1] (see [AK1] for more background). Therefore, in principle, the norm square of the moment map should induce a stratification of $\mathcal{I}_\omega$ with critical points being the extremal Kähler metrics compatible with the symplectic form. By work of Calabi [C1], for rational ruled surfaces these metrics correspond

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to a finite collection of Hirzebruch surfaces. In particular, this suggests that each stratum in $\mathcal{I}_\omega$ should be homotopy equivalent to a single orbit of the symplectomorphism group. A stratification of $\mathcal{J}_\omega$ with similar properties had been established by Abreu and studied in detail by McDuff [McD1]. This indicated that the hypothetical stratification on $\mathcal{I}_\omega$ determined by the moment map could be the one induced via the inclusion $\mathcal{I}_\omega \subset \mathcal{J}_\omega$.

We begin our study of the space $\mathcal{I}_\omega$ by analyzing the stratification induced by the above inclusion. We show that each stratum V contains an orbit of the symplectomorphism group, corresponding to a Hirzebruch surface F with a standard Kähler metric, which is weakly equivalent to it. The stratification of $\mathcal{I}_\omega$ has the advantage that its gluing data can be easily understood via Kodaira-Spencer deformation theory. By comparing our stratification of $\mathcal{I}_\omega$ with that of Abreu-McDuff, we prove:

Theorem 1.1. *The inclusion of the space of compatible integrable complex structures into the space of all compatible almost complex structures, $\mathcal{I}_\omega(M) \subset \mathcal{J}_\omega(M)$, is a weak homotopy equivalence for a rational ruled surface M. In particular, the space $\mathcal{I}_\omega(M)$ is weakly contractible.*

As far as we are aware, and besides the obvious case of real 2-dimensional surfaces, this is the first example where the topology of the space of compatible integrable complex structures on a symplectic manifold has been understood.

Recall that any rational ruled surface is diffeomorphic to either $S^2 \times S^2$, the trivial S^2-bundle over S^2, or $S^2 \tilde{\times} S^2$, the non trivial S^2-bundle over S^2. Work of Taubes, Liu-Li and Lalonde-McDuff (see [LM] for detailed references) implies that any symplectic form on one of these smooth manifolds is "standard", i.e. diffeomorphic to a scalar multiple of $\omega_\lambda = \lambda\sigma \oplus \sigma$, $1 \le \lambda \in \mathbb{R}$, on $S^2 \times S^2$, or to any chosen symplectic form ω_λ, $0 < \lambda \in \mathbb{R}$, on $S^2 \tilde{\times} S^2$ such that $[\omega_\lambda](E) = \lambda$ and $[\omega_\lambda](F) = 1$, where E denotes the homology class of the exceptional divisor under the natural identification $S^2 \tilde{\times} S^2 \cong \mathbb{P}^2 \sharp \overline{\mathbb{P}^2}$ and F denotes the homology class of the fiber.

Complex deformation theory gives us a good understanding of the way the strata of $\mathcal{I}_\omega$ glue and allows us to express the symplectomorphism group as an iterated homotopy pushout of certain compact subgroups (see Theorem 5.5). This, in turn immediately gives the integral cohomology groups of the symplectomorphism groups of rational ruled surfaces. Let G_λ denote the group of symplectomorphisms of $S^2 \times S^2$, with the symplectic form $\omega_\lambda = \lambda\sigma \oplus \sigma$ where $1 < \lambda \in \mathbb{R}$ lies between the integers $0 < \ell < \lambda \le \ell + 1$. We have:

Theorem 1.2. *The integral cohomology groups of BG_λ in the untwisted case are given by:*

$$H^*(BG_\lambda;\mathbb{Z}) = H^*(BSO(3) \times BSO(3));\mathbb{Z}) \oplus \bigoplus_{i=1}^{\ell} \Sigma^{4i-2} H^*(BS^1 \times BSO(3);\mathbb{Z}),$$

where Σ denotes the suspension of graded abelian groups.

The relevance of the compact Lie groups appearing in the previous statement will become clear in sections 3–5. If we work away from the prime 2, we can compute the ring structure:

Theorem 1.3. *The cohomology of BG_λ in the untwisted case with coefficients in the ring $R = \mathbb{Z}[1/2]$ is given by the the following free module over the ring $R[x, y]$ on generators*

b_i, a_j, $0 \leq i < l$, $0 \leq j \leq \ell$:

$$H^*(BG_\lambda; R) = R[x,y]\langle a_0, b_0, a_1, b_1, a_2, \dots a_l\rangle,$$

where $a_0 = 1$, the degree of the elements x, y is 4, the degree of b_k is $4k+2$, and that of a_k is $4k$. Moreover, as a ring, we may identify $H^*(BG_\lambda, R)$ as the subring of

$$H^*(BG_\lambda; \mathbb{Q}) = \frac{\mathbb{Q}[x,y,z]}{\langle z\prod_{i=1}^{\ell}(z^2 + i^4x - i^2y)\rangle}$$

where the degree of z is 2, and the elements b_k and a_k are identified respectively to the elements:

$$\frac{z}{(2k+1)!}\prod_{i=1}^{k}(z^2 + i^4x - i^2y), \quad \frac{z^2}{(2k)!}\prod_{i=1}^{k-1}(z^2 + i^4x - i^2y)$$

If G_λ denotes the group of symplectomorphisms of $S^2\tilde{\times}S^2$, with symplectic form ω_λ as above, where $0 < \lambda \in \mathbb{R}$ lies between the integers $0 \leq \ell < \lambda \leq \ell+1$, we have:

Theorem 1.4. *The cohomology groups of the space BG_λ in the twisted case are given by:*

$$H^*(BG_\lambda; \mathbb{Z}) = \bigoplus_{i=0}^{\ell}\Sigma^{4i}H^*(BU(2); \mathbb{Z}).$$

In particular we see that the cohomology of BG_λ is torsion free in the twisted case.

Theorem 1.5. *The rational cohomology ring of BG_λ in the twisted case is given by:*

$$H^*(BG_\lambda; \mathbb{Q}) = \frac{\mathbb{Q}[x,y,z]}{\langle\prod_{i=0}^{\ell}(z^2 + (2i+1)^4x - (2i+1)^2y)\rangle}$$

where the degree of the class z is 2, and that of the classes x and y is 4.

We bring to the attention of the reader the relations in the rational cohomology of BG_λ. In [AM], both in the twisted and untwisted cases, the decomposable term z^2 was missing in each of the linear factors that make up the relation. In addition, in the twisted case, the nondecomposable terms in the linear factors have powers of odd integers as coefficients instead of all integers. The source of these inaccuracies in [AM] is discussed in Remark 5.18.

Regarding previous papers on the topology of symplectomorphism groups of rational ruled surfaces, the results of this paper depend logically only on the analysis of the stratification of $\mathcal{J}_\omega$ in [McD1], the computation of the homotopy type of the strata in [Ab] (or [AM]) and (just for the computation of the ring structures) the fact, proved in [AM] (and [McD2]), that the homotopy colimit of certain inclusions $BG_\lambda \to BG_{\lambda+\epsilon}$ is the group of fiberwise diffeomorphisms.

Organization of the Paper:
The body of the paper is divided into four sections and four appendices.

Section 2 is further divided into four subsections. In the first part, we begin by developing our framework to study complex deformation theory of a complex 4-manifold M. In the next part, we specialize to a symplectic 4-manifold (M,ω). Let $\mathcal{J}_\omega$ be the space of almost complex structures on M compatible with ω, and let $\mathcal{I}_\omega$ be the compatible integrable structures. Given a Kähler structure $J_0 \in \mathcal{I}_\omega$, we interpret the normal bundle in $\mathcal{I}_\omega$ to the space of equivalent Kähler structures, in terms of deformation theory. In the next subsection, we study the action of the diffeomorphism group on the space of compatible complex structures for an arbitrary symplectic manifold (M, ω). Using this we show that, under certain conditions, the space of diffeomorphic compatible structures is homotopy equivalent to an orbit of the symplectomorphism group. Finally, in the last part of this section, we analyze the inclusion $\iota : \mathcal{I}_\omega \subset \mathcal{J}_\omega$. The space $\mathcal{J}_\omega$ admits a map from the moduli space $\mathcal{M}$ of J-holomorphic curves. We derive sufficient cohomological conditions for $\mathcal{M}$ to meet ι transversally. In the case of rational ruled surfaces, this implies that the map ι is transverse to the strata introduced by Abreu. The technical lemmas of section 2 have been banished to Appendices A through C.

In section 3, we study rational ruled surfaces in detail. This section is divided into two parts. In the first part, we describe a stratification on the space $\mathcal{J}_\omega$ of compatible almost complex structures, for a fixed symplectic form ω, previously studied in [Ab, AM, McD1]. In the second part, we use results from the previous section to show that this induces an analogous stratification on the corresponding space $\mathcal{I}_\omega$ of the compatible complex structures. We then conclude with the proof of Theorem 1.1.

Section 4 is dedicated to studying the deformation theory of Hirzebruch surfaces. We fix a "standard" toric symplectic form and use the fixed point formula for elliptic complexes to determine the isotropy representations of the symplectomorphism group on the normal bundle to the various strata in $\mathcal{I}_\omega$. This is applied in the next section to study the topology of the symplectomorphism group.

In section 5, we use the results of sections 3 and 4 to express the classifying space of the symplectomorphism group as a finite iterated homotopy pushout of classifying spaces of certain Kähler isometry subgroups (leaving some technicalities for Appendix D). Theorems 1.2 and 1.4 follow immediately. We then use the loop maps from G_λ to the group of fiberwise diffeomorphisms defined in [McD2], together with the classification of Hamiltonian S^1-actions on four-manifolds [Ka] to compute the rational cohomology rings described in Theorems 1.3 and 1.5 (see Theorem 5.16 and Remark 5.17). Finally, we use the computation in [HHH] of the T^2-equivariant cohomology groups of $\Omega SU(2)$ to find the cohomology ring of the classifying space of the fiberwise diffeomorphism group away from 2 and use this to complete the proof of Theorem 1.3.

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Conventions:
Throughout this paper, we work with infinite dimensional Fréchet manifolds which are locally modeled on an inverse limit of Banach spaces. Standard theorems such as the inverse function theorem do not hold automatically in the Fréchet setting. Therefore, we need to say a few words about the context in which the transversality arguments in this paper need to be interpreted. All the Fréchet manifolds we work with can naturally be interpreted as inverse limits of Banach manifolds. For example, the space of smooth sections of a bundle over a smooth manifold is the intersection over k of the corresponding Banach manifolds of C^k-sections. For

each individual k, all the transversality arguments we use hold in the infinite dimensional context. The validity of the results stated in the smooth setting should therefore be interpreted as the validity of the corresponding result for each Banach manifold indexed by k. Statements about the homotopy type of the corresponding Fréchet manifold can be derived from the fact that the successive inclusions between the Banach manifolds are weak equivalences (see [P]). We shall illustrate this with an example in Remark 2.2.

2. General facts on compatible complex structures

The goal of this section is to set up a geometric framework and establish some facts regarding the space of compatible integrable complex structures $\mathcal{I}_\omega$ and its inclusion in the space of compatible almost complex structures $\mathcal{J}_\omega$ on a symplectic 4-manifold (M, ω). In particular, we describe cohomological conditions under which the space $\mathcal{I}_\omega$ is a submanifold of $\mathcal{J}_\omega$.

Complex structures:
Given an almost complex manifold (M, J), the Nijenhuis tensor $N_J \in \Omega^{0,2}_J(TM) = \Omega^{0,2}_J(M) \otimes \Omega^0(TM)$ is a $(0, 2)$-form on (M, J) with values in (TM, J) that measures the non-integrability of J (see Definition A.3 in Appendix A). If one considers the space $\mathcal{J}$ of all almost complex structures on the manifold M, the Nijenhuis tensor N can be seen as a section of the natural vector bundle $\Omega^{0,2}(TM)$ over $\mathcal{J}$, whose fiber over a point $J \in \mathcal{J}$ is $\Omega^{0,2}_J(TM)$:

$$N : \mathcal{J} \to \Omega^{0,2}(TM)\ , \quad J \mapsto (J, N_J).$$

The space $\mathcal{I}$ of integrable complex structures on M is the zero-set of this Nijenhuis section N. As usual, it will be a submanifold of $\mathcal{J}$ if the Nijenhuis section N is transversal to the zero section.

The vector bundle $\Omega^{0,2}(TM)$ is a canonical summand of the trivial bundle over $\mathcal{J}$ with fiber $\Omega^2(TM)$:

$$\Omega^{0,2}(TM) \oplus \left(\Omega^{2,0}(TM) \oplus \Omega^{1,1}(TM)\right) \equiv \Omega^2(TM) \times \mathcal{J} \to \mathcal{J}\ .$$

This means in particular that $\Omega^{0,2}(TM)$ has a natural connection ∇, given by projection of the trivial connection on $\Omega^2(TM) \times \mathcal{J}$:

$$\nabla \cdot = (d\cdot)^{0,2}\ .$$

Since $T\mathcal{J} \cong \Omega^{0,1}(TM)$, we have that ∇N can be regarded as a bundle map

$$\nabla N : \Omega^{0,1}(TM) \to \Omega^{0,2}(TM)\ .$$

In particular, if for a given $J \in \mathcal{I}$, i.e. a $J \in \mathcal{J}$ for which $N_J \equiv 0$, the map

$$\nabla N_J : \Omega^{0,1}_J(TM) \to \Omega^{0,2}_J(TM)$$

is surjective, then the Nijenhuis section is transversal to the zero section at this $J \in \mathcal{I}$.

As proved in Corollary B.4, the map ∇N is essentially the $\bar\partial$-operator (see Definition A.8). More precisely,

$$\nabla N_J = (-2J)\bar\partial_J\ ,\ \forall\, J \in \mathcal{J}\ .$$

The following proposition is then immediate.

Proposition 2.1. *If M is a 4-dimensional manifold, $J \in \mathcal{I}$ is an integrable complex structure on M and the cohomology group $H^{0,2}_J(TM) = 0$, then $\mathcal{I}$ is a submanifold of $\mathcal{J}$ in the neighborhood of J, with tangent space*

$$T_J\mathcal{I} = \ker\left\{\bar\partial : \Omega^{0,1}_J(TM) \to \Omega^{0,2}_J(TM)\right\} \subset \Omega^{0,1}_J(TM) = T_J\mathcal{J} .$$

The group $\mathrm{Diff}(M)$ of diffeomorphisms of M acts naturally on $\mathcal{J}$ via

$$\varphi^*(J) := (d\varphi)^{-1}\, J\, (d\varphi)\ ,\ \forall\, \varphi \in \mathrm{Diff}(M),\ J \in \mathcal{J} .$$

The induced infinitesimal action is given by the Lie derivative

$$\mathcal{L} : \Omega^0(TM) \to \Omega^{0,1}(TM) = T\mathcal{J}$$

which, as proved in Proposition A.7, can be written for a given $X \in \Omega^0(TM)$ and at a given $J \in \mathcal{J}$ as

$$\mathcal{L}_X J = (2J)(\bar\partial X) + \frac{1}{2} J(X \lrcorner N_J) \in \Omega^{0,1}_J(TM) = T_J\mathcal{J} .$$

The action of $\mathrm{Diff}(M)$ on $\mathcal{J}$ preserves $\mathcal{I}$. Since $\bar\partial$ commutes with J (see Proposition A.6), the previous formula implies that the tangent space to an orbit $\mathrm{Diff}(M)\cdot J$ at a point $J \in \mathcal{I}$, is given by

$$T_J(\mathrm{Diff}(M)\cdot J) = \mathrm{im}\left\{\bar\partial : \Omega^0_J(TM) \to \Omega^{0,1}_J(TM)\right\} .$$

It then follows from Proposition 2.1 that the moduli space of infinitesimal deformations of $J \in \mathcal{I}$ is given by $H^{0,1}_J(TM)$ and

$$T_J\mathcal{I} \cong T_J(\mathrm{Diff}(M)\cdot J) \oplus H^{0,1}_J(TM) .$$

Remark 2.2. *As promised, we elaborate on the transversality argument here:*

We may see N as a section of the bundle $\Omega^{0,2}_k(TM)$ with base being the space $\mathcal{J}_{k+1}$ (the space of C^{k+1}-almost complex structures), and fiber $\Omega^{0,2}_k(TM, J)$ (the space of C^k-forms of type (0,2) with values in TM) over $J \in \mathcal{J}_{k+1}$. This bundle supports a natural connection and the same proof shows that ∇N can be identified with the $\bar\partial$ operator

$$\nabla N = \bar\partial : \Omega^{0,1}_{k+1}(TM, J) \longrightarrow \Omega^{0,2}_k(TM, J)$$

Let $H^{0,n}_k(TM, J)$ be the cohomology groups of the Dolbeaut complex

$$\Omega^0_{k+2}(TM, J) \xrightarrow{\bar\partial} \Omega^{0,1}_{k+1}(TM, J) \xrightarrow{\bar\partial} \Omega^{0,2}_k(TM, J) \longrightarrow 0$$

Notice that $H^{0,n}_k(TM, J)$ is isomorphic to $H^{0,n}(TM, J)$, for $J \in \mathcal{I}$, and for all $n, k \geq 0$ since the complex above is a resolution of the sheaf of holomorphic vector fields on M by fine sheaves. Hence we may apply the implicit function theorem for each k to show that the zero locus of N is a Banach manifold $\mathcal{I}_{k+1} \subset \mathcal{J}_{k+1}$ modeled on the Banach space of closed C^{k+1}-forms of type $(0,1)$.

Now let $\mathrm{Diff}_{k+2}(M)$ denote the group of C^{k+2} diffeomorphisms of M. The action map of $\mathrm{Diff}_{k+2}(M)$ on $\mathcal{J}_{k+1}$ is C^1, so we may differentiate the action. Repeating the proof of the previous claim, the tangent space to the orbit of the group $\mathrm{Diff}_{k+2}(M)$ at $J \in \mathcal{I}\cap\mathcal{J}_{k+1}$ can be identified with the space of exact C^{k+1}-forms of type $(0,1)$. Therefore, the space of infinitesimal deformations of J inside $\mathcal{I}_{k+1}$ can be naturally identified with the cohomology group $H^{0,1}(TM, J)$ (which is independent of k).

Compatible complex structures:
Now let (M,ω) be a symplectic 4-manifold and $\mathcal{J}_\omega$ the contractible submanifold of $\mathcal{J}$ consisting of almost complex structures on M compatible with ω. Given $J \in \mathcal{J}_\omega$, denote by

$$h_J(\cdot,\cdot) \equiv \omega(\cdot, J\cdot) - i\omega(\cdot,\cdot)$$

the hermitian metric on TM induced by the pair (ω, J). We may use h_J to identify $T_J\mathcal{J} = \Omega^{0,1}_J(TM)$ with the space $T^{0,2}_J(M) \equiv \Omega^{0,1}_J(M) \otimes \Omega^{0,1}_J(M)$ of complex $(0,2)$-tensors, via

$$\Omega^{0,1}_J(TM) \ni A(\cdot) \leftrightarrow \theta_A(\cdot,\cdot) \equiv h_J(A\cdot,\cdot) \in T^{0,2}_J\,.$$

Under this identification, the subspace $T_J\mathcal{J}_\omega = S\Omega^{0,1}_J(TM) \subset \Omega^{0,1}_J(TM) = T_J\mathcal{J}$ can be identified with the subspace of complex symmetric $(0,2)$-tensors $S^{0,2}_J(M) \subset T^{0,2}_J(M)$:

$$A \in T_J\mathcal{J}_\omega \Leftrightarrow AJ + JA = 0 \quad \text{and} \quad \omega(A\cdot,\cdot) = -\omega(\cdot, A\cdot) \Leftrightarrow \theta_A \in S^{0,2}_J(M)\,.$$

The quotient may therefore be identified with the space of $(0,2)$-forms on M:

$$T_J\mathcal{J}/T_J\mathcal{J}_\omega = \Omega^{0,1}_J(TM)/S\Omega^{0,1}_J(TM) \cong T^{0,2}_J(M)/S^{0,2}_J(M) = \Omega^{0,2}_J(M)\,.$$

Given a compatible integrable complex structure $J \in \mathcal{I}_\omega \subset \mathcal{J}_\omega$, consider the following sequence of chain complexes with exact columns, where the above identifications are taken into account and commutativity of the lower left corner is proved in Appendix C:

$$\begin{array}{ccccccc}
0 & \longrightarrow & S\Omega^{0,1}_J(TM) & \xrightarrow{\bar\partial} & \Omega^{0,2}_J(TM) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega^0(TM) & \xrightarrow{\bar\partial} & \Omega^{0,1}_J(TM) & \xrightarrow{\bar\partial} & \Omega^{0,2}_J(TM) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega^{0,1}_J(M) & \xrightarrow{\bar\partial} & \Omega^{0,2}_J(M) & \xrightarrow{\bar\partial} & 0 & \longrightarrow & 0
\end{array}$$

The snake lemma yields a long exact sequence of holomorphic cohomology groups:

$$0 \longrightarrow H^0_J(TM) \longrightarrow cl\Omega^{0,1}_J(M) \xrightarrow{\delta} clS\Omega^{0,1}_J(TM) \longrightarrow H^{0,1}_J(TM) \longrightarrow$$
$$\longrightarrow H^{0,2}_J(M) \longrightarrow SH^{0,2}_J(TM) \longrightarrow H^{0,2}_J(TM) \longrightarrow 0$$

where $cl\Omega^{0,1}_J(M)$ and $clS\Omega^{0,1}_J(TM)$ denote the closed $(0,1)$-forms in the respective spaces and the symmetric cohomology $SH^{0,2}_J(TM)$ is given by the cokernel of $\bar\partial$ restricted to the symmetric $(0,2)$-tensors $S\Omega^{0,1}_J(TM)$. A geometric interpretation of the maps in the above long exact sequence can be given as follows.

By analysing the Nijenhuis tensor and its covariant derivative as before, one sees that $\mathcal{I}_\omega$ is a submanifold of $\mathcal{J}_\omega$ in the neighborhood of $J \in \mathcal{I}_\omega$ if the symmetric cohomology group $SH^{0,2}_J(TM)$ vanishes. Under this assumption, $clS\Omega^{0,1}_J(TM)$ can be identified with the tangent space $T_J\mathcal{I}_\omega$, and the image of $\delta : cl\Omega^{0,1}_J(M) \to clS\Omega^{0,1}_J(TM)$ identifies the tangent space to the intersection of the orbit $(\mathrm{Diff}(M)\cdot J)$ with $\mathcal{I}_\omega$. Moreover, the image of the cokernel of δ in $H^{0,1}_J(TM)$ identifies the the moduli of infinitesimal deformations of $J \in \mathcal{I}_\omega \subset \mathcal{I}$ that can be realized in an ω-compatible way. We then have the following theorem.

Theorem 2.3. *If (M,ω) is a symplectic 4-dimensional manifold, $J \in \mathcal{I}_\omega$ is a compatible integrable complex structure on (M,ω) and the cohomology groups $H^{0,2}_J(TM)$ and $H^{0,2}_J(M)$ are zero, then $\mathcal{I}_\omega$ is a submanifold of $\mathcal{J}_\omega$ in the neighborhood of J, with tangent space*

$$T_J\mathcal{I}_\omega = \ker\left\{\bar\partial : S\Omega^{0,1}_J(TM) \to \Omega^{0,2}_J(TM)\right\} \subset S\Omega^{0,1}_J(TM) = T_J\mathcal{J}_\omega\,.$$

Moreover, the moduli space of infinitesimal compatible deformations of J in $\mathcal{I}_\omega$ coincides with the moduli space of infinitesimal deformations of J in $\mathcal{I}$, i.e. it is given by $H^{0,1}_J(TM)$ and

$$T_J\mathcal{I}_\omega \cong T_J((\mathrm{Diff}(M)\cdot J)\cap \mathcal{I}_\omega) \oplus H^{0,1}_J(TM)\,.$$

Remark 2.4. *If $H^1(M,\mathbb{R}) = 0$, then $cl\Omega^{0,1}_J(M) = \bar\partial\Omega^0(M,\mathbb{C}) = \Omega^0(M,\mathbb{C})/\mathbb{C}$ is naturally identified with the complexified Lie algebra of the symplectomorphism group of (M,ω). Moreover, the kernel of δ can be identified with the vector space of holomorphic vector fields on (M,J). This nicely agrees with Donaldson's formal picture where*

$$(\mathrm{Diff}(M)\cdot J)\cap\mathcal{I}_\omega = (\mathrm{Symp}(M)^{\mathbb{C}}\cdot J)\,.$$

Diffeomorphic compatible complex structures:
In this subsection (M,ω) will be an arbitrary symplectic manifold, not necessarily of dimension 4, and $\mathcal{I}_\omega$ will denote again the space of complex structures on M compatible with ω. We write

$$\mathrm{Diff}_{[\omega]}(M) = \{\varphi \in \mathrm{Diff}(M) \mid \varphi^*([\omega]) = [\omega] \in H^2(M;\mathbb{R})\} \tag{1}$$

for the subgroup of diffeomorphisms of M preserving the cohomology class of the symplectic form.

Let $J_0 \in \mathcal{I}_\omega$ denote a fixed complex structure compatible with ω. Define

$$U = \{J \in \mathcal{I}_\omega \mid J_0 = \varphi^*J \text{ for some } \varphi\in\mathrm{Diff}_{[\omega]}(M)\} \subset \mathcal{I}_\omega$$
$$\mathcal{K} = \{\eta \in \Omega^2(M) \mid d\eta = 0, [\eta]=[\omega] \text{ and } \eta \text{ is compatible with } J_0\}\,.$$

$\mathcal{K}$ is a contractible convex subset in $\Omega^2(M)$ and we want to describe the topology of U.

Define a map from $\mathcal{K}$ to the identity component $\mathrm{Diff}_0(M)$ of the diffeomorphism group of M,

$$\Psi : \mathcal{K} \to \mathrm{Diff}_0(M)\,,$$

as follows. Given $\eta \in \mathcal{K}$, let $\psi_t \in \mathrm{Diff}_0(M)$ be the isotopy satisfying $\psi_t^*((1-t)\omega + t\eta) = \omega$ which is canonically determined by Moser's method and the Riemannian metric given by (ω, J_0). Then

$$\Psi(\eta) := \psi_1\,.$$

Note that, if $K = \mathrm{Iso}(\omega, J_0)$ and $g\in K$ then we have $\psi_t(g^*\eta) = g\psi_t(\eta)$ and, in particular, $\Psi(g^*\eta) = g\Psi(\eta)\,,\ \forall\,\eta\in\mathcal{K}$. Moreover, $\Psi(\eta)^*(\eta) = \omega\,,\ \forall\,\eta\in\mathcal{K}$.

Denote by $\mathrm{Hol}_{[\omega]}(J_0)$ the group of complex automorphisms of (M,J_0) that preserve the cohomology class $[\omega]\in H^2(M;\mathbb{R})$.

Proposition 2.5. *The map $\mu : \mathrm{Symp}(\omega)\times\mathcal{K} \to U$ defined by*

$$\mu(\phi,\eta) = (\phi^{-1})^*\Psi(\eta)^*J_0$$

is a principal $\mathrm{Hol}_{[\omega]}(J_0)$-bundle.

Proof. Suppose $J \in U$ and let $\varphi \in \mathrm{Diff}_{[\omega]}(M)$ be such that $\varphi^*(J) = J_0$. Setting $\eta = \varphi^*\omega$, we derive:

$$J_0^*\eta = J_0^*\varphi^*\omega = (d\varphi \circ J_0)^*\omega = (J \circ d\varphi)^*\omega = \varphi^* J^*\omega = \varphi^*\omega = \eta$$
$$\eta(X, J_0X) = \omega((d\varphi)X, (d\varphi)J_0X) = \omega((d\varphi)X, J(d\varphi)X) > 0\,.$$

so we conclude that η is compatible with J_0. Since $[\eta] = [\omega]$ we see that $\eta \in \mathcal{K}$. Moreover, $\varphi\Psi(\eta) \in \mathrm{Symp}(\omega)$, since $(\varphi\Psi(\eta))^*\omega = \Psi(\eta)^*\eta = \omega$, and we have

$$\mu(\varphi\Psi(\eta), \eta) = (\Psi(\eta)^{-1}\varphi^{-1})^*\Psi(\eta)^*J_0 = (\varphi^{-1})^*J_0 = J\,.$$

Hence, the map μ is surjective.

Now, given $J \in U$ and an element $(\phi, \eta) \in \mu^{-1}(J)$, consider $\varphi = \phi\Psi(\eta)^{-1}$. Then $\varphi \in \mathrm{Diff}_{[\omega]}(M)$ and

$$\mu(\phi, \eta) = (\phi^{-1})^*\Psi(\eta)^*J_0 = J \Rightarrow J_0 = \varphi^*J\,.$$

Conversely, given $\varphi \in \mathrm{Diff}_{[\omega]}(M)$ such that $J_0 = \varphi^*J$, we have that $(\varphi\Psi(\eta), \eta) \in \mu^{-1}(J)$, where $\eta = \varphi^*\omega$. Hence

$$\mu^{-1}(J) \cong \{\varphi \in \mathrm{Diff}_{[\omega]}(M) \;\mid\; J_0 = \varphi^*J\}\,,$$

i.e. the fibers of the map μ are torsors on the group $\mathrm{Hol}_{[\omega]}(J_0)$. In fact, defining a right action of $\mathrm{Hol}_{[\omega]}(J_0)$ on $\mathrm{Symp}(\omega) \times \mathcal{K}$ by

$$(\phi, \eta) \cdot \varphi = (\phi\Psi(\eta)^{-1}\varphi\Psi(\varphi^*(\eta)), \varphi^*(\eta))\,,$$

we see that the fibers of the map μ are free orbits of this action. □

Corollary 2.6. *If $J_0 \in \mathcal{I}_\omega$ is such that the inclusion $\mathrm{Iso}(\omega, J_0) \hookrightarrow \mathrm{Hol}_{[\omega]}(J_0)$ is a weak homotopy equivalence, then the inclusion of the $\mathrm{Symp}(M)$-orbit of J_0 in U, i.e.*

$$\mathrm{Symp}(\omega)/\,\mathrm{Iso}(\omega, J_0) \hookrightarrow U\,,$$

is also a weak homotopy equivalence.

Proof. Indeed, we have

$$\mathrm{Symp}(M) \overset{\sim}{\hookrightarrow} \mathrm{Symp}(M) \times \{\omega\} \subset \mathrm{Symp}(M) \times \mathcal{K} \overset{\mu}{\to} U\,,$$

which induces

$$\mathrm{Symp}(M)/\,\mathrm{Iso}(\omega, J_0) \overset{\sim}{\hookrightarrow} (\mathrm{Symp}(M) \times \mathcal{K})\,/\,\mathrm{Iso}(\omega, J_0) \overset{\overline{\mu}}{\to} U\,,$$

where the fiber of $\overline{\mu}$ is weakly contractible by assumption. □

Remark 2.7. *According to Calabi* [C1]*, a source of examples for the previous corollary are J_0's determining extremal Kähler metrics, at least on manifolds where $\mathrm{Hol}(J_0)$ and $\mathrm{Iso}(\omega, J_0)$ are both connected.*

Transversality:
In this subsection we study transversality properties of certain strata of compatible almost complex structures on a compact symplectic manifold (M, ω), with respect to the inclusion $\mathcal{I}_\omega \subset \mathcal{J}_\omega$. These strata are characterized by the existence of certain pseudo-holomorphic curves.

Recall that if (M, J) is an almost complex manifold and (Σ, j) is a Riemann surface, a map $u \in \mathrm{Map}(\Sigma, M)$ is called a J-holomorphic curve if

$$J \circ du = du \circ j \,,$$

and a simple J-holomorphic curve if, in addition, it is not multiply covered (see [MS]).

Given a compact symplectic manifold (M, ω), a compact Riemann surface (Σ, j) and a homology class $A \in H_2(M, \mathbb{Z})$, consider the space

$$\mathcal{M}(A, \Sigma) = \{(u, J) \in \mathrm{Map}(\Sigma, M) \times \mathcal{J}_\omega \,:\, u \text{ is a simple } J\text{-holomorphic curve with } u_*([\Sigma]) = A\}\,.$$

Denote by $\Omega^{0,1}(\Sigma, TM)$ the vector bundle over $\mathrm{Map}(\Sigma, M) \times \mathcal{J}_\omega$ whose fiber over (u, J) is

$$\Omega^{0,1}_J(\Sigma, u^*TM) \equiv \; J \text{ anti-linear 1-forms on } \Sigma \text{ with values in } u^*TM.$$

This vector bundle has a natural section, denoted by $\bar\partial$, given at $(u, J) \in \mathrm{Map}(\Sigma, M) \times \mathcal{J}_\omega$ by

$$\bar\partial_{(u,J)} \equiv \bar\partial_J(u) \equiv \frac{1}{2}(du + J \circ du \circ j) \in \Omega^{0,1}_J(\Sigma, u^*TM)\,.$$

The space $\mathcal{M}(A, \Sigma)$ is the zero set of this section.

The tangent space to $\mathrm{Map}(\Sigma, M) \times \mathcal{J}_\omega$ at (u, J) is given by

$$T_{(u,J)}(\mathrm{Map}(\Sigma, M) \times \mathcal{J}_\omega) = T_u \mathrm{Map}(\Sigma, M) \oplus T_J \mathcal{J}_\omega = \Omega^0(\Sigma, u^*TM) \oplus S\Omega^{0,1}_J(TM)\,.$$

At $(u, J) \in \bar\partial^{-1}(0) = \mathcal{M}(A, \Sigma)$, the vertical component of the derivative of the section $\bar\partial$,

$$D(\bar\partial)_{(u,J)} : \Omega^0(\Sigma, u^*TM) \oplus S\Omega^{0,1}_J(TM) \to \Omega^{0,1}_J(\Sigma, u^*TM)\,,$$

is surjective and given by

$$D(\bar\partial)_{(u,J)}(\xi, \alpha) = \bar\partial_J(\xi) + u^*(\alpha)\,,$$

where $\bar\partial_J(\xi)$ is given as in Definition A.5 of Appendix A (see [MS], Remark 3.1.2 and Proposition 3.2.1). Hence, the following holds.

Proposition 2.8. *$\mathcal{M}(A, \Sigma)$ is an infinite-dimensional submanifold of $\mathrm{Map}(\Sigma, M) \times \mathcal{J}_\omega$, with tangent space given by*

$$T_{(u,J)}\mathcal{M}(A, \Sigma) = \{(\xi, \alpha) \in \Omega^0(\Sigma, u^*TM) \oplus S\Omega^{0,1}_J(TM) \,:\, \bar\partial_J(\xi) + u^*(\alpha) = 0\}\,.$$

The image of the projection

$$\pi : \mathcal{M}(A, \Sigma) \to \mathcal{J}_\omega\,, \quad (u, J) \mapsto J$$

defines a subset $U_A \subset \mathcal{J}_\omega$ characterized by

$$J \in U_A \Leftrightarrow A \in H_2(M, \mathbb{Z}) \text{ can be represented by a simple } J\text{-holomorphic curve with domain } (\Sigma, j).$$

We now want to find conditions ensuring that the image U_A of π is transversal to $\mathcal{I}_\omega \subset \mathcal{J}_\omega$ at points $(u, J) \in \mathcal{M}(A, \Sigma)$ with $J \in \mathcal{I}_\omega$. It follows from the previous proposition that

$$\pi_*(T_{(u,J)}\mathcal{M}(A,\Sigma)) = \{\alpha \in S\Omega^{0,1}_J(TM) \,:\, [u^*\alpha] = 0 \in H^{0,1}_J(\Sigma, u^*TM)\}\,.$$

Assume that the restriction map

$$u^* : clS\Omega^{0,1}_J(TM) \to H^{0,1}_J(\Sigma, u^*TM)$$

is surjective. Then, given any $\gamma \in T_J\mathcal{J}_\omega = S\Omega^{0,1}_J(TM)$, there exists $\beta \in T_J\mathcal{I}_\omega = clS\Omega^{0,1}_J(TM)$ such that $(\gamma - \beta) \in \pi_*(T_{(u,J)}\mathcal{M}(A,\Sigma))$. Combining this with the reasoning leading to Theorem 2.3, one gets the following result.

Theorem 2.9. *Let $(M, \omega, J \in \mathcal{I}_\omega)$ be a Kähler 4-manifold such that the cohomology groups $H^{0,2}_J(M)$ and $H^{0,2}_J(TM)$ are zero. Suppose that $(u, J) \in \mathcal{M}(A, \Sigma)$ is such that $u^* : H^{0,1}_J(TM) \to H^{0,1}_J(u^*(TM))$ is an isomorphism. Then $\pi : \mathcal{M}(A, \Sigma) \to \mathcal{J}_\omega$ is transversal at (u, J) to $\mathcal{I}_\omega \subset \mathcal{J}_\omega$ and the infinitesimal complement to the image U_A of π in a neighborhood of J can be identified with the moduli space of infinitesimal deformations $H^{0,1}_J(TM)$.*

3. Compatible Complex Structures on Rational Ruled Surfaces

In this section we describe the stratification on the space $\mathcal{J}_\omega$ of compatible almost complex structures on a rational ruled surface, previously studied in [Ab, AM, McD1], and use results from the previous section to show that it induces an analogous stratification on the corresponding space $\mathcal{I}_\omega$ of compatible complex structures. We then conclude with the proof of theorem 1.1.

Compatible almost complex structures:
Consider $S^2 \times S^2$ with split symplectic form $\omega_\lambda = \lambda\sigma \oplus \sigma$, for some $1 \leq \lambda \in \mathbb{R}$. Denote by $\mathcal{J}_\lambda$ its contractible space of compatible almost complex structures and by G_λ its symplectomorphism group.

The following theorem plays a fundamental role in the results obtained in [Ab] and [AM] regarding the topology of G_λ. It will also play a fundamental role here. The most technical point, listed as (v) in the statement, was proved in [McD1] using gluing techniques for pseudo-holomorphic spheres.

Theorem 3.1. *Given $1 \leq \lambda \in \mathbb{R}$, there is a stratification of the contractible space $\mathcal{J}_\lambda$ of the form*

$$\mathcal{J}_\lambda = U_0 \sqcup U_1 \sqcup \cdots \sqcup U_\ell\,,$$

with $\ell \in \mathbb{N}_0$ such that $\ell < \lambda \leq \ell + 1$ and where:

(i)

$$U_k \equiv \{J \in \mathcal{J}_\lambda :\ (1, -k) \in H_2(S^2 \times S^2; \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere}\}\,.$$

(ii) U_0 *is open and dense in* $\mathcal{J}_\lambda$*. For* $k \geq 1$*,* U_k *has codimension* $4k - 2$ *in* $\mathcal{J}_\lambda$*.*

(iii) $\overline{U_k} = U_k \sqcup U_{k+1} \sqcup \cdots \sqcup U_\ell$*.*

(iv) The inclusion

$$(G_\lambda / K'_k) \longrightarrow U_k\ , \quad [\psi] \longmapsto \psi_*(J'_k)$$

is a weak homotopy equivalence, where $J'_k \in U_k$ is any compatible almost complex structure with $K'_k = \mathrm{Iso}(\omega_\lambda, J'_k)$ such that

$$K'_k \cong \begin{cases} \mathbb{Z}/2 \ltimes (SO(3) \times SO(3))\,, & \text{if } \lambda = 1 \text{ and } k = 0; \\ SO(3) \times SO(3)\,, & \text{if } \lambda > 1 \text{ and } k = 0; \\ S^1 \times SO(3)\,, & \text{if } k \geq 1. \end{cases}$$

(v) *Each U_k has a tubular neighborhood $NU_k \subset \mathcal{J}_\lambda$ which fibers over U_k as a ball bundle.*

The situation is completely analogous for $S^2 \tilde{\times} S^2$, the non trivial S^2-bundle over S^2. For any $0 < \lambda \in \mathbb{R}$, let ω_λ be a symplectic form on $S^2 \tilde{\times} S^2$ such that $[\omega_\lambda](E) = \lambda$ and $[\omega_\lambda](F) = 1$, where E and F are the homology classes described in the introduction. Identify $H_2(S^2 \tilde{\times} S^2; \mathbb{Z})$ with $\mathbb{Z} \times \mathbb{Z}$ via

$$H_2(S^2 \tilde{\times} S^2; \mathbb{Z}) \ni mE + nF \mapsto (m, n) \in \mathbb{Z} \times \mathbb{Z}\,.$$

We again denote by $\mathcal{J}_\lambda$ the contractible space of compatible almost complex structures on $(S^2 \tilde{\times} S^2, \omega_\lambda)$ and by G_λ its symplectomorphism group. The analogue of theorem 3.1 is then the following (see [AM, McD1]).

Theorem 3.2. *Given $0 < \lambda \in \mathbb{R}$, there is a stratification of the contractible space $\mathcal{J}_\lambda$ of the form*

$$\mathcal{J}_\lambda = U_0 \sqcup U_1 \sqcup \cdots \sqcup U_\ell\,,$$

with $\ell \in \mathbb{N}_0$ such that $\ell < \lambda \leq \ell + 1$ and where:

(i)

$$U_k \equiv \{J \in \mathcal{J}_\lambda : \ (1, -k) \in H_2(S^2 \tilde{\times} S^2; \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere}\}\,.$$

(ii) *U_k has codimension $4k$ in $\mathcal{J}_\lambda$.*

(iii) *$\overline{U_k} = U_k \sqcup U_{k+1} \sqcup \cdots \sqcup U_\ell$.*

(iv) *The inclusion*

$$(G_\lambda / K'_k) \longrightarrow U_k\,, \quad [\psi] \longmapsto \psi_*(J'_k)$$

is a weak homotopy equivalence, where $J'_k \in U_k$ is any compatible almost complex structure with $K'_k = \mathrm{Iso}(\omega_\lambda, J'_k) \cong U(2)$.

(v) *Each U_k has a tubular neighborhood $NU_k \subset \mathcal{J}_\lambda$ which fibers over U_k as a ball bundle.*

Remark 3.3. *In* [AM], *Section 2, such $(S^2 \times S^2, \omega_\lambda, J'_k)$ and $(S^2 \tilde{\times} S^2, \omega_\lambda, J'_k)$ were explicitly constructed as Kähler reductions of $\mathbb{C}^4$. This is reviewed below in the proof of the Proposition 5.11.*

Compatible complex structures:

Our goal now is to show that an analogous theorem holds for the space $\mathcal{I}_\lambda \subset \mathcal{J}_\lambda$ of compatible integrable complex structures, both on $(S^2 \times S^2, \omega_\lambda)$, $1 \leq \lambda \in \mathbb{R}$, and $(S^2 \tilde{\times} S^2, \omega_\lambda)$, $0 < \lambda \in \mathbb{R}$. We will denote by (M, ω_λ) either one of these symplectic manifolds.

For each $k \in \{0, 1, \ldots, \ell\}$, with $\ell \in \mathbb{N}_0$ such that $\ell < \lambda \leq \ell + 1$, define

$$V_k \equiv U_k \cap \mathcal{I}_\lambda = \{J \in \mathcal{I}_\lambda : \ (1, -k) \in H_2(M; \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere}\}\,.$$

It follows from standard complex geometry (see [BPV] or [Ca]) that any $J \in V_k$ is complex isomorphic to the Hirzebruch surface

$F_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))\,,$ with $n = 2k$ if $M = S^2 \times S^2$, and $n = 2k + 1$ if $M = S^2 \tilde{\times} S^2$, by a diffeomorphism of M that acts as the identity in homology (here, $\mathcal{O}(-1)$ denotes the tautological line bundle over $\mathbb{P}^1$ and $\mathbb{P}(E)$ the projectivization of a vector bundle E). Moreover, Calabi proved in [C1] that there is a complex structure $J_k \in V_k$, unique up to the action of G_λ, for which $g_{\lambda,k} \equiv \omega_\lambda(\cdot, J_k\cdot)$ is an extremal Kähler metric, with Kähler isometry group $K_k \equiv \mathrm{Iso}(\omega_\lambda, J_k)$ as in Theorems 3.1 and 3.2. These two facts together imply that

$$V_k = \{J \in \mathcal{I}_\lambda \mid J_k = \varphi^* J \text{ for some } \varphi \in \mathrm{Diff}_{[\omega_\lambda]}(M)\} \subset \mathcal{I}_\lambda$$

Theorem 3.4. *The inclusions*

$$G_\lambda/K_k \hookrightarrow V_k \hookrightarrow U_k$$

are weak homotopy equivalences.

Proof. Corollary 2.6 says that the left map is a weak homotopy equivalence while Theorems 3.1(iv) and 3.2(iv) say that the composite is a weak homotopy equivalence. It follows that the map $V_k \hookrightarrow U_k$ is also a weak homotopy equivalence. □

Any Hirzebruch surface F_n satisfies $H^{0,2}(F_n) = H^{0,2}(TF_n) = 0$ [Ko]. Hence, it follows from Theorem 2.3 that $\mathcal{I}_\lambda$ is an infinite dimensional submanifold of $\mathcal{J}_\lambda$. Since U_0 is open in $\mathcal{J}_\lambda$, we have that V_0 is also open in $\mathcal{I}_\lambda$.

For each $k \in \{1, \ldots, \ell\}$, consider the space

$$\mathcal{M}_k = \{(u, J) \in \mathrm{Map}(S^2, M) \times \mathcal{J}_\lambda \ : u \text{ is a simple } J\text{-holomorphic sphere with } u_*([S^2]) = (1, -k) \in H_2(M; \mathbb{Z})\}\,,$$

which by Proposition 2.8 is an infinite dimensional submanifold of $\mathrm{Map}(S^2, M) \times \mathcal{J}_\lambda$. Positivity of intersections and the adjunction inequality for pseudo-holomorphic curves in almost complex 4-manifolds (see Theorems 2.6.3 and 2.6.4 in [MS]) imply that:

- if $(u, J) \in \mathcal{M}_k$ then the simple J-holomorphic map $u : S^2 \to M$ is an embedding;
- if $(u_1, J), (u_2, J) \in \mathcal{M}_k$ then the simple J-holomorphic maps $u_1, u_2 : S^2 \to M$ have exactly the same image, i.e. they differ only by an holomorphic reparametrization of S^2 given by an element in $PSL(2, \mathbb{C})$.

This means that the projection

$$\pi : \mathcal{M}_k \to U_k \subset \mathcal{J}_\lambda \ , \quad (u, J) \mapsto J$$

is a principal $PSL(2, \mathbb{C})$-bundle map over U_k.

Proposition 3.5. *For any $(u, J) \in \mathcal{M}_k$, the map $u^* : H^{0,1}(TF_n) \to H^{0,1}(u^*(TF_n))$ is an isomorphism, where $n = 2k$ if $M = S^2 \times S^2$ and $n = 2k+1$ if $M = S^2 \tilde{\times} S^2$.*

Proof. Recall that $F_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))$. The inclusion map $u : \mathbb{P}^1 \to F_n$ corresponds to the zero section $\mathbb{P}(0 \oplus \mathcal{O}(-n))$. Let $v : \mathbb{P}^1 \to F_n$ denote the section at infinity: $\mathbb{P}(\infty \oplus \mathcal{O}(-n))$. Let $i : Z(0) \subset F_n$ be the complement of u, and $j : Z(\infty) \subset F_n$ be the complement of v. Notice that the space $Z(0)$ is equivalent to the total space of the bundle $\mathcal{O}(n)$ over v, and $Z(\infty)$ is equivalent to $\mathcal{O}(-n)$ over u. We have a short exact sequence of sheaves of $\mathcal{O}_{F_n}$-modules:

$$0 \to \mathcal{O}_{F_n} \to j_* j^* \mathcal{O}_{F_n} \to v_* v^! \mathcal{O}_{F_n} \to 0$$

where $v^!$ is a functor from $\mathcal{O}_{F_n}$-modules to $\mathcal{O}_{\mathbb{P}^1}$-modules such that the stalk of $v^!\mathcal{S}$ at $x \in \mathbb{P}^1$ (identified with the image of v) is given by the quotient:

$$\mathcal{S}_x \to (j_*j^*\mathcal{S})_x \to (v^!\mathcal{S})_x \to 0$$

Hence, $v^!\mathcal{S}$ may be seen as the higher residues along the normal. Identifying $Z(0)$ with $\mathcal{O}(n)$, it follows that $v^!\mathcal{O}_{F_n} = \mathrm{Sym}_+\mathcal{O}(n)$, where $\mathrm{Sym}_+\mathcal{S}$ stands for the augmentation ideal in the symmetric algebra on $\mathcal{S}$. We may now tensor the above short exact sequence with TF_n to get:

$$0 \to \mathcal{O}(TF_n) \to j_*j^*\mathcal{O}(TF_n) \to v_*v^!\mathcal{O}(TF_n) \to 0$$

In cohomology, we get the following exact sequence:

$$\ldots \to H^1(TF_n) \to H^1(Z(\infty), j^*TF_n) \to H^1(\mathbb{P}^1, v^!TF_n) \to \ldots$$

Since $v^!TF_n$ is the bundle $(\mathcal{O}(2) \oplus \mathcal{O}(n)) \otimes \mathrm{Sym}_+\mathcal{O}(n)$, the last term in the above sequence is trivial. Therefore the restriction map:

$$j^* : H^1(TF_n) \to H^1(Z(\infty), j^*TF_n)$$

is an epimorphism. Now recall that $Z(\infty)$ is the total space of the bundle $\mathcal{O}(-n)$ over u. Let $\pi : Z(\infty) \to \mathbb{P}^1$ be the projection map for this bundle. We get an isomorphism:

$$\pi_* : H^*(Z(\infty), j^*TF_n) \to H^*(\mathbb{P}^1, \pi_*j^*TF_n) = H^*(\mathbb{P}^1, u^*TF_n \otimes \mathrm{Sym}\,\mathcal{O}(n))$$

The long exact sequence in cohomology now gives us an epimorphism:

$$u^* : H^1(Z(\infty), j^*TF_n) \to H^1(\mathbb{P}^1, u^*TF_n \otimes \mathrm{Sym}\,\mathcal{O}(n)) \to H^1(\mathbb{P}^1, u^*TF_n)$$

Composing the above sequence of epimorphisms, we see that the required map u^* in the statement of the proposition is an epimorphism. Finally, notice that $u^*TF_n = \mathcal{O}(2) \oplus \mathcal{O}(-n)$, hence the dimension of $H^1(\mathbb{P}^1, u^*TF_n)$ and $H^1(TF_n)$ both equal $n-1$ (see for example [Ko, Example 6.2(b)(4), p.309]. The proof follows. □

It follows from Theorem 2.9 that, for $k \in \{1, \ldots, \ell\}$, each strata $U_k \subset \mathcal{J}_\lambda$ is transversal to $\mathcal{I}_\lambda \subset \mathcal{J}_\lambda$. Hence, the stratification of $\mathcal{J}_\lambda$ induces by intersection a stratification of $\mathcal{I}_\lambda$ of the form

$$\mathcal{I}_\lambda = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_\ell\,,$$

which satisfies the direct analogues of items (i), (ii), (iii) and (iv) in Theorems 3.1 and 3.2.

Since each stratum V_k, $k \in \{1, \ldots, \ell\}$, is the transversal intersection of U_k and $\mathcal{I}_\lambda$, the tubular neighborhood $NU_k \subset \mathcal{J}_\lambda$ of Theorem 3.1-(v) gives rise to a tubular neighborhood $NV_k \equiv NU_k \cap \mathcal{I}_\lambda$ of V_k in $\mathcal{I}_\lambda$, which fibers over V_k as a ball bundle. By Theorem 2.9, each of these balls can be identified with a neighborhood of zero in

$$H^{0,1}(TF_n) \cong \mathbb{C}^{n-1} \quad \text{[Ko, Example 6.2(b)(4), p.309].}$$

Proof of Theorem 1.1:
It follows from the results stated in the previous two subsections that the inclusion $\mathcal{I}_\lambda \hookrightarrow \mathcal{J}_\lambda$ is transversal to the stratification

$$\mathcal{J}_\lambda = U_0 \sqcup U_1 \sqcup \cdots \sqcup U_\ell \,.$$

By Theorem 3.4, the induced stratification

$$\mathcal{I}_\lambda = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_\ell$$

is such that the inclusions

$$V_k \hookrightarrow U_k \quad \text{and} \quad NV_k \setminus V_k \hookrightarrow NU_k \setminus U_k$$

are weak homotopy equivalences. Writing $U_{0i} = U_0 \sqcup \cdots \sqcup U_i$ and $V_{0i} = V_0 \sqcup \cdots \sqcup V_i$, assume inductively that the inclusion

$$U_{0(i-1)} \to V_{0(i-1)}$$

is a weak equivalence. Then we have a map of excisive triads

$$(V_{0i}, NV_i, V_{0(i-1)}) \to (U_{0i}, NU_i, U_{0(i-1)})$$

which is a weak equivalence when restricted to $NV_i, V_{0(i-1)}$ and their intersection $NV_i \setminus V_i$. It follows (see for instance [May, p. 80]) that

$$V_{0i} \to U_{0i}$$

is a weak equivalence and the result now follows by induction.

Since $\mathcal{J}_\lambda$ is contractible, saying that $\mathcal{I}_\lambda \subset \mathcal{J}_\lambda$ is a weak equivalence is of course equivalent to the statement that $\mathcal{I}_\lambda$ is weakly contractible.

Remark 3.6. *Note that the arguments above also apply to the space of complex structures tamed by a symplectic form. Thus Theorem 1.1 still holds with the word compatible replaced by tame.*

4. Deformation of Hirzebruch Surfaces

In this section, we would like to calculate the deformation space $H^{0,1}(TF_n)$ for a Hirzebruch surface F_n. This will be used in the next section to compute the integral cohomology groups of the symplectomorphism groups of rational ruled surfaces.

Notice that this deformation space is naturally a representation of the group of holomorphic transformations of F_n, which is clear from the geometric picture. Let $G(n)$ denote the group of holomorphic transformations of the Hirzebruch surface F_n. Notice that $G(n)$ need not be a reductive group. However, the maximal reductive subgroup is a rank two Lie group making F_n a toric surface. We will use two pieces of information. Firstly, it is known that $H^{0,2}(TF_n) = 0$, and secondly $dim_{\mathbb{C}}(H^{0,1}(TF_n)) = n-1$, for $n > 0$ and is zero if $n = 0$ (see [Ko, Example 6.2(b)(4), p.309]).

For our calculation ,we will use the $G(n)$ equivariant elliptic complex:

$$\Omega^0(TF_n) \longrightarrow \Omega^{0,1}(TF_n) \longrightarrow \Omega^{0,2}(TF_n)$$

and calculate its index as a virtual representation of $G(n)$, using the Index Theorem. The advantage of having a toric structure, is that we may restrict our attention to the fixed points under the torus action, and use the fixed point formula of Atiyah and Bott [AB2]. The details are as follows.

The toric structure of F_n may be represented by the Delzant polygon in $\mathbb{R}^2$ with vertices $(0,0),(0,1),(1,1),(n+1,0)$, labeled A,B,C,D respectively. We identify $\mathbb{R}^2$ with the Lie algebra of the maximal compact torus $\mathbb{T}$, with the standard orientation. Using the theory of toric manifolds, we may calculate the (multiplicative) weights of T on the tangent space of F_n restriced to the fixed points A,B,C,D. This is given by the following table:

Wt	A	B	C	D
w_1	$1/x$	$1/x$	x	x
w_2	$1/y$	y	y/x^n	x^n/y

To express the topological index of this complex, we define $\sigma_i(w_1,w_2)$ to be the elementary symmetric polynomial in variables w_1,w_2. By the Atiyah-Bott fixed point theorem, the topological index $I(n)$ is given by the character of $\mathbb{T}$:

$$I(n)=\sum\frac{\sigma_1(w_1,w_2)\sigma_2(w_1,w_2)}{1-\sigma_1(w_1,w_2)+\sigma_2(w_1,w_2)}=\sum\frac{\sigma_1(w_1,w_2)\sigma_2(w_1,w_2)}{(1-w_1)(1-w_2)}$$

where the sum is being taken over the four fixed points of $\mathbb{T}$ described above. One may express the above sum in terms of the weights x,y to get the following terms:

$$\frac{(1/x+1/y)1/(xy)}{(1-1/x)(1-1/y)}+\frac{(1/x+y)y/x}{(1-1/x)(1-y)}+\frac{(x+y/x^n)xy/x^n}{(1-x)(1-y/x^n)}+\frac{(x+x^n/y)x^{n+1}/y}{(1-x)(1-x^n/y)}$$

writing the above expression in linearly independent monomials we get:

$$I(n)=2+x+\frac{1}{x}+\frac{1+x+\ldots+x^n}{y}-\frac{y(1+x+\ldots+x^{n-2})}{x^{n-1}},\quad n>1$$

$$I(0)=2+x+\frac{1}{x}+y+\frac{1}{y}\quad,\qquad I(1)=2+x+\frac{1}{x}+\frac{1+x}{y}$$

Notice that the negative term has complex dimension $n-1$. Since we know that this index is given by the virtual representation $Ad(G(n))-H^{0,1}(TF_n)$, it follows that the character of $H^{0,1}(TF_n)$ must be the negative term. As a consequence, we also get the character of the adjoint representation of $G(n)$. In particular the real dimension of the group $G(n)$ is $2n+10$ for $n>0$.

The maximal compact subgroup $K(n)$ of $G(n)$ is well known. For even values of n, it is given by $S^1\times SO(3)$ if $n>0$, and $SO(3)\times SO(3)$ if $n=0$. For odd values of n, this group is $U(2)$. Moreover, we may identify the weights x,y in terms of these groups as follows. If $n=2k$, then the weight x corresponds to the standard torus in $SO(3)$, and the weight y/x^k corresponds to the factor S^1. If $n=2k+1$, then the weights x^{k+1}/y and x^k/y may be seen as the characters of the standard torus in $U(2)$. From the calculation above, we get:

Theorem 4.1. *The representation of $K(n)$ on $H^{0,1}(TF_n)$ is given by*

$$det\otimes sym^{2k-2}(\mathbb{C}^2)\quad \textit{if n=2k},$$

where det represents the standard representation of S^1, and $sym^{2k-2}(\mathbb{C}^2)$ denotes the $2k-2$ symmetric power of the canonical projective representation of $SO(3)$ on $\mathbb{C}^2$. Similarly, if $n=2k+1$, we get the representation

$$det^{-k}\otimes sym^{2k-1}(\mathbb{C}^2)\,,$$

where det denotes the determinant representation of $U(2)$ and $sym^{2k-1}(\mathbb{C}^2)$ denotes the $2k-1$ symmetric power of the canonical representation of $U(2)$ on $\mathbb{C}^2$.

Remark 4.2. *An alternate proof of the previous theorem can be obtained by computing explicitly the action of the Lie algebra of global holomorphic vector fields $H^0(F_n;\Theta)$ on $H^1(F_n;\Theta)$ using explicit bases which can be found in* [Ca, p. 19]. *This approach also permits a better understanding of the normal links of the stratifications of $\mathcal{I}$ and $\mathcal{J}$, a problem considered in* [McD1]. *The induced stratification on the link is precisely the cone on the usual stratification of the appropriate $\mathbb{P}^n$ by the secant varieties of the rational normal curve*[1].

5. The homotopy type of the symplectomorphism group

In this section we apply the results of the previous sections to study the topology of the symplectomorphism groups of rational ruled surfaces.

Let G_λ denote the symplectomorphism group of the standard symplectic form ω_λ on $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$, where $\lambda > 1$ in the untwisted case and $\lambda > 0$ in the twisted case.

If G is a topological group and X is a G-space, we write

$$X_{hG} = EG \times_G X$$

for the homotopy orbits (or Borel construction) of the G-action on X.

The homotopy decomposition of G_λ:
In this section we write $\mathcal{J}_\lambda = \mathcal{J}_{\omega_\lambda}$ and $\mathcal{I}_\lambda = \mathcal{I}_{\omega_\lambda}$. Let $\mathcal{A}_\lambda$ denote the space of almost complex structures tamed by ω_λ, $\mathcal{A}^i_\lambda \subset \mathcal{A}_\lambda$ the subspace of complex structures and Ω_λ the space of symplectic forms cohomologous to ω_λ.

We will need to use the spaces of tame complex structures because the maps between the classifying spaces BG_λ for different values of λ are best understood in terms of these spaces (see (2) below, cf. [McD2]).

We write $\mathcal{J}_{[\lambda]}, \mathcal{I}_{[\lambda]}, \mathcal{A}_{[\lambda]}, \mathcal{A}^i_{[\lambda]}$ for the analogous spaces where the symplectic form ω_λ is replaced with all symplectic forms in the cohomology class of ω_λ.

Let

$$\mathcal{K}_{[\lambda]} = \{(J,\omega) \in \mathcal{J}_{[\lambda]} \times \Omega_\lambda \colon J \text{ is compatible with } \omega\}.$$

$\mathcal{K}^t_{[\lambda]}$ denotes the analogous space with J tamed by ω and $\mathcal{K}^i_{[\lambda]}$ and $\mathcal{K}^{i,t}_{[\lambda]}$ the analogous subspaces where J is required to be integrable.

The above spaces have a natural action of the subgroup of diffeomorphisms preserving the cohomology class $[\omega_\lambda]$ (cf. (1)). In our situation this is the subgroup of diffeomorphisms inducing the identity on homology which we denote by $\mathrm{Diff}_{[0]}$.

Finally, if S is a subset of the isomorphism classes of complex structures, we will decorate the above spaces with a superscript S to indicate that the corresponding complex structures belong to S. In the case of not necessarily integrable complex structures the superscript S indicates representability of the appropriate homology class by an embedded J-holomorphic curve.

Proposition 5.1. *Let S be a set of isomorphism classes of complex structures. The canonical maps*

$$(\mathcal{J}^S_\lambda)_{hG_\lambda} \to (\mathcal{J}^S_{[\lambda]})_{h\,\mathrm{Diff}_{[0]}}$$

[1]The authors thank Barbara Fantechi for bringing this stratification of $\mathbb{P}^n$ to our attention.

are weak equivalences. Similarly for tame and/or integrable complex structures.

Proof. We have a $\mathrm{Diff}_{[0]}$ equivariant bundle

$$\mathcal{J}^S_\lambda \to \mathcal{K}^S_{[\lambda]} \xrightarrow{\pi_2} \Omega_\lambda.$$

As $\Omega_\lambda = \mathrm{Diff}_{[0]}/G_\lambda$ by [LM], this implies that

$$(\mathcal{J}^S_\lambda)_{hG_\lambda} \to (\mathcal{K}^S_{[\lambda]})_{h\,\mathrm{Diff}_{[0]}}$$

is a weak equivalence. We have a commutative diagram

$$\begin{array}{ccc} & & \mathcal{K}^S_{[\lambda]} \\ & \nearrow & \downarrow \pi_2 \\ \mathcal{J}^S_\lambda & \hookrightarrow & \mathcal{J}^S_{[\lambda]} \end{array}$$

and π_2 is a $\mathrm{Diff}_{[0]}$ equivariant map with contractible fibers so the result follows. □

Remark 5.2. *Using the arguments in the proof of Theorem 1.1 (which corresponds to the case when S is the set of all complex structures) one can show that in fact all the inclusions between the four different spaces of complex structures associated to a symplectic form restricted to a set S of isomorphism classes induce weak equivalences. Hence, for each S, the eight spaces mentioned in the statement of Proposition 5.1 are all weakly equivalent. We will not need this, however.*

Lemma 5.3. *Let S denote a set of complex structures. Then $\mathcal{A}^{i,S}_{[\lambda]} = \mathcal{I}^S_{[\lambda]}$.*

Proof. Since $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$ are simply connected, any complex structure is compatible with some symplectic form. Thus each of the sets is a union of the same $\mathrm{Diff}_{[0]}$-orbits indexed by S. □

Recall from [McD2, Lemma 2.2] that $\mathcal{A}_{[\lambda]} \subset \mathcal{A}_{[\lambda+\epsilon]}$ for $\epsilon > 0$. The loop maps $G_\lambda \to G_{\lambda+\epsilon}$ are defined by taking the homotopy orbits of $\mathrm{Diff}_{[0]}$ along this inclusion:

$$BG_\lambda \simeq (\mathcal{A}_{[\lambda]})_{h\,\mathrm{Diff}_{[0]}} \subset (\mathcal{A}_{[\lambda+\epsilon]})_{h\,\mathrm{Diff}_{[0]}} \simeq BG_{\lambda+\epsilon}. \tag{2}$$

Given Theorem 1.1 or, more precisely, Remark 3.6, we may replace $\mathcal{A}_{[\lambda]}$ by the subspaces $\mathcal{A}^i_{[\lambda]}$ of integrable complex structures. Since this space does not vary in intervals of the form $]k, k+1]$ we have the following result of McDuff's.

Proposition 5.4. [McD2, Theorem 1.4] *For $\lambda, \mu \in]k, k+1]$, $BG_\lambda \simeq BG_\mu$.*

We can now prove our main result concerning the homotopy type of BG_μ.

As in the previous section we write $K(n)$ for the connected component of the identity of the Kähler isometry group of the Hirzebruch surface F_n. Thus

$$K(n) = \begin{cases} SO(3) \times SO(3) & \text{if } n = 0 \text{ in the untwisted case,} \\ S^1 \times SO(3) & \text{if } n > 0 \text{ is even,} \\ U(2) & \text{otherwise.} \end{cases} \tag{3}$$

For the sake of simplifying the statement of the following theorem, in the untwisted case, for $0 < \lambda \leq 1$ we write G_λ for the connected component of the identity of the symplectomorphism group G_1. By a theorem of Gromov [Gr], the inclusion $K(0) \subset G_1$ is a weak equivalence and, in the twisted case, $K(1) \subset G_\lambda$ is a weak equivalence for $0 < \lambda \leq 1$.

Theorem 5.5. *Let $\lambda > 0$ and let ℓ be the integer such that $\ell < \lambda \leq \ell + 1$. There is a homotopy pushout square*

(4)
$$\begin{array}{ccc} S^{2m-3}_{hK(m)} & \xrightarrow{\pi} & BK(m) \\ {\scriptstyle j}\downarrow & & \downarrow \\ BG_\lambda & \xrightarrow[i]{} & BG_{\lambda+1} \end{array}$$

where

$$m = \begin{cases} 2\ell + 2 & \text{in the untwisted case,} \\ 2\ell + 3 & \text{in the twisted case,} \end{cases}$$

S^{2m-3} is the unit sphere of the representation of $K(m)$ described in Theorem 4.1, π is the canonical projection and i the map described in (2).

Proof. Consider the usual stratification of $\mathcal{I}_{\lambda+1}$:

$$\mathcal{I}_{\lambda+1} = V_0 \cup \ldots \cup V_{\ell+1}.$$

Writing

$$V_{0k} = V_0 \cup \ldots \cup V_k$$

and $NV_{\ell+1}$ for a tubular neighborhood of $V_{\ell+1}$. We have a homotopy pushout decomposition

(5)
$$\begin{array}{ccc} NV_{\ell+1} \setminus V_{\ell+1} & \longrightarrow & V_{\ell+1} \\ \downarrow & & \downarrow \\ V_{0\ell} & \longrightarrow & \mathcal{I}_{\lambda+1} \end{array}$$

By Theorems 3.4 and 4.1, the projection $NV_{\ell+1} \setminus V_{\ell+1} \to V_{\ell+1}$ has the (weak) homotopy type of the projection $G_{\lambda+1} \times_{K(m)} S^{2m-3} \to G_{\lambda+1}/K(m)$ (see Proposition D.5 for more details). By Proposition 5.1 and Lemma 5.3 the map

$$(V_{0\ell})_{hG_{\lambda+1}} \simeq (\mathcal{I}_{[\lambda]})_{h\,\mathrm{Diff}_{[0]}} \simeq BG_\lambda$$

and the inclusion

$$(V_{0\ell})_{hG_{\lambda+1}} \subset (\mathcal{I}_{\lambda+1})_{hG_{\lambda+1}}$$

is the map (2).

The only reason why the Theorem doesn't follow immediately by taking homotopy orbits of $G_{\lambda+1}$ on (5) is that $NV_{\ell+1}$ is not invariant under the $G_{\lambda+1}$-action. Nevertheless the slice theorem implies that for each compact set $W \subset G$ there is a smaller tubular neighborhood $NV'_{\ell+1}$ sent by the action of W into $NV_{\ell+1}$. This allows us to define an A_∞ action of $G_{\lambda+1}$ on $NV_{\ell+1}$ which, as such, is equivalent to the left $G_{\lambda+1}$-action on $G_{\lambda+1} \times_{K(m)} S^{2m-3}$ (see Appendix D (20) for the details). The result follows. □

We note the immediate consequence of the previous Theorem.

Corollary 5.6. [McD2, Theorem 1.4] *With m as in the statement of Theorem 5.5, for all $\mu > \lambda$, the map $BG_\lambda \to BG_\mu$ is $(2m-3)$-connected.*

In the rest of the paper we will use Theorem 5.5 to obtain results concerning the cohomology of BG_λ. First we need to compute the effect of the map π on cohomology.

There are natural choices of generators under which

$$H^*(BU(2);\mathbb{Z}) = \mathbb{Z}[T,U], \quad H^*(BS^1 \times BSO(3);\mathbb{Z}[1/2]) = \mathbb{Z}[1/2][T,U]$$

with $|T| = 2$ and $|U| = 4$ while

$$H^*(BS^1 \times BSO(3);\mathbb{Z}/2) = \mathbb{Z}/2[T, w_2, w_3]$$

with $|T| = |w_2| = 2$ and $|w_3| = 3$.

Lemma 5.7. *For $n > 1$, the Euler class e_n of the isotropy representation of $K(n)$ is given by*

$$e_n = \begin{cases} T\prod_{i=1}^{\frac{n}{2}-1}(T^2 - i^2U) \in H^*(BK(n);\mathbb{Z}[\frac{1}{2}]) & \text{if } n \text{ is even,} \\ \prod_{i=1}^{\frac{n-1}{2}}\left((2i-1)^2U - i(i-1)T^2\right) \in H^*(BK(n);\mathbb{Z}) & \text{if } n \text{ is odd.} \end{cases}$$

With any coefficients R, the Euler class is a non-zero divisor in $H^*(BK(n);R)$.

Proof. The inclusion of the maximal torus

$$BS^1 \times BS^1 \to BK(n)$$

identifies $H^*(BK(n);R)$ with the ring of symmetric polynomials in $H^*(BS^1 \times BS^1) = R[T_1,T_2]$ (where $R = \mathbb{Z}$ or $\mathbb{Z}[\frac{1}{2}]$). The computation of the Euler class of the representation of Theorem 4.1 restricted to the maximal torus is immediate and the formulas for the Euler class follow.

Since $H^*(BK(n);R)$ is a polynomial ring for any n and R, it remains to check that $e_n \neq 0$ for any coefficients. As $((2i-1)^2, i(i-1)) = 1$, this is the case for e_{2n+1}. At any prime (including 2), the coefficient of T^{2n-1} in the formula for e_{2n} is 1 so the Euler class doesn't vanish. □

Proof of Theorems 1.2 and 1.4:

Consider the diagram (4). The previous lemma says that given any coefficient ring R, the map $H^*(\pi;R)$ is the quotient map

$$H^*(BK(m);R) \to H^*(BK(m);R)/\langle e_m\rangle.$$

Thus the Mayer-Vietoris sequence associated to (4) splits into a short exact sequence

$$0 \to \langle e_m\rangle = \Sigma^{2m-2}H^*(BK(m);R) \to H^*(BG_{\lambda+1};R) \to H^*(BG_\lambda;R) \to 0.$$

When R is a field this splits and since the groups in the extension are finitely generated, it follows that the same is true over $\mathbb{Z}$. The statements of Theorems 1.2 and 1.4 follow by induction.

Computation of the rational cohomology ring of BG_λ:

Let FDiff denote the group of fiber preserving diffeomorphisms of an S^2 bundle over S^2 inducing the identity on homology. It will be clear from the context whether the bundle in question is trivial or not. To compute the cohomology ring of BG_λ we will make use of the loop maps

$$G_\lambda \to \text{FDiff}$$

defined in [McD2].

Writing $\mathcal{A}_{[\infty]}$ for the space of almost complex structures compatible with some symplectic form, McDuff shows [McD2, Proof of Prop 1.1] that the space $\mathcal{A}_{[\infty]}$ is $\mathrm{Diff}_{[0]}$-equivariantly weakly equivalent to the space $\mathrm{Diff}_{[0]} / \mathrm{FDiff}$ and, hence, taking homotopy orbits we have canonical maps

$$BG_\lambda \simeq (\mathcal{A}_{[\lambda]})_{h\,\mathrm{Diff}_{[0]}} \subset (\mathcal{A}_{[\infty]})_{h\,\mathrm{Diff}_{[0]}} \simeq B\,\mathrm{FDiff}\,.$$

By Theorem 1.1 we can replace $\mathcal{A}_{[\infty]}$ with its subspace $\mathcal{I}_{[\infty]}$ of complex structures.

We begin by noting the following consequence of Theorem 5.5.

Proposition 5.8. *For all λ, the map $BG_\lambda \to B\,\mathrm{FDiff}$ induces a surjection on cohomology with any coefficients. Moreover*

$$H^*(B\,\mathrm{FDiff}) = \lim_{\lambda} H^*(BG_\lambda).$$

Proof. By Lemma 5.7 (cf. the proof of Theorems 1.2 and 1.4) the map $H^*(BG_\mu) \to H^*(BG_\lambda)$ is surjective for all $\lambda < \mu$. Since

$$B\,\mathrm{FDiff} = \mathrm{hocolim}_\lambda\, BG_\lambda$$

Lemma 5.6 implies that the connectivity of the maps $BG_\mu \to B\,\mathrm{FDiff}$ tends to ∞ with μ. This proves the first statement. The second statement is obvious. □

The problem that must be overcome in order to compute the cohomology ring of G_λ is that of understanding the effect of the map j in (4) on cohomology since, by Lemma 5.7,

$$\begin{array}{ccc} H^*S^{2m-3}_{hK(m)} & \xleftarrow{\pi^*} & H^*BK(m) \\ {\scriptstyle j^*}\uparrow & & \uparrow \\ H^*BG_\lambda & \xleftarrow[i^*]{} & H^*BG_{\lambda+1} \end{array} \tag{6}$$

is a pushout (or equivalently pullback) square of graded abelian groups and hence a pullback square of graded rings.

We will do this, for cohomology with rational coefficients, by making use of the commutative diagram

$$\begin{array}{ccccc} S^{2m-3}_{hK(m)} & \xrightarrow{\pi} & BK(m) & & \\ {\scriptstyle j}\downarrow & & \downarrow & \searrow & \\ BG_\lambda & \xrightarrow{i} & BG_{\lambda+1} & \searrow & \\ & \searrow & & & B\,\mathrm{FDiff}\,. \end{array}$$

We begin by analyzing $H^*(B\,\mathrm{FDiff};\mathbb{Q})$. By a Theorem of Smale, the inclusion $SO(3) \subset \mathrm{Diff}^+(S^2)$ is a weak equivalence so we will not distinguish between the two. There is a short exact sequence

$$\mathcal{S} \to \mathrm{FDiff} \xrightarrow{e_1} SO(3)$$

where the map e takes a diffeomorphism to the projection of its action on the base and $\mathcal{S}$ denotes the gauge group of the appropriate S^2 bundle over S^2. This yields a fiber sequence

$$B\mathcal{S} \to B\,\mathrm{FDiff} \xrightarrow{Be_1} BSO(3). \tag{7}$$

Now $B\mathcal{S}$ is the component of the null map in

$$\mathrm{Map}(S^2, BSO(3))$$

in the untwisted case, while in the twisted case it is the component of the essential map corresponding to the generator of $\pi_2(BSO(3)) = \mathbb{Z}/2$. The two components are equivalent away from the prime 2 (and hence rationally) since the degree 2 map of S^2 induces an equivalence

$$\mathrm{Map}_1(S^2, BSO(3)) \to \mathrm{Map}_0(S^2, BSO(3)).$$

Lemma 5.9. *$H^*(B\,\mathrm{FDiff}; \mathbb{Q}) = \mathbb{Q}[A, X, Y]$ with $|A| = 2$ and $|X| = |Y| = 4$, i.e. the rationalization of the classifying space $B\,\mathrm{FDiff}_{\mathbb{Q}} \simeq K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 4)$.*

Proof. We have a fiber sequence

$$\Omega S^3 \simeq \Omega_0^2 BSO(3) \to \mathrm{Map}_0(S^2, BSO(3)) \to BSO(3).$$

Since $H^*(\Omega S^3; \mathbb{Q})$ is polynomial generated by a class in degree 2 and $H^*(BSO(3); \mathbb{Q})$ is a polynomial ring generated by a class in degree 4, the Serre spectral sequence collapses and so $H^*(B\mathcal{S}; \mathbb{Q}) = \mathbb{Q}[A, Y]$. Note that by the discussion preceding the statement this is true in both twisted and untwisted cases. Finally, the spectral sequence of (7) also collapses. □

There is a canonical choice for the degree 2 generator and one of the degree 4 generators, namely the pullback of the generator of $H^4(BSO(3); \mathbb{Q})$ under Be_1 but this is *not* the case for the remaining degree 4 generator. In order to continue the computation we must choose well defined generators in $H^*(B\,\mathrm{FDiff}; \mathbb{Q})$.

By Lemma 5.6, in both twisted and untwisted cases, the map

$$BG_2 \to B\,\mathrm{FDiff}$$

is at least 5-connected and hence induces an isomorphism on cohomology in degrees ≤ 4. We will use this fact and the pushout decomposition in Theorem 5.5 to pick the generators of $H^4(B\,\mathrm{FDiff}; \mathbb{Q})$.

By Theorem 4.1, the groups $K(2)$ and $K(3)$ act transitively on the unit sphere of their isotropy representation with isotropy groups $SO(3) \subset K(2) = S^1 \times SO(3)$ and $S^1 \subset K(3) = U(2)$ the circle fixing the second axis in the standard representation of $U(2)$. Thus, according to Theorem 5.5, BG_2 is obtained by the homotopy pushouts

$$\begin{array}{ccc} BSO(3) & \xrightarrow{\pi} & BS^1 \times BSO(3) \\ {\scriptstyle j}\downarrow & & \downarrow \\ BSO(3) \times BSO(3) & \longrightarrow & BG_2 \end{array} \qquad \begin{array}{ccc} BS^1 & \xrightarrow{\pi} & BU(2) \\ {\scriptstyle j}\downarrow & & \downarrow \\ BU(2) & \longrightarrow & BG_2 \end{array}$$

in the untwisted and twisted cases respectively, with π the maps induced by the inclusions of the isotropy groups. The map j in the left square is the inclusion of the diagonal by Iglesias' classification of $SO(3)$-equivariant symplectic four-manifolds [I] (cf. also [AG, Theorem 1.1 (ii)]) while the map j on the right will be described below (Proposition 5.15). Regardless of what these maps are, the Mayer-Vietoris sequence of the above diagrams above together with the connectivity result implies that the inclusions

$$(BSO(3)\times BSO(3))\vee BS^1 \xrightarrow{\psi} B\,\mathrm{FDiff} \quad \text{and} \quad BK(1)\vee BSU(2)\xrightarrow{\psi} B\,\mathrm{FDiff},$$

where $S^1\subset K(2)=S^1\times SO(3)$ is the inclusion of the first factor and $SU(2)\subset K(3)=U(2)$ is the standard inclusion, induce isomorphisms on cohomology (even with integral coefficients) in degrees ≤ 4.

For the rest of this section we will use the following notation for the standard generators in $H^*(BK(n);\mathbb{Q})$:

$$H^*(BK(n);\mathbb{Q})=\begin{cases}\mathbb{Q}[X_0,Y_0] & \text{if } n=0,\\ \mathbb{Q}[A_n,X_n] & \text{if } n>0.\end{cases} \tag{8}$$

with $|A_n|=2$ and $|X_n|=|Y_0|=4$.

Definition 5.10. *Let $X,Y\in H^4(B\,\mathrm{FDiff};\mathbb{Q})$ denote the unique classes such that*

$$\psi^*(X)=\begin{cases}X_0 & \text{in the untwisted case}\\ X_1 & \text{in the twisted case.}\end{cases}\qquad \psi^*(Y)=\begin{cases}Y_0+A_2^2 & \text{in the untwisted case,}\\ X_3 & \text{in the twisted case.}\end{cases}$$

and $T\in H^2(B\,\mathrm{FDiff};\mathbb{Q})$ the unique class such that $\psi^*(T)=A_k$, with $k=2$ in the untwisted case and $k=1$ in the twisted case.

We now need to compute the effect of the inclusions

$$BK(n)\xrightarrow{\psi_n} B\,\mathrm{FDiff}$$

on rational cohomology. By definition of the generators of $H^*(B\,\mathrm{FDiff};\mathbb{Q})$ we know the answer for $n=0,1$ and partly for $n=2,3$. To complete the computation we will use the fact that for each n,m with the same parity there are S^1's inside $K(n)$ and $K(m)$ which are conjugate inside FDiff. This should have an elementary proof but we have only been able to obtain one for $(n,m)=(0,2k)$ (which suffices to compute the cohomology ring in the untwisted case). In order to handle the twisted and untwisted cases uniformly we will take a different tack and use Karshon's classification of S^1-actions to find the conjugate circles inside G_λ.

We will use the standard basis for the maximal tori

$$S^1\times S^1\subset K(n).$$

Our choice of maximal torus $S^1\subset SO(3)$ is the image of $U(1)\times 1$ under the projection $U(2)\to SO(3)$. A circle in $K(n)$ is now described by an integer vector $(a,b)\in\mathbb{Z}^2$.

Proposition 5.11. *Given λ such that there are complex structures compatible with ω_λ isomorphic to F_k and F_l, there are S^1-equivariant symplectomorphisms between*

- *For k and l odd:*
  - *F_k with the S^1-action given by $(\frac{l+1}{2},\frac{l-1}{2})$,*
  - *F_l with the S^1-action given by $(\frac{k+1}{2},\frac{k-1}{2})$.*
- *For k and l even:*
  - *F_k with the S^1-action given by $(\frac{l}{2},1)$,*
  - *F_l with the S^1-action given by $(\frac{k}{2},1)$.*

Proof. The Hirzebruch surfaces F_k with symplectic form in the cohomology class $(\lambda, 1)$ can be obtained (see for instance [AM, Section 2.3]) from Kähler reduction of $\mathbb{C}^4$ by the action of the torus T^2_k

$$(s,t)\cdot(z_1,\ldots,z_4) = (s^k t z_1, t z_2, s z_3, s z_4)$$

at the values

$$\begin{cases}(\lambda + \frac{k}{2}, 1) & \text{if } k \text{ is even,}\\ (\lambda + \frac{k+1}{2}, 1) & \text{if } k \text{ is odd.}\end{cases}$$

The inclusion of the torus T^2_k in the standard torus $T^4 \subset U(4)$ is given by the matrix

$$\begin{bmatrix} k & 1\\ 0 & 1\\ 1 & 0\\ 1 & 0\end{bmatrix}$$

The isometry group of F_k is

$$K(k) = Z_{U(4)}(T^2_k)/T^2_k = \begin{cases}(U(2)\times U(2))/T^2_0 & \text{if } k = 0,\\ (T^2 \times U(2))/T^2_k & \text{if } k \geq 1,\end{cases}$$

(cf. (3)). The maximal torus of $K(k)$ is T^4/T^2_k. A basis is given by projection of the following two vectors $(0,1,0,0)$ and $(0,0,1,0)$ in T^4. The image of the moment map determined by this choice of basis for the maximal torus of $K(k)$ is the standard picture for the Hirzebruch surface with vertices $(0,0), (1,0), (1,\mu)$ and $(0, \mu - k)$ in $\mathbb{R}^2$ with

$$\mu = \begin{cases}\lambda + \frac{k}{2} & \text{if } k \text{ is even,}\\ \lambda + \frac{k+1}{2} & \text{if } k \text{ is odd.}\end{cases}$$

By an appropriate change of basis, from these standard pictures, it is easy to check from [Ka, Theorem 4.1] that (with k, l of the same parity) the S^1's determined by the vectors

$$\begin{bmatrix}\frac{l-k}{2}\\ 1\end{bmatrix} \text{ on } F_k \qquad \begin{bmatrix}\frac{k-l}{2}\\ 1\end{bmatrix} \text{ on } F_l$$

produce equivariantly symplectomorphic manifolds (for instance, the two polygons on the left in [Ka, Figure 4, p.9] correspond to the case $(k,l) = (0,2)$).

For k odd, $K(k) = U(2)_{z_3,z_4}/(\mathbb{Z}/k) \simeq U(2)$ and it is not hard to check that with respect to the standard bases, the k-fold covering map $U(2)_{z_3,z_4} \to K(k)$ is given on the maximal tori by the matrix

$$\begin{bmatrix}\frac{k+1}{2} & \frac{k-1}{2}\\ \frac{k-1}{2} & \frac{k+1}{2}\end{bmatrix}$$

It follows that the image of the circle $(0,1,0,0)$ is the diagonal $(1,1)$ and the circle $(0,0,1,0)$ maps to $(\frac{k+1}{2}, \frac{k-1}{2})$. Applying this change of basis to the maximal torus of $K(k)$ completes the proof of the Proposition for k odd.

For $k = 0$, $K(0) = SO(3) \times SO(3)$ and with respect to the standard bases we clearly have that $(0,1,0,0)$ is sent to $(-1,0)$ and $(0,0,1,0)$ to $(0,1)$. For $k > 0$ even, $K(k) = U(2)_{z_3,z_4}/(\mathbb{Z}/k) \simeq S^1 \times SO(3)$ and one sees that the k-fold covering $U(2)_{z_3,z_4} \to K(k)$ is given on the maximal tori by the matrix

$$\begin{bmatrix}\frac{k}{2} & \frac{k}{2}\\ 1 & -1\end{bmatrix}$$

Thus $(0,1,0,0)$ is sent to $(1,0)$ and $(0,0,1,0)$ is sent to $(\frac{k}{2},1)$. Applying this change of basis completes the proof. □

The following is a simple exercise.

Lemma 5.12. *Writing $H^*(BS^1) = \mathbb{Q}[T]$, $H^*(BS^1 \times BSO(3);\mathbb{Q}) = H^*(BU(2);\mathbb{Q}) = \mathbb{Q}[A,X]$ and $H^*(BSO(3)\times BSO(3);\mathbb{Q}) = \mathbb{Q}[X,Y]$, the map $S^1 \xrightarrow{(a,b)} S^1 \times SO(3)$ induces the map*

$$A \mapsto aT, \qquad X \mapsto b^2T^2,$$

$S^1 \xrightarrow{(a,b)} U(2)$ induces

$$A \mapsto (a+b)T, \qquad X \mapsto (ab)T^2.$$

and $S^1 \xrightarrow{(a,b)} SO(3)\times SO(3)$ induces

$$X \mapsto a^2T^2, \qquad Y \mapsto b^2T^2.$$

Proposition 5.13. *Consider the inclusions $BK(n) \xrightarrow{\psi_n} B\,\mathrm{FDiff}$.*

- *If $n > 0$ is even then*

$$\begin{aligned}\psi_n^*(T) &= \frac{n}{2}A_n\\ \psi_n^*(X) &= X_n\\ \psi_n^*(Y) &= A_n^2 + \frac{n^2}{4}X_n.\end{aligned}$$

- *If $n > 0$ is odd then*

$$\begin{aligned}\psi_n^*(T) &= nA_n\\ \psi_n^*(X) &= \frac{n^2-1}{4}A_n^2 + (1-\frac{n^2-1}{8})X_n\\ \psi_n^*(Y) &= \frac{n^2-1}{8}X_n.\end{aligned}$$

Proof. For $n = 2$, we need to compute the coefficients of X_2 in $\psi_2^*(X)$ and $\psi_2^*(Y)$. By Proposition 5.11 we have a homotopy commutative diagram

$$\begin{CD} BS^1 @>{B(0,1)}>> BK(2)\\ @V{B(1,1)}VV @VV{\psi_2}V\\ BK(0) @>>{\psi_0}> B\,\mathrm{FDiff}\end{CD}$$

and so Lemma 5.12 implies that these coefficients are both 1 as the statement indicates.

For $n = 3$, we only need to find the coefficient of A_3^2 in $\psi_3^*(Y)$ and $\psi_3^*(T)$. By Proposition 5.11 we have a commutative diagram

(9)
$$\begin{CD} BS^1 @>{B(1,0)}>> BK(3)\\ @V{B(2,1)}VV @VV{\psi_3}V\\ BK(1) @>>{\psi_1}> B\,\mathrm{FDiff}\end{CD}$$

and so Lemma 5.12 implies that $\psi_3^*(T) = 3A_3$ and the coefficient of A_3^2 is 0 as the statement indicates.

For all higher n, Proposition 5.11 and the previous arguments tell us the effect on cohomology of $BS^1 \xrightarrow{(a,b)} BK(n) \xrightarrow{\psi_n} B\,\mathrm{FDiff}$ for two independent vectors (a,b) and so simple algebra yields the remaining formulas. □

Corollary 5.14. *The kernel of the map $\psi_n^* : H^*(B\,\mathrm{FDiff};\mathbb{Q}) \to H^*(BK(n);\mathbb{Q})$ is the ideal*

(i) $\langle T\rangle$ *if* $n = 0$,

(ii) $\langle \frac{n^4}{16}X - \frac{n^2}{4}Y + T^2\rangle$ *if n is even,*

(iii) $\langle \frac{(n^2-1)n^2}{8}(X+Y) - n^2Y - \frac{(n^2-1)^2}{32}T^2\rangle$ *if n is odd.*

We begin by proving the analog of [AG, Theorem 1.1(ii) and Corollary 4.5] in the twisted case.

Proposition 5.15. *In the twisted case*

$$BG_2 = \mathrm{hocolim}\left(BU(2) \xleftarrow{B(2,1)} BS^1 \xrightarrow{B(1,0)} BU(2)\right) \tag{10}$$

Moreover

$$H^*(BG_2;\mathbb{Q}) = \mathbb{Q}[T,X,Y]/Y(9X - 2T^2)$$

and $H^*(BG_2;\mathbb{Z})$ is the subring generated[2] *over $\mathbb{Z}$ by T, X, Y and $\frac{TY}{3}$.*

Proof. The map $BS^1 \xrightarrow{B(0,1)} BU(2)$ is (homotopic to) the inclusion of the isotropy group of the representation of $K(3)$ on the normal slice. Proposition 5.11 thus identifies the map j in Theorem 5.5 with $BS^1 \xrightarrow{B(2,1)} BU(2)$ and the first statement follows.

Using the Mayer-Vietoris sequence of (10) one checks that

$$T = (A_1, 3A_3), \quad X = (X_1, 2A_3^2), \quad Y = (0, X_3) \in H^*(BK(1);\mathbb{Z}) \times H^*(BK(3);\mathbb{Z})$$

generate $H^*(BG_2;\mathbb{Z}) \subset H^*(BK(1);\mathbb{Z}) \times H^*(BK(3);\mathbb{Z})$ over $\mathbb{Q}$ and these together with $(0, A_3X_3)$ generate over $\mathbb{Z}$. The result follows. □

The following result collects the statements concerning the rational cohomology ring in Theorems 1.3 and 1.5:

Theorem 5.16. *Let $\ell < \lambda \le \ell + 1$. With the choice of generators indicated in Definition 5.10, the map $H^*(B\,\mathrm{FDiff};\mathbb{Q}) \to H^*(BG_\lambda;\mathbb{Q})$ is the quotient map*

$$\mathbb{Q}[T,X,Y] \longrightarrow \mathbb{Q}[T,X,Y]/(R_\ell(T,X,Y))$$

where

$$R_k(T,X,Y) = \begin{cases} T(X - Y + T^2)\ldots(k^4X - k^2Y + T^2) & \textit{in the untwisted case,}\\ Y\ldots\left((2k+1)^2(\frac{k(k+1)}{2}U - Y) - \frac{k^2(k+1)^2}{2}T^2\right) & \textit{in the twisted case.}\end{cases}$$

and $U = X + Y$.

[2] There are infinitely many relations on the generators of $H^*(BG_2;\mathbb{Z})$ corresponding to the fact that some elements in $\langle Y(9X - 2T^2)\rangle$ are divisible by powers of 3. These divided classes have to be included as relations so as to not introduce torsion.

Proof. The proof is by induction. The result is clear for $\ell = 0$. Assume $\ell \geq 0$ and the result holds for λ. Then by Theorem 5.5 and Lemma 5.7 we have a pullback diagram of rings

$$\begin{array}{ccc} \mathbb{Q}[A_m, X_m]/\langle e_m \rangle & \xleftarrow{\quad \pi^* \quad} & \mathbb{Q}[A_m, X_m] \\ {\scriptstyle j^*}\uparrow & & \uparrow \\ \mathbb{Q}[T, X, Y]/\langle R_\ell(T, X, Y)\rangle & \longleftarrow & H^*(BG_{\lambda+1}; \mathbb{Q}) = \mathbb{Q}[T, X, Y]/I_{\ell+1} \end{array} \tag{11}$$

where we have used Proposition 5.8 to express $H^*(BG_\lambda; \mathbb{Q})$ as a quotient of $H^*(B\,\mathrm{FDiff}; \mathbb{Q})$ by an ideal $I_{\ell+1}$ and e_m denotes the Euler class calculated in Lemma 5.7.

We must have $I_{\ell+1} \subset \langle R_\ell \rangle$. On the other hand, by Corollary 5.14 we also have

$$I_{\ell+1} \subset \begin{cases} \langle \frac{m^4}{16}X - \frac{m^2}{4}Y + T^2 \rangle & \text{if } m \text{ is even,} \\ \langle \frac{(m^2-1)m^2}{8}(X+Y) - m^2 Y - \frac{(m^2-1)^2}{32}T^2 \rangle & \text{if } m \text{ is odd.} \end{cases} \tag{12}$$

Since R_ℓ and the polynomials $k_m(T, X, Y)$ appearing in (12) are coprime it follows that

$$I_{\ell+1} \subset \langle R_\ell k_m \rangle = \langle R_{\ell+1} \rangle.$$

Denoting by d the degree of R_ℓ, the pullback square (11) gives the following generating function for the graded ring $H^*(BG_{\lambda+1}; \mathbb{Q})$:

$$\chi = \frac{1}{(1-t^2)(1-t^4)} + \frac{1-t^d}{(1-t^2)(1-t^4)^2} - \frac{1-t^d}{(1-t^2)(1-t^4)}.$$

This simplifies to

$$\chi = \frac{1-t^{d+4}}{(1-t^2)(1-t^4)^2},$$

which is the generating function for $\mathbb{Q}[T, X, Y]/\langle R_{\ell+1}(T, X, Y)\rangle$. Hence $I_{\ell+1} = \langle R_{\ell+1} \rangle$ [3]. □

Remark 5.17. *It is easy to check that we may rewrite the relations in the cohomology of BG_k in a form similar to the untwisted case by making the following change of variables:*

$$z = T, \quad x = 4U + T^2, \quad y = 4U + 32Y + 2T^2$$

This change of variables gives us the equality:

$$\prod_{i=0}^{k}((2k+1)^2(\frac{k(k+1)}{2}U - Y) - \frac{k^2(k+1)^2}{2}T^2) = 2^{-5(k+1)} \prod_{i=0}^{k}(z^2 + (2i+1)^4 x - (2i+1)^2 y)$$

hence we have:

$$H^*(BG_k, \mathbb{Q}) = \frac{\mathbb{Q}[x, y, z]}{\langle \prod_{i=0}^{k}(z^2 + (2i+1)^4 x - (2i+1)^2 y)\rangle}$$

Remark 5.18. *The ring structure obtained in Theorem 5.16 and Remark 5.17 differs from the one previously calculated in* [AM, Theorem 1.2 and Theorem 1.5]. *There are two different reasons for the difference.*

Regarding the multiples of T^2 in the factors that make up the relation, the problem can be traced to a misapplication of [AA, Theorem 5.4] *in* [AM, p. 1007]. *Using the notation*

[3]Checking that ψ_m^* does indeed send R_ℓ to the ideal generated by e_m (so that j^* is well defined) is a recommended confidence building activity.

of [AA], *$\overline{K}(d\mu)$ depends only on the value of $d\mu$ in the associated graded vector space of the filtration of the Sullivan model by word length. Thus the higher Whitehead products provide information only on the image of the relation in this associated graded vector space and have no bearing on the coefficients of the decomposable T^2 terms (which have higher filtration).*

In this way one sees that the higher Whitehead products in $\pi_*(G_\lambda) \otimes \mathbb{Q}$ can not be used exclusively to compute the ring structure in $H^*(BG_\lambda;\mathbb{Q})$ but only to yield some information on these relations.

The explanation for the remaining difference in the twisted case (regarding the coefficients of X and Y in the factors that make up the relation) lies in a mistake in [AM, Lemma 2.11]. *Indeed, it follows easily from Proposition 5.13 that (with the notation of* [AM]*) we have*

$$\begin{aligned} \alpha_k &= (2k+1)\alpha_0 \in \pi_1(G^1_\lambda), \\ \xi_k &= \xi_0 + \frac{k(k+1)}{2}\eta \in H_3(G^1_\lambda;\mathbb{Z}). \end{aligned}$$

Computation of the cohomology ring away from 2 in the untwisted case:
In this section, we calculate $H^*(BG_\lambda;\mathbb{Z}[1/2])$ in the untwisted case. This will be done by combining Proposition 5.8 with the calculation of $H^*(B\,\mathrm{FDiff};\mathbb{Z}[1/2])$. Henceforth, all spaces will be localized away from the prime 2. We will write $R = \mathbb{Z}[1/2]$.

Since we are working away from the prime 2, it is easy to see that, in the untwisted case, FDiff is equivalent to the semi-direct product

$$\mathcal{G} = SO(3) \ltimes \mathrm{Map}(S^2, SU(2))$$

where $SO(3)$ acts on $\mathrm{Map}(S^2, SU(2))$ by pre-composition of its standard action on $S^2 = \mathbb{C}P^1$.

Theorem 5.19. *Let $R = \mathbb{Z}[1/2]$. $H^*(B\mathcal{G};R)$ is a free module over $R[x,y]$ on generators a_k, b_k with $k \geq 0$:*

$$H^*(B\mathcal{G},R) = R[x,y]\langle a_0, b_0, a_1, b_1, a_2, \ldots\rangle,$$

where $a_0 = 1$, $|x| = |y| = 4$, $|b_k| = 4k+2$, and $|a_k| = 4k$. Moreover, $H^*(B\mathcal{G},R)$ is isomorphic to the subring of $\mathbb{Q}[x,y,z]$, with $|z| = 2$, when b_k and a_k are identified respectively with:

$$\frac{z}{(2k+1)!}\prod_{i=1}^{k}(z^2 + i^4x - i^2y), \quad \frac{z^2}{(2k)!}\prod_{i=1}^{k-1}(z^2 + i^4x - i^2y).$$

Remark 5.20. *One can see that the groups of fiber preserving diffeomorphisms for the twisted and untwisted bundles are equivalent away from the prime 2, and so the previous Theorem describes $H^*(B\,\mathrm{FDiff};\mathbb{Z}[1/2])$ also in the twisted case. We will not use this, however.*

Proof of Theorem 5.19:
Recall that $\mathcal{G} = SO(3) \ltimes \mathrm{Map}(S^2, SU(2))$. $\mathcal{G}$ contains a subgroup $G = SO(3) \times SU(2)$ extending the group of constant maps. Let $\mathbb{T} \times S^1$ be the maximal torus of G. Notice that $\mathbb{T}$ acts on $\mathrm{Map}(S^2, SU(2))$ by pre-composition with the action of $\mathbb{T}$ on S^2 given by rotation about the vertical axis. S^1 is seen as the subgroup of constant maps with value in the maximal torus of $SU(2)$. Let τ_1 and τ_2 be elements in each factor of G that map to generators of the Weyl group. Notice also that $\mathcal{G}$

contains the $\mathbb{T}$ invariant subgroup $\Omega^2 SU(2) \subset \mathrm{Map}(S^2, SU(2))$ consisting of maps that take the north pole to the identity element. Here and henceforth, we will fix the north pole of S^2 as the basepoint.

The proof of Theorem 5.19 uses a sequence of inclusions of subgroups:

$$\Omega^2 SU(2) \subset \mathcal{K} \subset \mathcal{H} \subset \mathcal{G}$$

that induces maps of classifying spaces:

$$\Omega SU(2) \longrightarrow B\mathcal{K} \longrightarrow B\mathcal{H} \longrightarrow B\mathcal{G}$$

with $B\mathcal{H}$ equivalent to the homotopy orbit space of a $\mathbb{Z}/2\langle\tau_2\rangle$-action on $B\mathcal{K}$, and $B\mathcal{G}$ being equivalent to the homotopy orbit space of the $\mathbb{Z}/2\langle\tau_1\rangle$-action on $B\mathcal{H}$.

Definition 5.21. *$\mathcal{H} \subset \mathcal{G}$ is defined as the subgroup*

$$\mathcal{H} = \mathbb{T} \ltimes \mathrm{Map}(S^2, SU(2)) \subset SO(3) \ltimes \mathrm{Map}(S^2, SU(2)) = \mathcal{G}$$

The element τ_1 acts on $\mathcal{H}$ by conjugation in $\mathcal{G}$. This action is given by inversion on the $\mathbb{T}$ factor, and by the action induced on $\mathrm{Map}(S^2, SU(2))$ via the action of $\tau_1 \in SO(3)$ by left multiplication on $S^2 = SO(3)/\mathbb{T}$. This induces a τ_1-action on $B\mathcal{H}$.

Definition 5.22. *$\mathcal{K} \subset \mathcal{H}$ is defined as the subgroup*

$$\mathcal{K} = (\mathbb{T} \times S^1) \ltimes \Omega^2 SU(2) \subset (\mathbb{T} \times SU(2)) \ltimes \Omega^2 SU(2) = \mathbb{T} \ltimes \mathrm{Map}(S^2, SU(2)) = \mathcal{H}$$

where $S^1 \ltimes \Omega^2 SU(2)$ may be seen as the subspace of maps from S^2 to $SU(2)$ that map the basepoint of S^2 to S^1. The τ_2-action on $\mathcal{K}$ is given by conjugation in $\mathcal{H}$. This action preserves the $\mathbb{T}$ factor, acts by inversion on the S^1-factor, and acts by pointwise conjugation with $\tau_2 \in SU(2)$ on $\Omega^2 SU(2)$. As before, this induces an action of τ_2 on $B\mathcal{K}$.

From the above descriptions, we can describe the homotopy type of the respective classifying spaces away from the prime 2:

$$B\mathcal{K} = E(\mathbb{T} \times S^1) \times_{\mathbb{T}\times S^1} \Omega^2 BSU(2) \tag{13}$$

$$B\mathcal{H} = E\mathbb{T} \times_{\mathbb{T}} \mathrm{Map}(S^2, BSU(2)) \tag{14}$$

$$B\mathcal{G} = ESO(3) \times_{SO(3)} \mathrm{Map}(S^2, BSU(2)) \tag{15}$$

It is a standard argument to identify the invariant cohomology rings

$$H^*(B\mathcal{K}, R)^{\tau_2} = H^*(B\mathcal{H}, R), \quad \text{and} \quad H^*(B\mathcal{H}, R)^{\tau_1} = H^*(B\mathcal{G}, R).$$

The action of τ_1 is subtle. To understand this action, we start by $\mathbb{T}$- equivariantly decomposing S^2 as a pushout of two hemispheres intersecting over the equator. From (14) we get a pullback diagram:

$$\begin{array}{ccc} B\mathcal{H} & \xrightarrow{ev_N} & B\mathbb{T} \times BSU(2) \\ \downarrow{\scriptstyle ev_S} & & \downarrow \\ B\mathbb{T} \times BSU(2) & \longrightarrow & E\mathbb{T} \times_{\mathbb{T}} LBSU(2) \end{array}$$

where ev_N and ev_S denote evaluation at the north and south pole. Also, the notation $LBSU(2)$ refers to the free loop space of $BSU(2)$ i.e. $\mathrm{Map}(S^1, BSU(2))$. We note that the τ_1-action on $B\mathcal{H}$ described above has the property of swapping the corners of the pullback diagram since it interchanges the north and south pole of S^2, and inducing an action on the equator given by inversion.

Notice that (13) shows that the cohomology of $B\mathcal{K}$ is equivalent to the equivariant cohomology of $\Omega SU(2)$. This calculation has been made in [HHH].

Before we give a description of this cohomology, let us set some notation. Let $u, v \in H^2(B\mathbb{T} \times BS^1, \mathbb{Z})$ be the canonical generators corresponding to the two factors respectively. It also follows from an easy spectral sequence argument that $H^2(B\mathcal{G}, \mathbb{Z})$ is a free $\mathbb{Z}$ module generated by a unique class w that restricts to the generator of $H^2(\Omega SU(2), \mathbb{Z})$. Moreover, w restricts trivially to the cohomology of $B(\mathbb{T} \times S^1)$ since the inclusion of $\mathbb{T} \times S^1 \subset \mathcal{G}$ factors through the group $SU(2) \times SU(2)$. Hence, w restricts to the canonical generator of $\mathbb{T} \times S^1$-equivariant cohomology of $\Omega SU(2)$ described in [HHH]. Therefore, we have:

Theorem 5.23. [HHH] *Let u, v and w be the classes defined above. Then, the cohomology of $B\mathcal{K}$ with coefficients in the ring $\mathbb{Z}$ is given by the the following free module over the ring $\mathbb{Z}[u, v]$ on generators f_k, g_k, $k \geq 0$:*

$$H^*(B\mathcal{K}, \mathbb{Z}) = \mathbb{Z}[u, v]\langle g_0, f_0, g_1, f_1, g_2, \ldots \rangle, \quad \textit{where} \quad g_0 = 1$$

where the degree of f_k is $4k + 2$, and that of g_k is $4k$. Moreover, as a ring, we may identify $H^*(B\mathcal{K}, \mathbb{Z})$ as the subring of $\mathbb{Q}[u, v, w]$, where the degree of Z is 2, and the elements f_k and g_k are identified respectively to the elements:

$$\frac{w}{(2k+1)!} \prod_{i=1}^{k} ((w + i^2 u)^2 - 4i^2 v^2), \quad \frac{w(w + k^2 u + 2kv)}{(2k)!} \prod_{i=1}^{k-1} ((w + i^2 u)^2 - 4i^2 v^2).$$

From the previous remark, and the description in [HHH] we see that the τ_2 action has the property:

$$\tau_2(u) = u, \quad \tau_2(v) = -v, \quad \tau_2(w) = w$$

Filtering $H^*(B\mathcal{K}, R)$ by powers of v, and taking $\mathbb{Z}/2$ invariants, we easily derive the following result

Proposition 5.24. *The cohomology of $B\mathcal{H}$ with coefficients in the ring R is given by the the following free module over the ring $R[u, v^2]$ on generators b_k, a_k, $k \geq 0$:*

$$H^*(B\mathcal{H}, R) = R[u, v^2]\langle a_0, b_0, a_1, b_1, a_2, \ldots \rangle, \quad \textit{where} \quad a_0 = 1$$

where the degree of b_k is $4k + 2$, and that of a_k is $4k$. Moreover, as a ring, we may identify $H^*(B\mathcal{H}, R)$ as the subring of $\mathbb{Q}[u, v, w]$, where the degree of w is 2, and the elements b_k and a_k are identified respectively to the elements:

$$\frac{w}{(2k+1)!} \prod_{i=1}^{k} ((w + i^2 u)^2 - 4i^2 v^2), \quad \frac{w^2}{(2k)!} \prod_{i=1}^{k-1} ((w + i^2 u)^2 - 4i^2 v^2).$$

The hard part now is to identify the action of τ_1 on $H^*(B\mathcal{H}, R)$.

Proposition 5.25. *The action of τ_1 on $H^*(B\mathcal{H}, R)$ is given by*

$$\tau_1(w) = w, \quad \tau_1(u) = -u, \quad \tau_1(v^2) = v^2 - uw$$

Before we proceed with the proof of the Proposition 5.25, let us see how we may derive the Theorem 5.19 for the cohomology of $B\mathcal{G}$ from this action. Observe that the following elements are invariant under τ_1:

$$w, \quad u^2, \quad 2v^2 - uw$$

Notice also that we have the following equality:

$$(w+i^2u)^2 - 4i^2v^2 = w^2 + i^4u^2 - i^2 2(2v^2 - uw) \tag{16}$$

It follows that all the elements a_k, b_k are invariant under τ_1. We claim:

Proposition 5.26.

$$H^*(B\mathcal{G}, R) = H^*(B\mathcal{H}, R)^{\mathbb{Z}/2} = R[u^2, 2v^2 - uw]\langle a_0, b_0, a_1, b_1, \dots\rangle$$

Proof. Since u, w are elements of $H^*(B\mathcal{H}, R)$, we may replace the element v^2 by $2v^2 - uw$ to get:

$$H^*(B\mathcal{H}, R) = R[u, 2v^2 - uw]\langle a_0, b_0, a_1, b_1, \dots\rangle$$

The proof follows on filtering $H^*(B\mathcal{H}, R)$ by powers of a, and taking invariants. □

Over R we can replace the generator $2v^2 - uw$ with $2(2v^2 - uw)$ hence, taking note of (16) and setting

$$x = u^2, \quad y = 2(2v^2 - uw), \quad z = w \tag{17}$$

we obtain

$$H^*(B\mathcal{G}, R) = R[x, y]\langle a_0, b_0, a_1, b_1, a_2, \dots\rangle, \quad \text{where} \quad a_0 = 1.$$

This completes the proof of Theorem 5.19 assuming Proposition 5.25.

Proof of Proposition 5.25. It is clear from the definition that the action must preserve w. It also follows from Definition 5.21 that the action reverses the sign of u. Hence, only the action on v^2 needs to be described. Notice that $v^2 = ev_N^*(\sigma)$, where $\sigma \in H^4(BSU(2), \mathbb{Z})$ is a generator. Recall that the $\mathbb{Z}/2\langle\tau_1\rangle$ action on the diagram defining $B\mathcal{H}$ as a pullback has the property of switching the corners. Hence, it follows that $\tau_1(v^2) = ev_S^*(\sigma)$. We reconsider the pullback:

$$\begin{array}{ccc} B\mathcal{H} & \xrightarrow{ev_N} & B\mathbb{T} \times BSU(2) \\ \downarrow{\scriptstyle ev_S} & & \downarrow \\ B\mathbb{T} \times BSU(2) & \longrightarrow & E\mathbb{T} \times_{\mathbb{T}} LBSU(2) \end{array}$$

and consider the Serre spectral sequence for the right vertical map, seen as a fibration, whose fiber is $\Omega SU(2)$. Let α be the element in the E_2-term representing a generator of $H^2(\Omega SU(2), \mathbb{Z})$. It is well known [K, ABKS] that the integral cohomology of $E\mathbb{T} \times_{\mathbb{T}} LBSU(2)$ is free in degree 2, generated by u, and trivial in degree 5. Hence the class $\pm\alpha u$ represents σ in the cohomology of the total space $B\mathbb{T} \times BSU(2)$. Recall that the class $w \in H^2(B\mathcal{H}, R)$ is represented by α in the Serre spectral sequence for the left vertical fibration (up to an indeterminacy given by a multiple of u). Hence, by choosing a suitable sign for w, we notice that $\tau_1(v^2) = -uw$ modulo lower filtration in the Serre spectral sequence of the left vertical map, seen as a fibration. We may therefore write:

$$\tau_1(v^2) = -uw + av^2 + bu^2$$

applying τ_1 again to this equation tells us that $a = 1, b = 0$. □

We can now prove Theorem 1.3.

Proof of Theorem 1.3:
The canonical projection map

$$B\phi : B\mathcal{G} \longrightarrow B\,\mathrm{FDiff}$$

is an isomorphism on cohomology with $\mathbb{Z}[1/2]$ coefficients. Comparing the definition of the classes x, y and z in the rational cohomology of $B\mathcal{G}$ given in (17) below, with that of X, Y and T in Definition 5.10, we see that $B\phi^*(X) = x, B\phi^*(Y) = y$ and $B\phi^*(T) = z$. Henceforth we identify these graded rings by the map $B\phi^*$.

Recall that we have an inclusion map

$$BK(2k) \xrightarrow{\psi_{2k}} B\,\mathrm{FDiff}$$

whose effect on rational cohomology was described in Proposition 5.13. We recall that

$$\psi_{2k}^*(X) = X_{2k}, \quad \psi_{2k}^*(Y) = k^2 X_{2k} + A_{2k}^2, \quad \psi_{2k}^*(T) = kA_{2k}$$

where we are using the notation established in (8). Consider the classes a_k, b_{k-1} in the cohomology of $B\,\mathrm{FDiff}$ defined in Theorem 5.19. Then

$$\psi_{2k}^*(b_{k-1}) = A_{2k} \prod_{i=1}^{k-1} (A_{2k}^2 - i^2 X_{2k}) = e_{2k}, \qquad \psi_{2k}^*(a_k) = \frac{A_{2k}\, e_{2k}}{2}$$

where e_{2k} denotes the Euler class calculated in Lemma 5.7. Hence, $B\psi_{2k}^*$ maps the submodule $R[x, y]\langle b_{k-1}, a_k\rangle$ isomorphically onto the ideal generated by the Euler class e_{2k}. Moreover, it is also clear that the classes b_i and a_j map to zero if $i \geq k$ and $j > k$. It now follows by induction using Theorem 5.5 and Proposition 5.8 that the kernel of the map

$$H^*(B\,\mathrm{FDiff}, R) \to H^*(BG_l, R)$$

is the submodule generated over $R[x, y]$ by the elements b_i, a_j where $i \geq l$ and $j > l$. Furthermore, one has the following identification:

$$H^*(BG_\lambda, \mathbb{Q}) = \mathbb{Q}[x, y]\langle a_0, b_0, a_1, \ldots a_l\rangle = \frac{\mathbb{Q}[x, y, z]}{\langle z \prod_{i=1}^{l} (z^2 + i^4 x - i^2 y)\rangle}$$

and so we may identify $H^*(BG_\lambda, R) = R[x, y]\langle a_0, b_0, a_1, \ldots a_l\rangle$ naturally as a subring of the above quotient. This completes the proof of Theorem 1.3.

APPENDIX A. $\bar{\partial}$-OPERATORS ON ALMOST-COMPLEX MANIFOLDS

Let (M, J) be an almost-complex manifold. As usual, we identify TM with the $+i$ eigenspace of the action of J on $TM \otimes \mathbb{C}$. For a complex valued function $f \in \Omega^0(M)$ we have that $\bar{\partial} f \in \Omega^{0,1}_J(M)$ is given by

$$\bar{\partial} f = \frac{1}{2}(d + iJd)f \,,$$

where $(Jdf)(X) = df(JX)$ for any vector field $X \in \Omega^0(TM)$. This formula for $\bar{\partial} f$ reflects the decomposition $\Omega^1(M) = \Omega^{1,0}_J(M) \oplus \Omega^{0,1}_J(M)$, where

$$\alpha \in \Omega^{1,0}_J(M) \Leftrightarrow J\alpha = i\alpha \Leftrightarrow \alpha(JX) = i\alpha(X) \,,\ \forall X \in \Omega^0(TM)$$

while

$$\alpha \in \Omega^{0,1}_J(M) \Leftrightarrow J\alpha = -i\alpha \Leftrightarrow \alpha(JX) = -i\alpha(X) \,,\ \forall X \in \Omega^0(TM) \,.$$

Any 1-form $\alpha \in \Omega^1(M)$ can be written uniquely as $\alpha = \alpha^{1,0} + \alpha^{0,1}$, with $\alpha^{1,0} \in \Omega^{1,0}_J(M)$ and $\alpha^{0,1} \in \Omega^{0,1}_J(M)$, where

$$\alpha^{1,0} = \frac{1}{2}(\alpha - iJ\alpha) \quad \text{and} \quad \alpha^{0,1} = \frac{1}{2}(\alpha + iJ\alpha) \,.$$

For 2-forms we have the decomposition $\Omega^2(M) = \Omega^{2,0}_J(M) \oplus \Omega^{0,2}_J(M) \oplus \Omega^{1,1}_J(M)$, where

$$\begin{aligned}
&\alpha \in \Omega^{2,0}_J(M) \Leftrightarrow \alpha(JX, \cdot) = i\alpha(X, \cdot) \,,\ \forall X \in \Omega^0(TM) \,,\\
&\alpha \in \Omega^{0,2}_J(M) \Leftrightarrow \alpha(JX, \cdot) = -i\alpha(X, \cdot) \,,\ \forall X \in \Omega^0(TM) \,,\ \text{and}\\
&\alpha \in \Omega^{1,1}_J(M) \Leftrightarrow \alpha(J\cdot, J\cdot) = \alpha(\cdot, \cdot) \,.
\end{aligned}$$

Any 2-form $\alpha \in \Omega^2(M)$ can be written uniquely as $\alpha = \alpha^{2,0} + \alpha^{0,2} + \alpha^{1,1}$, with $\alpha^{2,0} \in \Omega^{2,0}_J(M)$, $\alpha^{0,2} \in \Omega^{0,2}_J(M)$ and $\alpha^{1,1} \in \Omega^{1,1}_J(M)$, where

$$\begin{aligned}
\alpha^{2,0}(X, Y) &= \frac{1}{4}\left\{\alpha(X, Y) - \alpha(JX, JY) - i\left(\alpha(X, JY) + \alpha(JX, Y)\right)\right\} \,,\\
\alpha^{0,2}(X, Y) &= \frac{1}{4}\left\{\alpha(X, Y) - \alpha(JX, JY) + i\left(\alpha(X, JY) + \alpha(JX, Y)\right)\right\} \ \text{and}\\
\alpha^{1,1}(X, Y) &= \frac{1}{2}\left(\alpha(X, Y) + \alpha(JX, JY)\right) \,,\ \forall X, Y \in \Omega^0(TM) \,.
\end{aligned}$$

Definition A.1. *Given $\alpha \in \Omega^{0,1}_J(M)$, the form $\bar{\partial}\alpha \in \Omega^{0,2}_J(M)$ is defined by*

$$\bar{\partial}\alpha = (d\alpha)^{0,2} \,.$$

Remark A.2. *It is immediate from this definition that this $\bar{\partial}$-operator, $\bar{\partial} : \Omega^{0,1}_J(M) \to \Omega^{0,2}_J(M)$, satisfies the Leibnitz rule*

$$\bar{\partial}(f\alpha) = (\bar{\partial} f) \wedge \alpha + f\bar{\partial}\alpha \,,\ \forall f \in \Omega^0(M) \,,\ \alpha \in \Omega^{0,1}_J(M) \,.$$

Definition A.3. *Given an almost-complex manifold (M, J), the Nijenhuis tensor N_J is defined by*

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \,,\ \forall X, Y \in \Omega^0(TM) \,.$$

Note that $N_J \in \Omega^{0,2}_J(TM) = \Omega^{0,2}_J(M) \otimes \Omega^0(TM)$, i.e.

$$N_J(JX, Y) = N_J(X, JY) = -JN_J(X, Y) \,,\ \forall X, Y \in \Omega^0(TM) \,.$$

Proposition A.4. *For $\alpha \in \Omega^{0,1}_J(M)$, the form $\bar\partial\alpha \in \Omega^{0,2}_J(M)$ is given by*

$$(\bar\partial\alpha)(X,Y) = \bar\partial_X(\alpha(Y)) - \bar\partial_Y(\alpha(X)) + \frac{\alpha\left\{[JX,JY]-[X,Y]-(1/2)N_J(X,Y)\right\}}{2}\,,$$

for all $X,Y \in \Omega^0(TM)$.

Proof. For any $\alpha \in \Omega^1(M)$, the exterior derivative $d\alpha \in \Omega^2(M)$ is given by

$$(d\alpha)(X,Y) = X\cdot\alpha(Y) - Y\cdot\alpha(X) - \alpha([X,Y])\,.$$

Using this formula and the fact that $\alpha(JX) = -i\alpha(X)$, we get that

$$\begin{aligned}
4(\bar\partial\alpha)(X,Y) &= 4(d\alpha)^{0,2}(X,Y)\\
&= (d\alpha)(X,Y) - (d\alpha)(JX,JY) + i[(d\alpha)(X,JY) + (d\alpha)(JX,Y)]\\
&= X\cdot\alpha(Y) - Y\cdot\alpha(X) - \alpha([X,Y]) - (JX)\cdot\alpha(JY)\\
&\quad + (JY)\cdot\alpha(JX) + \alpha([JX,JY]) + i\left\{X\cdot\alpha(JY) - (JY)\cdot\alpha(X)\right.\\
&\quad \left.-\alpha([X,JY]) + (JX)\cdot\alpha(Y) - Y\cdot\alpha(JX) - \alpha([JX,Y])\right\}\\
&= 2[X\cdot\alpha(Y) + i(JX)\cdot\alpha(Y))] - 2[Y\cdot\alpha(X) + i(JY)\cdot\alpha(X))]+\\
&\quad + \alpha([JX,JY] + J[X,JY] + J[JX,Y] - [X,Y])\\
&= 4[\bar\partial_X(\alpha(Y)) - \bar\partial_Y(\alpha(X))] + \alpha\left(2([JX,JY]-[X,Y]) - N_J(X,Y)\right)\,.
\end{aligned}$$

□

Definition A.5. *Define the operator $\bar\partial : \Omega^0(TM) \to \Omega^1(TM)$, $Y \mapsto \bar\partial Y$, by*

$$(\bar\partial Y)(X) \equiv \bar\partial_X Y \equiv \frac{1}{2}\left\{[X,Y] + J[JX,Y] + \frac{1}{2}N_J(X,Y)\right\}\,,\ \forall X,Y \in \Omega^0(TM)\,.$$

Proposition A.6. *The operator $\bar\partial : \Omega^0(TM) \to \Omega^1(TM)$ has the following properties:*

(i) *$(\bar\partial Y)(X)$ is a tensor in X.*
(ii) *$(\bar\partial Y) \in \Omega^{0,1}_J(TM)$.*
(iii) *$\bar\partial(f\cdot Y) = (\bar\partial f)\otimes Y + f\cdot(\bar\partial Y)$, for any function $f \in \Omega^0(M)$.*
(iv) *$\bar\partial_X(JY) = J\bar\partial_X Y$.*

Proof. Property (i) follows from the fact that, for any $f \in \Omega^0(M)$ and $X,Y \in \Omega^0(TM)$, we have

$$\begin{aligned}
2(\bar\partial Y)(fX) &= [fX,Y] + J[J(fX),Y] + \frac{1}{2}N_J(fX,Y)\\
&= f[X,Y] - df(Y)X + fJ[JX,Y] - Jdf(Y)JX\frac{1}{2}fN_J(X,Y)\\
&= f\left([X,Y] + J[JX,Y] + \frac{1}{2}N_J(X,Y)\right)\\
&= 2f(\bar\partial Y)(X)\,.
\end{aligned}$$

To prove property (ii) note that

$$\begin{aligned}
2(\bar\partial Y)(JX) &= [JX,Y] + J[J^2X,Y] + \frac{1}{2}N_J(JX,Y)\\
&= (-J)J[JX,Y] + (-J)[X,Y] + \frac{1}{2}(-J)N_J(X,Y)\\
&= 2(-J)(\bar\partial Y)(X)\,.
\end{aligned}$$

Property (iii) follows from the following calculation:

$$\begin{aligned} 2\bar\partial(fY)(X) &= [X, fY] + J[JX, fY] + \frac{1}{2}N_J(X, fY) \\ &= (X\cdot f)Y + J[(JX)\cdot f]Y + 2f\bar\partial_X(Y) \\ &= [(df)(X) + i(df)(JX)]Y + 2f\bar\partial_X(Y) \\ &= [2(\bar\partial f)(X)]Y + 2f\bar\partial_X(Y)\,. \end{aligned}$$

To prove property (iv) note that

$$\begin{aligned} 2[\bar\partial_X Y + J\bar\partial_X(JY)] &= [X, Y] + J[JX, Y] + \frac{1}{2}N_J(X, Y) \\ &\quad + J[X, JY] - [JX, JY] + \frac{1}{2}JN_J(X, JY) \\ &= [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \\ &\quad + \frac{1}{2}(N_J(X, Y) - J^2N_J(X, Y)) \\ &= -N_J(X, Y) + N_J(X, Y) \\ &= 0\,. \end{aligned}$$

□

Proposition A.7. *On an almost complex manifold (M, J), the Lie derivative of J with respect to a vector field $Y \in \Omega^0(TM)$ is given by*

$$\mathcal{L}_Y J = (2J)(\bar\partial Y) + \frac{1}{2}J(Y \lrcorner N_J) \in \Omega^{0,1}_J(TM)\,.$$

Proof. We have that

$$\begin{aligned} (\mathcal{L}_Y J)(X) &= [Y, JX] - J[Y, X] = J[X, Y] - [JX, Y] \\ &= (2J)\frac{1}{2}\left([X, Y] + J[JX, Y] + \frac{1}{2}(N_J(X, Y) - N_J(X, Y))\right) \\ &= (2J)(\bar\partial_X Y) + \frac{1}{2}JN_J(Y, X)\,. \end{aligned}$$

□

Definition A.8. *Define the operator $\bar\partial : \Omega^{0,1}_J(TM) \to \Omega^{0,2}_J(TM)$, $A \mapsto \bar\partial A$, as the unique linear operator which is given on elements of the form $A = \alpha \otimes Z \in \Omega^{0,1}_J(M) \otimes \Omega^0(TM) = \Omega^{0,1}_J(TM)$ by*

$$\bar\partial A = \bar\partial(\alpha \otimes Z) = \bar\partial\alpha \otimes Z - \alpha \wedge \bar\partial Z\,.$$

Proposition A.9. *For $A \in \Omega^{0,1}_J(TM)$, the form $\bar\partial A \in \Omega^{0,2}_J(TM)$ is given by*

$$(\bar\partial A)(X, Y) = \bar\partial_X(AY) - \bar\partial_Y(AX) + \frac{A\{[JX, JY] - [X, Y] - (1/2)N_J(X, Y)\}}{2}\,, \tag{18}$$

for all $X, Y \in \Omega^0(TM)$.

Proof. One just needs to check that the $\bar\partial$-operator defined by (18) has the characterizing property of Definition A.8. This follows by direct calculation from the definitions and Propositions A.4 and A.6. □

APPENDIX B. THE DERIVATIVE OF THE NIJENHUIS TENSOR

Let $\mathcal{J}$ denote the space of almost-complex structures on the manifold M. The Nijenhuis tensor can be seen as a map

$$N : \mathcal{J} \to \Omega^2(TM)\,,$$

with derivative

$$dN : T\mathcal{J} \to \Omega^2(TM)\,.$$

Given $J \in \mathcal{J}$ we have that

$$T_J\mathcal{J} = \{A \in \mathrm{Aut}(TM)\ :\ AJ + JA = 0\} \equiv \Omega^{0,1}_J(TM)\,,$$

wich means that

$$dN_J : \Omega^{0,1}_J(TM) \to \Omega^2(TM)\,.$$

Lemma B.1. *Given $J \in \mathcal{J}$ and $A \in \Omega^{0,1}_J(TM)$, we have that $dN_J(A) \in \Omega^2(TM)$ is given by*

$$dN_J(A)(X,Y) = [AX, JY]+[JX, AY]-J([X, AY]+[AX, Y])-A([X, JY]+[JX, Y])\,,$$

for any $X, Y \in \Omega^0(TM)$.

Proof. Consider a one parameter family of almost complex structures $J_t \in \mathcal{J}$ given by

$$J_t = J + tA + O(t^2)\,, \quad \text{with } A \in \Omega^{0,1}_J(TM).$$

Then

$$\begin{aligned} N_{J_t}(X,Y) &= [(J+tA)X, (J+tA)Y] - (J+tA)[X, (J+tA)Y] \\ &\quad - (J+tA)[(J+tA)X, Y] - [X,Y] + O(t^2) \\ &= N_J(X,Y) + t\,\{[AX, JY] + [JX, AY] - J[X, AY] \\ &\quad -A[X, JY] - J[AX, Y] - A[JX, Y]\} + O(t^2)\,, \end{aligned}$$

and so

$$\begin{aligned} \frac{d}{dt}\left(N_{J_t}(X,Y)\right)|_{t=0} &= [AX, JY] + [JX, AY] - J([X, AY] + [AX, Y]) \\ &\quad - A([X, JY] + [JX, Y])\,. \end{aligned}$$

□

Proposition B.2. *Given $J \in \mathcal{J}$ and $A \in \Omega^{0,1}_J(TM)$, we have that $dN_J(A) \in \Omega^2(TM) = \Omega^{2,0}_J(TM) \oplus \Omega^{0,2}_J(TM) \oplus \Omega^{1,1}_J(TM)$ can be decomposed as*

$$dN_J(A) = (dN_J(A))^{0,2} + (dN_J(A))^{2,0} + (dN_J(A))^{1,1}\,,$$

where

$$\begin{aligned} (dN_J(A))^{0,2}(X,Y) &= (-2J)(\bar\partial A)(X,Y)\,, \\ (dN_J(A))^{2,0}(X,Y) &= \frac{1}{2} JA(N_J(X,Y)) \quad \textit{and} \\ (dN_J(A))^{1,1}(X,Y) &= \frac{1}{2} J[N_J(X, AY) + N_J(AX, Y)]\,, \end{aligned}$$

for any $X, Y \in \Omega^0(TM)$.

Proof. Using Lemma B.1, the fact that $A \in \Omega^{0,1}_J(TM)$, i.e. $AJ + JA = 0$, and Proposition A.9, we have that

$$\begin{aligned} dN_J(A)(X,Y) &= [AX, JY] + [JX, AY] - J([X, AY] + [AX, Y]) \\ &\quad - A([X, JY] + [JX, Y]) \\ &= (-J)([X, AY] + J[JX, AY]) - (-J)([Y, AX] + J[JY, AX]) \\ &\quad - A([X, JY] + [JX, Y]) \\ &= (-2J)\left(\bar\partial_X(AY) - \frac{1}{4}N_J(X, AY) - \bar\partial_Y(AX) + \frac{1}{4}N_J(Y, AX)\right) \\ &\quad + (-2J)\left(\frac{1}{2}A(J[X, JY] + J[JX, Y])\right) \\ &= (-2J)\left((\bar\partial A)(X,Y) - \frac{1}{4}(N_J(X, AY) + N_J(AX, Y)\right) \\ &\quad - JA\left(J[X, JY] + J[JX, Y] - [JX, JY] + [X, Y] + \frac{1}{2}N_J(X,Y)\right) \\ &= (-2J)(\bar\partial A)(X,Y) + \frac{1}{2}J[N_J(X, AY) + N_J(AX, Y)] \\ &\quad + \frac{1}{2}JA(N_J(X,Y)) \,. \end{aligned}$$

One easily checks that each of the final three terms has the right (p,q)-type. □

Remark B.3. *When $J \in \mathcal{I} \subset \mathcal{J}$ is an integrable complex structure, Proposition B.2 tells us that*

$$dN_J = (dN_J)^{0,2} = (-2J)\bar\partial$$

Let $\Omega^{0,2}(TM)$ denote the vector bundle over $\mathcal{J}$ whose fiber over a point $J \in \mathcal{J}$ is given by

$$\Omega^{0,2}(TM)|_J = \Omega^{0,2}_J(TM) \,.$$

Since $\Omega^{0,2}(TM)$ is a canonical summand of the trivial bundle $\Omega^2(TM) \times \mathcal{J}$ over $\mathcal{J}$, it carries a natural connection ∇ defined by projection:

$$\nabla \cdot = (d\cdot)^{0,2} \,.$$

The Nijenhuis tensor N can be seen as a natural section of this vector bundle:

$$N : \mathcal{J} \to \Omega^{0,2}(TM) \,.$$

Proposition B.2 immediately implies the following generalization of Remark B.3.

Corollary B.4. *For any $J \in \mathcal{J}$ we have that*

$$\nabla N_J = (dN_J)^{0,2} = (-2J)\bar\partial \,.$$

Appendix C. A commutation relation for Kähler manifolds

Our goal here is to prove that on a Kähler manifold (M, J, ω), with Riemannian metric given by $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, the diagram

$$\begin{array}{ccc} X \in \Omega^0(TM) & \xrightarrow{\bar\partial} & \Omega^{0,1}(TM) \ni \Gamma \\ \big\downarrow & & \big\downarrow \\ \alpha_X \in \Omega^{0,1}(M) & \xrightarrow{\bar\partial} & \Omega^{0,2}(M) \ni \alpha_\Gamma \end{array}$$

commutes, where

$$\alpha_X(Y) = g(X,Y) - i\omega(X,Y)$$

and

$$\alpha_\Gamma(Y,Z) = [g(\Gamma(Y),Z) - g(\Gamma(Z),Y)] - i[\omega(\Gamma(Y),Z) - \omega(\Gamma(Z),Y)]\,.$$

In other words, we need to show that

$$\bar\partial(\alpha_X) - \alpha_{\bar\partial X} = 0\,,\ \forall X \in \Omega^0(TM)\,. \tag{19}$$

Lemma C.1. *The map*

$$\begin{aligned} \Omega^0(TM) &\to \Omega^{0,2}_J(M) \\ X &\mapsto \bar\partial(\alpha_X) - \alpha_{\bar\partial X} \end{aligned}$$

is a tensor.

Proof. As usual, it suffices to show that the given map is $\Omega^0(M)$-linear. For any function $f \in \Omega^0(M)$ and vector field $X \in \Omega^0(TM)$, we have that

$$\bar\partial(\alpha_{fX}) = \bar\partial(f\alpha_X) = (\bar\partial f)\wedge\alpha_X + f\bar\partial\alpha_X$$

while

$$\bar\partial(fX) = (\bar\partial f)\otimes X + f\bar\partial X \quad\Rightarrow\quad \alpha_{\bar\partial(fX)} = \alpha_{(\bar\partial f)\otimes X} + f\alpha_{\bar\partial X}\,.$$

Since

$$\begin{aligned} \alpha_{(\bar\partial f)\otimes X}(Y,Z) &= [g((\bar\partial f)(Y)X,Z) - g((\bar\partial f)(Z)X,Y)] \\ &\quad - i[\omega((\bar\partial f)(Y)X,Z) - \omega((\bar\partial f)(Z)X,Y)] \\ &= (\bar\partial f)(Y)[g(X,Z) - i\omega(X,Z)] \\ &\quad - (\bar\partial f)(Z)[g(X,Y) - i\omega(X,Y)] \\ &= (\bar\partial f)(Y)\alpha_X(Z) - (\bar\partial f)(Z)\alpha_X(Y) \\ &= \left(\bar\partial f\wedge\alpha_X\right)(Y,Z)\,, \end{aligned}$$

we conclude that

$$\begin{aligned} \bar\partial(\alpha_{fX}) - \alpha_{\bar\partial(fX)} &= \left(\bar\partial f\wedge\alpha_X + f\bar\partial\alpha_X\right) - \left(\bar\partial f\wedge\alpha_X + f\alpha_{\bar\partial X}\right) \\ &= f\left(\bar\partial(\alpha_X) - \alpha_{\bar\partial X}\right)\,, \end{aligned}$$

which proves the lemma. □

Lemma C.1 implies that it suffices to prove (19) at an arbitrary point. Since a Kähler manifold behaves up to first order at a point as flat $\mathbb{C}^n$, where (19) clearly holds, we can conclude that (19) is true on any Kähler manifold.

APPENDIX D. THE A_∞ ACTION OF G_λ ON THE TUBULAR NEIGHBORHOODS

In this section we fill in some details in the proof of Theorem 5.5. A friendly and elegant reference for homotopy limits and colimits is [HV] (see also [BK] and [V]). As usual, we regard posets as categories with at most one arrow between two objects.

If X is a space, we denote by $\mathcal{P}_X$ the poset of subspaces of X ordered by *reverse* inclusion.

Let G be a topological group. We denote by $\mathcal{K}_G$ the partially ordered set of compact subspaces of G ordered by inclusion. There is a canonical map

$$A_G = \operatorname{hocolim}_{K \in \mathcal{K}_G} K \xrightarrow{\phi} G$$

which is a weak homotopy equivalence.

There is a strictly associative multiplication on A_G induced by the functor

$$\begin{array}{rcl} \mathcal{K}_G \times \mathcal{K}_G & \to & \mathcal{K}_G \\ (K, L) & \mapsto & KL \end{array}$$

and the natural isomorphism [HV, Proposition 3.1(4)]

$$A_G \times A_G = \operatorname{hocolim}_{(K,L) \in \mathcal{K}_G \times \mathcal{K}_G} K \times L.$$

Note that ϕ is a strictly multiplicative weak equivalence.

Consider the set

$$\mathcal{K}S_G = \{(K_n, \dots, K_1) \mid n \geq 0, \quad K_i \subset G \text{ compact and } \neq \emptyset\}$$

(when $n = 0$ we mean the empty word) with the partial order defined by

$$(K_n, \dots, K_1) \leq (H_m, \dots, H_1) \text{ if } n \leq m \text{ and } K_i \subset H_i.$$

Given sequences $S, T \in \mathcal{K}S_G$ we write $S * T$ for their concatenation.

Definition D.1. *Let G be a topological group, X a G-space and $U \subset X$ a subspace (not necessarily G-invariant). A* near action *of G on U consists of a functor*

$$\begin{array}{rcl} \mathcal{K}S_G & \to & \mathcal{P}_U \\ (K_n, \dots, K_1) & \mapsto & U_{(K_n, \dots, K_1)} \end{array}$$

such that

(i) $U_\emptyset = U$ (where $\emptyset$ denotes the empty sequence),
(ii) For each $S \in \mathcal{K}$, the inclusion $U_S \to U$ is a weak equivalence,
(iii) Given $K \in \mathcal{K}_G$ and $T \in \mathcal{K}S_G$, the restriction of the G-action

$$K \times U_{(K)*T} \to X$$

has image contained in U_T.

A near G-equivariant map *is a natural transformation of functors commuting strictly with the action of the compact subsets.*

Lemma D.2. *A near action of G on U induces a canonical action of A_G on $T(U) = \operatorname{holim}_{S \in \mathcal{K}S_G} U_S$.*

Proof. There is an obvious action $K \times T(U) \to T(U)$ for each compact set and this extends canonically to the required action. □

Note that there is a canonical homotopy equivalence

$$T(U) \xrightarrow{\pi} U$$

induced by the inclusion of the empty sequence in $\mathcal{K}S_G$.

Definition D.3. *The* homotopy orbit space $\overline{U_{hG}}$ *of a near action of G on U is the realization of the semi-simplicial space*

$$n \mapsto A_G^n \times T(U)$$

A G-space U has a trivial near G-action where $U_S = U$ for all $S \in \mathcal{K}S_G$ and unless we specify otherwise we always give G-spaces this near G-action.

Lemma D.4. *If U is G-invariant there is a weak equivalence*

$$\overline{U_{hG}} \to U_{hG}.$$

Proof. For each compact set there is a canonical homotopy making the diagram

$$\begin{array}{ccc} K \times T(U) & \longrightarrow & K \times U \\ \downarrow & & \downarrow \\ T(U) & \longrightarrow & U \end{array}$$

and this yields a homotopy coherent weak equivalence between the two semi-simplicial spaces in question. The result follows by taking realization. □

We can now elaborate on the proof of Theorem 5.5. With reference to the notation in the statement of that theorem, in the remainder of this section we will write K for the isometry group $K(m)$, W for the representation of K on the normal to the corresponding stratum (described in Theorem 4.1), V for the stratum $V_{\ell+1}$, NV for its tubular neighborhood, X for $\mathcal{I}_{\lambda+1}$ and G for $G_{\lambda+1}$.

We'll fix a K-invariant metric on W and given a continuous function $\epsilon\colon G \to \mathbb{R}_+$ write

$$G \times_K (W \setminus 0)_\epsilon = \{g \cdot w \mid g \in G, 0 < |w| < \epsilon(g)\}$$

Proposition D.5. *There is a continuous function $\epsilon : G \to \mathbb{R}_+$ such that*

$$\begin{array}{ccc} G \times_K (W \setminus 0)_\epsilon & \overset{\psi}{\hookrightarrow} & NV \setminus V \\ \downarrow & & \downarrow \pi \\ G/K & \hookrightarrow & V \end{array}$$

commutes in the (weak) homotopy category. Moreover ψ is a weak equivalence.

Proof. The function ϵ exists by continuity of the action. The slice theorem for the action of the symplectomorphism group G on the space of compatible almost complex structures[4] together with the uniqueness of tubular neighborhoods give, for

[4]The construction of the slice for the action of the diffeomorphism group on the space of metrics in [Eb] works in this case.

each right K-invariant compact subset $L \subset G$, a homeomorphism ψ_L, homotopic to the inclusion, such that the diagram

$$\begin{array}{ccc} L \times_K (V \setminus 0) & \xrightarrow{\psi_L} & \pi^{-1}(L/K) \\ & \searrow \quad \swarrow_{\pi} & \\ & L/K & \end{array}$$

commutes. The uniqueness of tubular neighborhoods implies that if $L \subset L'$, $\psi_{L'|(L \times_K (V \setminus 0))}$ is homotopic over L/K to ψ_L. The result follows. □

Remark D.6. *The crucial point in the proof of the previous proposition is the existence of a slice for the action. One can apply the arguments in this section whenever this is the case.*

The subspace

$$G \times_K (W \setminus 0)_\epsilon \subset NV \setminus V$$

can be endowed with a near G-action, by choosing for each sequence $S \in \mathcal{K}S_G$ a continuous function $\epsilon_S \colon G \to \mathbb{R}_+$ with $\epsilon_\emptyset = \epsilon$ in such a way that for each compact subset $L \subset G$,

$$L \cdot (G \times_K (W \setminus 0)_{\epsilon_{(L)*S}}) \subset G \times_K (W \setminus 0)_{\epsilon_S}.$$

Giving a G-space U the trivial near G-action we have a pushout diagram of near G-spaces and near G-equivariant maps

$$G/K \longleftarrow G \times_K (W \setminus 0)_\epsilon \longrightarrow (X \setminus V).$$

Writing P for the homotopy pushout, there is an obvious near G-action on P together with a near G-equivariant map

$$P \to X$$

which is clearly a weak equivalence. Since the canonical map

$$\operatorname{hocolim}(T(G/K) \leftarrow T(G \times_K (W \setminus 0)_\epsilon) \to T(X \setminus V)) \to T(P)$$

is a weak equivalence, applying A_G homotopy orbits (in the sense of Definition D.3) and Lemma D.4 we get

$$\operatorname{hocolim}(BK \to (W \setminus 0)_{hK} \to (X \setminus V)_{hG}) \simeq X_{hG} \tag{20}$$

as required.

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