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A homotopy construction of the adjoint representation for Lie groups

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Abstract

Let G be a compact, simply-connected, simple Lie group and $T \subset G$ a maximal torus. The purpose of this paper is to study the connection between various fibrations over BG (where G is a compact, simply-connected, simple Lie group) associated to the adjoint representation and homotopy colimits over poset categories $\mathscr{C}$, $\mathrm{hocolim}_{\mathscr{C}} BG_I$ where G_I are certain connected maximal rank subgroups of G.

1. *Introduction*

The notion of a Lie group was introduced in the last century. Roughly speaking, a Lie group is a manifold with a group structure that locally corresponds to a Lie algebra. The classical examples are matrix groups such as the orthogonal, unitary or symplectic groups. The program for understanding the homotopy properties of compact Lie groups led to the concept of a p-compact group introduced by Dwyer and Wilkerson in 1994 [**3**]. p-Compact groups are p-local versions of finite loop spaces. Namely, a p-compact group is a triple $X = (X, BX, e)$ where BX is a p-complete pointed space, such that $H^*(X; \mathbb{F}_p)$ is finite and $e\colon \Omega BX \to X$ is a homotopy equivalence.

The obvious examples of p-compact groups are the p completions (in the sense of [**2**]) of compact connected Lie groups and their classifying spaces. Many properties of compact Lie group theory can be reinterpreted as homotopy theoretic properties of the classifying spaces (see [**4**]) in such a way that the concepts extend to the category of p-compact groups. For example, they admit the concept of a maximal torus and Weyl group. For these reasons, p-compact groups can be considered as homotopic versions of compact Lie groups. Notice, however, that p-compact groups do not possess analytic structures like Lie algebras. Since the adjoint representation of a Lie group is defined using the action of the group on its Lie algebra, there is a strong obstruction to finding a homotopic analogue to the adjoint representation. The main purpose of this paper is to obtain a homotopic construction of objects that are closely related to the adjoint representation.

Let G be a compact, simply-connected, simple Lie group and $T \subset G$ be a maximal torus of rank l. The corresponding Lie algebras are denoted by LG and LT resp. Let G_c denote the group G seen as a space with a G-action given by conjugation. Let $\mathrm{Ad}\colon G \to \mathrm{Aut}(LG)$ denote the adjoint representation. By fixing some invariant metric, one can assume that Ad is an orthogonal representation so that one obtains G-subspaces, D^{LG} and S^{LG} given by the unit disk and unit sphere within LG. We shall provide homotopy decompositions for the G-spaces G_c, D^{LG} and S^{LG} as follows.

Let $\mathscr{C}$ be the poset category of all proper subsets of the set $\{0, 1, \ldots, l\}$. Similarly, define $\mathscr{C}_0$ to be the poset category of all proper subsets of $\{1, \ldots, l\}$. We now present our main theorem. Let G be a compact, simply-connected, simple Lie group.

THEOREM 1·1. *There exist subgroups G_I of G, indexed by the poset category of all proper subsets of $\{0, 1, \ldots, l\}$, inducing homeomorphisms of G-spaces*

$$\Phi_1\colon \mathrm{hocolim}_{\mathscr{C}} G/G_{I\setminus 0} \longrightarrow D^{LG}$$

$$\Phi_2\colon \mathrm{hocolim}_{\mathscr{C}_0} G/G_I \longrightarrow S^{LG}$$

$$\Phi_3\colon \mathrm{hocolim}_{\mathscr{C}} G/G_I \longrightarrow G_c$$

where we are considering the explicit simplicial model of Bousfield–Kan for the homotopy colimit described in [**2**].

It is worth pointing out that the third homeomorphism is implicit in the work of Mitchell. Our methods are inspired by [**7**].

For a G-space X, let $EG \times_G X$ denote the Borel construction for X. On taking Borel constructions on either side of the equalities in the above theorem, we obtain:

COROLLARY 1·2. *There exist subgroups G_I of G, indexed by the poset category of all proper subsets of $\{0, 1, \ldots, l\}$, yielding homotopy decompositions*

$$(\Phi_1)_{hG}\colon \mathrm{hocolim}_{\mathscr{C}} (G/G_{I\setminus 0})_{hG} \longrightarrow EG \times_G D^{LG}$$

$$(\Phi_2)_{hG}\colon \mathrm{hocolim}_{\mathscr{C}_0} (G/G_I)_{hG} \longrightarrow EG \times_G S^{LG}$$

$$(\Phi_3)_{hG}\colon \mathrm{hocolim}_{\mathscr{C}} (G/G_I)_{hG} \longrightarrow EG \times_G G_c.$$

The spaces $(G/G_I)_{hG}$ are homeomorphic to EG/G_I and they have the homotopy type of BG_I.

Notice that the space $EG \times_G G_c$ is homotopy equivalent to $\Lambda(BG)$ [**8**], where $\Lambda(BG)$ denotes the space of free loops on the space BG.

The category $\mathscr{C}_0$ can be identified with the subcategory of $\mathscr{C}$ consisting of all proper subsets of $\{0, 1, \ldots, l\}$ containing 0. Under this identification $(\Phi_2)_{hG}$ can be seen as a restriction of $(\Phi_1)_{hG}$ to the homotopy colimit of the corresponding subdiagram. This corresponds to the obvious inclusion of $EG \times_G S^{LG}$ within $EG \times_G D^{LG}$. Consequently, we get a homotopy decomposition for the Thom space, $T(\mathrm{Ad})$, of the adjoint bundle over BG.

COROLLARY 1·3. *There exist subgroups G_I of G, indexed by the poset category of all proper subsets of $\{0, 1, \ldots, l\}$, yielding a homotopy equivalence*

$$\Phi\colon \mathrm{hocolim}_{\mathscr{C}} \tilde{B}G_I \longrightarrow T(\mathrm{Ad})$$

where $\tilde{B}G_I = *$ if $0 \in I$ and $\tilde{B}G_I = (G/G_I)_{hG}$ otherwise.

Our theorem provides a description of several objects associated to the adjoint representation in terms of subgroups and homotopy colimits. One may ask whether this can be generalized to p-compact groups. This question is still unanswered.

In Section 2 we review some standard facts on root systems and the adjoint representation of G. Section 3 is devoted to the description of the subgroups G_I and contains the proof of the main theorem. In Sections 4 and 5 some applications are indicated. In particular, we make essential use of Corollary 1·2 to determine the structure of the algebras $H^*(\Lambda(BG_2), \mathbb{F}_2)$ (Theorem 5·1) and $H^*(EG_2 \times_{G_2} S^{LG_2}, \mathbb{F}_2)$ (Corollary 5·2) where G_2 denotes the exceptional Lie group of rank 2.

2. *Notation and preliminaries*

This section contains a brief summary of the required concepts.

Let G be a compact, connected Lie group and let LG denote its Lie algebra (recall that $LG \cong \mathbb{R}^{\dim G}$). For each $g \in G$, define c_g as the inner automorphism of G, $c_g(x) = gxg^{-1}$. The adjoint representation of G is a morphism $\mathrm{Ad}\colon G \to \mathrm{Aut}(LG)$, $\mathrm{Ad}(g) = Lc_g$ where L means the differential at the unit. Observe that, as G is compact, there is a metric on LG such that all Lc_g are orthogonal. By considering the induced morphism on the corresponding Lie algebras $\mathrm{ad} \coloneqq L\mathrm{Ad}\colon LG \to L\mathrm{Aut}(LG) = \mathrm{End}(LG)$, we observe that $\mathrm{ad}(X)Y = [X, Y]$ where $[-, -]$ is the Lie bracket on the Lie algebra.

LG can be decomposed as $LG = LT \oplus (\oplus_{\alpha\in\Phi^+} Y_\alpha)$ where Y_α are the 'eigenspaces' for the action of LT on LG via the adjoint representation Ad. Each α is a function $LT \to \mathbb{R}$. These elements $\alpha \in \Phi^+$ are called positive roots. A simple system of roots $\{\alpha_1, \ldots, \alpha_l\} \subset \Phi^+$ is a set of positive roots that are not decomposable as sums of positive roots. If G is simple, there is a unique highest root α_0 characterized by the property that for every positive root α, $\alpha_0 + \alpha$ is not a root.

Now let $\mathscr{C}^+ = \{X \in LT\colon \alpha(X) > 0 \,\forall \alpha \in \Phi^+\}$ be the positive Weyl chamber. Consider $\{X \in \mathscr{C}^+\colon \alpha_0(X) < 1\}$. Its closure Δ is an l-simplex, called the Cartan simplex, whose walls are the hyperplanes $(\alpha_i = 0)$ $1 \leqslant i \leqslant l$ and $(\alpha_0 = 1)$. The last one is called the outer wall and will be denoted by Δ_0.

Let $\tilde{W}$ be the group generated by the reflections $\{s_1, \ldots, s_l\}$ on the hyperplanes $\alpha_i = 0$ $i = 1, \ldots, l$ and the reflection s_0 in the outer wall.

If $I \subset \tilde{S} \coloneqq \{s_0, s_1, \ldots, s_n\}$, define the I-face Δ_I of Δ as

$$\Delta_I = \{X \in \Delta\colon \alpha_i(X) = 0 \text{ if } i \in I, i \neq 0, \alpha_0(X) = 1 \text{ if } 0 \in I\}$$

Observe that the isotropy group in $\tilde{W}$ of any $X \in \dot{\Delta}_I$ is precisely $\tilde{W}_I$ (the subgroup of $\tilde{W}$ generated by I).

The following standard facts are fairly easy to prove and very useful for the subsequent discussion.

THEOREM 2·1 [**7**]. *Suppose $X, Y \in \Delta$ and $\sigma X = Y$ for some $\sigma \in \tilde{W}$. Then $X = Y$ and $\sigma \in \tilde{W}_I$, where $I = \{s \in \tilde{S}\colon sX = X\}$.*

Theorem 2·2 [**7**]. *Every element of G is conjugate to* $\exp X$ *for some $X \in \Delta$. If G is simply-connected then X is unique.*

Theorem 2·3. *Let D^{LT} and S^{LT} denote the unit disk and sphere within LT, then*

$$D^{LG} = \bigcup_{g\in G} \mathrm{Ad}_g(D^{LT}), \quad S^{LG} = \bigcup_{g\in G} \mathrm{Ad}_g(S^{LT}).$$

Proof. It is enough to prove that $LG = \bigcup_{g\in G} \mathrm{Ad}_g(LT)$. Now consider an element $v \in LG$. The one parameter subgroup of G given by $\exp(tv), t \in \mathbb{R}$ lies in some maximal torus of G. Since all maximal tori in a compact Lie group are conjugate, there is a fixed element $g \in G$ such that $\exp(\mathbb{R}v) \subseteq gTg^{-1}$. The result now follows on taking derivatives.

Corollary 2·4. *There are G-equivariant homeomorphisms*

$$D^{LG} \cong \bigcup_{g\in G} \mathrm{Ad}_g(\Delta), \quad S^{LG} \cong \bigcup_{g\in G} \mathrm{Ad}_g(\Delta_0).$$

Proof. As an easy consequence of the way in which $W = N_G(T)/T$ acts on the Weyl chambers in LT and consequently on the Cartan simplex Δ, one has the following W-equivariant homeomorphism [**7**], $D^{LT} \cong \bigcup_{\omega\in W} \mathrm{Ad}_\omega(\Delta)$. The proof is now clear using Theorem 2·3.

3. *Subgroups defined by the walls of the Cartan simplex Δ and the proof of the main theorem*

We will use the notation and conventions of the preceding section. Recall that for each positive root α, we have an eigenspace Y_α in LG. Each of these Y_αs generates a Lie subalgebra isomorphic to $\mathscr{SU}(2)$. In fact, since G is simply-connected, the corresponding subgroups G_α are isomorphic to $SU(2)$.

Definition 3·1. The subgroups G_I of G (where I is a subset of $\{0, 1, \ldots, l\}$) are the subgroups generated by T and the G_{α_i}, $i \in I$.

Notice that all these subgroups are connected and of maximal rank. In particular, for $I = \{1, \ldots, l\}$, the group G_I is the whole group G. The following theorem indicates their importance.

Theorem 3·2 [**7**]. *For any $X \in \dot{\Delta}_I$ (the interior of Δ_I), the isotropy subgroup at X for the adjoint action of G is given by $C_G(X) = G_{I\backslash 0}$. Similarly, for the conjugation action of G, we have $C_G(\exp(X)) = G_I$.*

Before proving the main theorem, it is worth pointing out the following description of a homotopy colimit over a poset category. Let G be a compact Lie group and $\mathscr{C}$ the poset of proper subgroups of the set $\{1, 2, \ldots, n\}$ including the empty set. Let F be a functor from $\mathscr{C}$ to *Groups* which takes values in subgroups of G where the order in the subgroups in G is given by inclusions, $\subseteq$.

Lemma 3·3. *The homotopy colimit* $\mathrm{hocolim}_{\mathscr{C}} G/F(I)$ *in the sense of Bousfield–Kan is homeomorphic to $(G/F(\emptyset) \times \Delta^n)/\sim$ where $(gF(\emptyset), X) \sim (hF(\emptyset), Y)$ if and only if $X = Y \in \dot{\Delta}_I$ and $g \in hF(I)$.*

Proof. The homotopy colimit in the sense of Bousfield–Kan is described as

$$(\bigcup_{m\geqslant 0}\bigcup_{I_0\to\cdots\to I_m} G/F(I_0)\times\Delta^m)/\sim$$

where the relation is given by the usual face and degeneracy maps, that is, for any $X\in\Delta^{m-1}$

$$(d_i(gF(I_1);I_1\subset\ldots\subset I_m),X)\sim((gF(I_1);I_1\subset\ldots\subset I_m),d^i(X))$$

and for any $X\in\Delta^{m+1}$

$$(s_i(gF(I_1);I_1\subset\ldots\subset I_m),X)\sim((gF(I_1);I_1\subset\ldots\subset I_m),s^i(X)),$$

where

$$d_i(gF(I_1);I_1\subset\cdots\subset I_m)=\begin{cases}(gF(I_2);I_2\subset\cdots\subset I_m) & i=0\\ (gF(I_1);I_1\subset\cdots\subset I_i\subset I_{i+2}\subset\cdots\subset I_m) & 0<i<m\\ (gF(I_1);I_1\subset\cdots\subset I_{m-1}) & i=m\end{cases}$$

$$s_i(gF(I_1);I_1\subset\cdots\subset I_m)=(gF(I_1);I_1\subset\cdots\subset I_i\subset I_i\subset\cdots\subset I_m).$$

First of all, it is easy to check that the geometrical realization of $\mathscr{C}$ corresponds to the barycentric subdivision of the standard simplex Δ^{n-1}, that is,

$$\Delta^{n-1}\cong(\bigcup_{m\geqslant 0}\bigcup_{I_0\to\cdots\to I_m}\Delta^m)/\sim.$$

Now consider the projection map onto the second factor:

$$(\bigcup_{m\geqslant 0}\bigcup_{I_0\to\cdots\to I_m} G/F(I_0)\times\Delta^m)/\sim\ \xrightarrow{\pi_2}\Delta^{n-1}.$$

Let X be any element within the interior of the face $\Delta_I\subseteq\Delta^{n-1}$ that is determined by the set I, then it is clear from general definitions that the preimage of X under π_2 is the space $G/F(I)$. Now notice that $\varnothing$ is an initial object in $\mathscr{C}$. Hence there is a continuous map

$$G/F(\varnothing)\times\Delta^{n-1}\xrightarrow{\phi}(\bigcup_{m\geqslant 0}\bigcup_{I_0\to\cdots\to I_m} G/F(I_0)\times\Delta^m)/\sim$$

defined as $\phi(gF(\varnothing),X)=[gF(I),X]$ where I is uniquely determined by demanding that X belong to the interior of the face Δ_I. It is clear that ϕ is exhaustive. Moreover, on composing ϕ with π_2, we notice that $\phi(g_1F(\varnothing),X)\sim\phi(g_2F(\varnothing),Y)$ if and only if $X=Y\in\dot{\Delta}_I$ and $g_1\in g_2F(I)$.

Since all the spaces involved are Hausdorff and compact, imposing the above equivalence relation on $G/F(\varnothing)\times\Delta^{n-1}$ makes it homeomorphic to the required homotopy colimit.

We are now ready to prove the main theorem. First consider the case of the G-space D^{LG}.

One has a well defined map $\pi\colon \mathrm{hocolim}_{\mathscr{C}}G/G_{I\backslash 0}\to LG$ given by $\pi(gT,X)=\mathrm{Ad}_g(X)$. Notice that by Corollary 2·4, the image of π can be identified with D^{LG}. Let Φ_1 be the obvious composite. Notice that Φ_1 is G-equivariant if we give $(G/T\times\Delta)/\sim$ a G-action via left translations and D^{LG} via the adjoint representation. As $G\times\Delta$

is compact, any quotient of it (with the induced topology) is also compact. Since D^{LG} is Hausdorff, it suffices to prove that Φ_1 is bijective in order to show that it is a homeomorphism. Observe that Φ_1 is exhaustive by construction and injective by Theorems 2·2 and 3·2. The proof for S^{LG} is similar and we omit it. For the case of the G-space G_c, one first identifies the homotopy colimit $\mathrm{hocolim}_{\mathscr{C}} G/G_I$ with the space

$$(G/T \times \Delta)/\sim, \quad \text{where} \quad (g_1T, X_1) \sim (g_2T, X_2) \Longleftrightarrow X_1 = X_2 \in \dot{\Delta}_I \quad \text{and} \quad g_1 \in g_2 G_I.$$

The map Φ_3 is induced by the map $(gT, X) \mapsto g(exp(X))g^{-1}$. The rest again follows from Theorems 2·2 and 3·2.

Proof of Corollary 1·3. Consider the following functors defined on the category $\mathscr{C}$:

$$\mathscr{S}, \mathscr{D}, \mathscr{P}, \mathscr{T} : \mathscr{C} \longrightarrow \text{Spaces}$$

in the following way,

$$\mathscr{D}(I) = (G/G_{I\backslash 0})_{hG}, \quad \mathscr{S}(I) = \begin{cases} (G/G_{I\backslash 0})_{hG} & I \in \mathscr{C}_0 \\ \varnothing & \text{otherwise} \end{cases}$$
$$\mathscr{T}(I) = \tilde{B}G_I, \qquad \mathscr{P}(I) = \begin{cases} * & I \in \mathscr{C}_0 \\ \varnothing & \text{otherwise.} \end{cases}$$

First of all, notice that $\mathrm{hocolim}_{\mathscr{C}} \mathscr{S} \simeq \mathrm{hocolim}_{\mathscr{C}_0} (G/G_{I\backslash 0})_{hG}$ since simplices in $\mathscr{C}$ not in $\mathscr{C}_0$ don't contribute to the homotopy colimit. Let $\mathscr{R}(I)$ be the functor on $\mathscr{C}$ defined as the homotopy pushout:

$$\begin{array}{ccc} \mathscr{S}(I) & \longrightarrow & \mathscr{D}(I) \\ \downarrow & & \downarrow \\ \mathscr{P}(I) & \longrightarrow & \mathscr{R}(I) \end{array}$$

It is clear that we have a natural homotopy equivalence $\mathscr{R}(I) \simeq \mathscr{T}(I)$. Hence we get the homotopy equivalence

$$\mathrm{hocolim}_{\mathscr{C}} \mathscr{R} \simeq \mathrm{hocolim}_{\mathscr{C}} \mathscr{T}.$$

On the other hand using the so-called Fubini theorem for homotopy colimits and Theorem 1·2, the above homotopy colimit is homotopy equivalent to the homotopy pushout of the diagram:

$$\begin{array}{ccc} (S^{LG})_{hG} & \longrightarrow & (D^{LG})_{hG} \\ \downarrow & & \\ B\mathscr{C}_0 & & \end{array}$$

which has the homotopy type of the Thom space since $B\mathscr{C}_0$ is contractible.

Remark 3·4. For $I \in \mathscr{C}$, the projection map $G/G_{I\backslash 0} \to G/G_I$ extends to a map

$$\varphi\colon \mathrm{hocolim}_{\mathscr{C}} G/G_{I\backslash 0} \to \mathrm{hocolim}_{\mathscr{C}} G/G_I$$

and it is easy to see that the following diagram commutes

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathscr{C}} G/G_{I\backslash 0} & \xrightarrow{\Phi_1} & D^{LG} \\ \Big\downarrow \varphi & & \Big\downarrow \exp \\ \mathrm{hocolim}_{\mathscr{C}} G/G_I & \xrightarrow{\Phi_3} & G_c \end{array}$$

where exp denotes the exponential map restricted to the subspace $\bigcup_{g \in G} \mathrm{Ad}_g(\Delta) \simeq D^{LG}$.

Remark 3·5. There exists a collapsing map $C\colon \Lambda(BG) \simeq EG \times_G G_c \to T(\mathrm{Ad})$ defined in the following way. Let U be an equivariant neighbourhood of the identity in G. Then

$$EG \times_G U \to EG_+ \wedge_G (U/\partial U) \simeq T(\mathrm{Ad}).$$

This map can now be described in terms of natural transformation between functors. It is clear that there exist a natural transformation T between the functors defining $\mathrm{hocolim}_{\mathscr{C}} BG_I$ and $\mathrm{hocolim}_{\mathscr{C}} \tilde{B}G_I$. If $0 \notin I$, consider the identity between in G_I. If $0 \in I$, consider the constant map. This natural transformation defines a map

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathscr{C}} BG_I & \xrightarrow{(\Phi_3)_{hG}} & EG \times_G G_c \\ \Big\downarrow T & & \Big\downarrow C \\ \mathrm{hocolim}_{\mathscr{C}} \tilde{B}G_I & \xrightarrow{\Phi} & T(\mathrm{Ad}) \end{array}$$

4. *Farjoun localization of homotopy colimits and applications*

Let $L_{B\mathbb{Z}/p}$ denote the Farjoun localization with respect to the map $B\mathbb{Z}/p \to *$ [**5**]. In this section we will show that $L_{B\mathbb{Z}/p}$ applied to the p-completion of the spaces $EG \times_G S^{LG}$ and $EG \times_G G_c$ yields a point up to homotopy.

Recall some of the basic facts about $L_{B\mathbb{Z}/p}$. Let BG denote the classifying space of a compact Lie group G, then it is well known that $L_{B\mathbb{Z}/p}(BG^\wedge_p)$ is contractible. On the other hand, for a p-complete finite complex X, $L_{B\mathbb{Z}/p}(X) \simeq X$. Another property of $L_{B\mathbb{Z}/p}$ which is perhaps not so well known is the behaviour of $L_{B\mathbb{Z}/p}$ on homotopy colimits

$$L_{B\mathbb{Z}/p}(\mathrm{hocolim}_{\mathscr{C}} \mathscr{F}) = L_{B\mathbb{Z}/p}(\mathrm{hocolim}_{\mathscr{C}} L_{B\mathbb{Z}/p} \mathscr{F})$$

where $\mathscr{F}$ is a functor from a small category $\mathscr{C}$ to spaces. A proof of this equality can be found in [**1**, section 9].

THEOREM 4·1. *The Farjoun localization of the spaces $EG \times_G S^{LG}$ and $EG \times_G G_c$ is equivalent to a point.*

Proof. Note that

$$L_{B\mathbb{Z}/p}((EG \times_G G_c)^{\wedge}_p) \simeq L_{B\mathbb{Z}/p}((\mathrm{hocolim}_{\mathscr{C}}(BG_I)^{\wedge}_p)^{\wedge}_p)$$

By [**1**, lemma 9·11], if $(L_{B\mathbb{Z}/p}(\mathrm{hocolim}_{\mathscr{C}} L_{B\mathbb{Z}/p}(BG_I)^{\wedge}_p))^{\wedge}_p \simeq *$ then

$$L_{B\mathbb{Z}/p}((\mathrm{hocolim}_{\mathscr{C}}(BG_I)^{\wedge}_p)^{\wedge}_p) \simeq *.$$

According to the above formula for the behaviour with respect to homotopy colimits, we have

$$L_{B\mathbb{Z}/p}(\mathrm{hocolim}_{\mathscr{C}} L_{B\mathbb{Z}/p}(BG_I)^{\wedge}_p) \simeq L_{B\mathbb{Z}/p}(\mathrm{hocolim}_{\mathscr{C}} *) \simeq L_{B\mathbb{Z}/p}(B\mathscr{C}) \simeq *$$

since $B\mathscr{C}$ is contractible. The proof for the other space is similar.

COROLLARY 4·2. *The fibrations*

$$G \longrightarrow EG \times_G G_c \longrightarrow BG$$

$$S^{LG} \longrightarrow EG \times_G S^{LG} \longrightarrow BG$$

are nontrivial when completed at any prime.

Proof. Note that if either of these fibrations were trivial at some prime p, then the corresponding Farjoun localization $L_{B\mathbb{Z}/p}$ of the total space would be nontrivial. But, by the above theorem, this cannot happen.

In particular, $\Lambda(BG)$ is not homotopy equivalent to $G \times BG$ when completed at any prime p.

This corollary provides an example of a fake $S^3 \times BSU(2)$ at $p = 2$. This means that the space $\Lambda(BSU(2))$ has the same mod 2 cohomology algebra (over the Steenrod algebra) as the space $S^3 \times BSU(2)$ but the two spaces are not homotopy equivalent even after 2-completion. It is worth pointing out that both the spaces $BSU(2)$ and S^3 satisfy the homotopy uniqueness property after 2-completion.

5. *Explicit computations with the exceptional Lie group G_2*

In this section we will use our homotopy decomposition to calculate the algebra $H^*(\Lambda(BG_2), \mathbb{F}_2)$ and $H^*(EG_2 \times_{G_2} S^{LG_2}, \mathbb{F}_2)$, where G_2 denotes the exceptional Lie group of rank 2. The following is the structure of the cohomology rings $H^*(G_2, \mathbb{F}_2)$ and $H^*(BG_2, \mathbb{F}_2)$ with the given action of the Steenrod algebra.

$$H^*(G_2, \mathbb{F}_2) = \mathbb{F}_2[x_3, x_5]/(x_3^4, x_5^2), \quad H^*(BG_2, \mathbb{F}_2) = \mathbb{F}_2[x_4, x_6, x_7] \tag{1}$$

	x_3	x_4	x_5	x_6	x_7
Sq^1	0	0	x_3^2	x_7	0
Sq^2	x_5	x_6	0	0	0
Sq^4	0	x_4^2	0	x_4x_6	x_4x_7

where the subscripts denote the degrees of the generators. Now recall that for any connected Lie group, G, the space $\Lambda(BG)$ can be identified with the Borel construction $EG \times_G G_c$. Thus $\Lambda(BG)$ fits into a fibration

$$G \longrightarrow \Lambda(BG) \longrightarrow BG. \tag{2}$$

This fibration corresponds to the one given by the evaluation map and whose fibre is $\Omega BG \simeq G$. Notice that the identity element in the group G is a fixed point for the G-action on G_c. Consequently, the above fibration admits a section which correspond to the one induced by the constant loop.

Let us fix G to be the exceptional Lie group G_2, and consider the Serre spectral sequence in mod-2 cohomology for the above fibration. Since this fibration has a section, all transgressions are trivial. Thus the class x_3 transgresses to zero. Since x_5 is connected to x_3 via a Sq^2, it also transgresses to zero and the spectral sequence collapses at E_2. The algebra E_∞ is given by

$$E_\infty = \mathbb{F}_2[x_3, x_4, x_5, x_6, x_7]/(x_3^4, x_5^2). \tag{3}$$

Notice that the classes x_3 and x_5 lift uniquely to classes of $H^*(\Lambda(BG_2), \mathbb{F}_2)$ in dimensions 3 and 5 resp. Similarly, the classes x_4, x_6, x_7 have obvious lifts given by the image of $H^*(BG_2, \mathbb{F}_2)$ via the map defined by the fibration (2). By abuse of notation, we shall denote the lifts by the same name. Since the classes x_3 and x_5 are detected on the fibre, it follows from (1) and the existence of a section to fibration (2) that

$$Sq^1 x_3 = 0, \quad Sq^2 x_3 = x_5, \quad Sq^1 x_5 = Sq^3 x_3 = x_3^2. \tag{4}$$

The action of the Steenrod algebra on the classes x_4, x_6 and x_7 follows from naturality and is given by (1). To completely understand the action of the Steenrod algebra on $H^*(\Lambda(BG_2), \mathbb{F}_2)$, it remains to calculate $Sq^4 x_5$. We leave this for later.

We now proceed to solve the algebraic extension problems. These extensions are nontrivial and they were first solved in [**6**]. Our homotopy decomposition simplifies these computations considerably.

Consider the homotopy decomposition given by Theorem 1·2. The Bousfield–Kan spectral sequence for this homotopy decomposition provides another means of calculating the cohomology. The E_2-term of this spectral sequence is given by

$$E_2^{p,q} = \lim_{\mathscr{C}}{}^p H^q(BG_I, \mathbb{F}_2).$$

Notice that the category $\mathscr{C}$ is the category of all proper subsets of $\{0, 1, 2\}$. Since $\mathscr{C}$ is a category in which the longest composable chain of nontrivial morphisms has length 2, the higher derived inverse limits are trivial for $p > 2$. Consequently the spectral sequence is concentrated in three columns. It follows that any element of $H^*(\Lambda(BG_2), \mathbb{F}_2)$ that is detected on ${\lim_{\mathscr{C}}}^p H^q(BG_I, \mathbb{F}_2)$ for $p > 0$ must be trivial when cubed. In particular, the class x_3 must be detected in ${\lim_{\mathscr{C}}}^0 H^q(BG_I, \mathbb{F}_2)$. Since the algebras appearing in this inverse limit are the cohomology rings of rank 2 compact connected Lie groups, the inverse limit contains no nilpotent elements. In other words, we have demonstrated that the class x_3 is essential!

Let us now consider the most general expression for x_5^2 in $H^*(\Lambda(BG_2), \mathbb{F}_2)$

$$x_5^2 = Ax_3x_7 + Bx_4x_3^2 + Cx_4x_6.$$

By applying Sq^1 to the above equality we observe that $C = 0$. Hence we reduce the problem to

$$x_5^2 = Ax_3x_7 + Bx_4x_3^2. \tag{5}$$

Applying Sq^2 to (5) and using (4) and (1), we obtain

$$x_3^4 = Ax_5x_7 + Bx_6x_3^2. \tag{6}$$

Applying Sq^1 to (6) we observe that $A = B$. Now since x_3 is essential, (6) forces $A = B = 1$ yielding the relations that solve the extension problem!

$$x_5^2 = x_3x_7 + x_4x_3^2, \quad x_3^4 = x_5x_7 + x_6x_3^2. \tag{7}$$

It remains to calculate Sq^4x_5. The most general expression is given by

$$Sq^4x_5 = \alpha x_3x_6 + \beta x_4x_5 + \gamma x_3^3.$$

Since by (3) the class x_5 restricts to the corresponding class in $H^*(G_2, \mathbb{F}_2)$, we notice using (1) that $\gamma = 0$. Now applying Sq^1 to the above equality we get an expression for $x_5^2 = Sq^1Sq^4x_5$, which when compared to the relation in (7) tells us that $\alpha = \beta = 1$. Summarizing the above calculations, we have:

THEOREM 5·1. *There is an isomorphism of algebras*

$$H^*(\Lambda(BG_2), \mathbb{F}_2) = \mathbb{F}_2[x_3, x_4, x_5, x_6, x_7]/(x_5^2 + x_3x_7 + x_4x_3^2, \quad x_3^4 + x_5x_7 + x_6x_3^2)$$

with the action of the Steenrod algebra given by the table

	x_3	x_4	x_5	x_6	x_7
Sq^1	0	0	x_3^2	x_7	0
Sq^2	x_5	x_6	0	0	0
Sq^4	0	x_4^2	$x_3x_6 + x_4x_5$	x_4x_6	x_4x_7

We finish these section with another example related to G_2. The computation of the mod 2 Euler class for the adjoint representation of G_2 on S^{LG_2}. We will make use of the decomposition of the total space of the spherical fibration

$$S^{LG_2} \longrightarrow \operatorname{hocolim}_{\mathscr{C}_0} B(G_2)_I \longrightarrow BG_2$$

where $\mathscr{C}_0$ is the poset category of all proper subsets of $\{1, 2\}$.

The mod 2 Euler class can be calculated using the Serre spectral sequence associated to the above fibration by computing the only possible differential d_{14}. Notice that we have several options for this differential, $d_{14}(y) = 0, x_7^2, x_4^2x_6$, where y is the generator of $H^{13}(S^{LG_2}, \mathbb{F}_2)$.

The Bousfield–Kan spectral sequence associated to the homotopy colimit with E_2-term

$$\lim_{\mathscr{C}_0}{}^p H^q(BG_I, \mathbb{F}_p)$$

is a two column spectral sequence and hence it collapses at E_2.

The element x_7 is an infinite cycle in the Serre spectral sequence so it has to appear in the E_2-page of the Bousfield–Kan spectral sequence. Moreover, x_7 must appear in the $\lim^1$ column because the $\lim^0$ column is concentrated in even degrees (see Definition 3·1, in fact, $G_1, G_2 \cong U(2)$). By the multiplicative properties of the Bousfield–Kan spectral sequence, x_7 is zero when squared.

That means x_7^2 must die in the Serre spectral sequence and consequently $d_{14}(y) = x_7^2$, where y is the generator of $H^{13}(S^{LG_2}, \mathbb{F}_2)$.

COROLLARY 5·2. *The* mod 2 *Euler class associated to the adjoint representation of* G_2 *is* x_7^2. *Consequently,* $H^*(EG_2 \times_{G_2} S^{LG_2}, \mathbb{F}_2)$ *is isomorphic to* $H^*(BG_2, \mathbb{F}_2)/(x_7^2)$ *as a* $H^*(BG_2, \mathbb{F}_2)$*-algebra.*

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