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Classifying spaces of Kač-Moody groups

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Abstract. We study the structure of classifying spaces of Kač-Moody groups from a homotopy theoretic point of view. They behave in many respects as in the compact Lie group case. The mod p cohomology algebra is noetherian and Lannes' T functor computes the mod p cohomology of classifying spaces of centralizers of elementary abelian p-subgroups. Also, spaces of maps from classifying spaces of finite p-groups to classifying spaces of Kač-Moody groups are described in terms of classifying spaces of centralizers while the classifying space of a Kač-Moody group itself can be described as a homotopy colimit of classifying spaces of centralizers of elementary abelian p-subgroups, up to p-completion. We show that these properties are common to a larger class of groups, also including parabolic subgroups of Kač-Moody groups, and centralizers of finite p-subgroups.

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1. Introduction

The representation theory of compact Lie groups extends naturally to the representation theory of a class of topological groups known as Kač-Moody groups [11, 10, 22]. Apart from simply-connected compact Lie groups, this class also contains Kač-Moody groups of affine type which are closely related to loop groups. In particular, Kač-Moody groups may be infinite dimensional in nature. The construction of Kač-Moody groups is motivated by the representation theory of infinite dimensional Lie algebras, and as such there is little reason to believe that Kač-Moody groups are indeed a

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legitimate extension of the class of simply connected compact Lie groups from the standpoint of topology. The object of this paper is to establish this fact. Namely, we intend to prove that the topological properties of Kač-Moody groups closely mimic those of compact Lie groups. For instance, the p-primary cohomology of the classifying space of a Kač-Moody group will be shown to be finitely generated as an algebra. More precisely we have the following

Theorem A [Theorem 4.8]. *For any Kač-Moody group G, the mod p cohomology ring $H^*(BG, \mathbb{F}_p)$ is finitely generated as an $\mathbb{F}_p$-algebra. Moreover, if F is a Kač-Moody subgroup of G, then $H^*(BF, \mathbb{F}_p)$ is a finitely generated $H^*(BG, \mathbb{F}_p)$-module via the restriction map $H^*(BG, \mathbb{F}_p) \longrightarrow \mathbb{H}^*(BF, \mathbb{F}_p)$.*

In fact we show that the Krull dimension of $H^*(BG, \mathbb{F}_p)$ is the rank of the maximal elementary abelian p-group in G (Corollary 4.2). The above theorem has been verified for affine Kač-Moody groups by A. Kono and K. Kozima [13–15] by a case-by-case analysis. Our approach is global and proves this for a much larger class of groups we now describe.

Let $\mathcal{X}$ be a class of compactly generated Hausdorff topological groups and let p be a fixed prime. Motivated by [16] define a new class of topological groups, $\mathcal{K}_1\mathcal{X}$, by demanding that a compactly generated, Hausdorff topological group G belongs to $\mathcal{K}_1\mathcal{X}$ if and only if there exists a finite G-CW-complex X with the following two properties:

(i) The isotropy subgroups of X belong to the class $\mathcal{X}$.
(ii) For every finite p-subgroup $\pi < G$, the fixed point space X^π is p-acyclic.

If a finite G-CW-complex X satisfies the conditions listed above, we shall call X a G-p-acyclic complex.

For the rest of this paper we fix $\mathcal{X}$ to be the class of compact Lie groups. Our goal is to understand the algebraic and geometric nature of the groups belonging to the class $\mathcal{K}_1\mathcal{X}$. Indeed, we show that groups in the class $\mathcal{K}_1\mathcal{X}$ share many of the properties enjoyed by compact Lie groups. We show in Sect. 5 that Kač-Moody groups belong to $\mathcal{K}_1\mathcal{X}$.

Among some of the other results for compact Lie groups that extend to groups in $\mathcal{K}_1\mathcal{X}$ is the result of Lannes [17] for which we have the generalization

Theorem B [Theorem 3.1]. *For any group G in the class $\mathcal{K}_1\mathcal{X}$ there is a natural isomorphism*

$$T_V(H^*(BG, \mathbb{F}_p)) \xrightarrow{\simeq} \prod_{\rho \in \mathrm{Rep}(V,G)} H^*(BC_G(\rho), \mathbb{F}_p)$$

where T_V stands for the Lannes T-functor corresponding to an elementary abelian p-group V, and $C_G(\rho)$ denotes the centralizer of the representation ρ in G.

More generally, we extend the result of Dwyer and Zabrodsky [5] on the nature of the centralizers of finite p-subgroups:

Theorem C [Corollary 3.3]. *If G is a group in $\mathcal{K}_1\mathcal{X}$ and π a finite p-group, then there is a natural homotopy equivalence*

$$\coprod_{\rho \in \mathrm{Rep}(\pi, G)} BC_G(\rho)^\wedge_p \xrightarrow{\simeq} \mathrm{Map}(B\pi, BG^\wedge_p).$$

Another result that extends to our context is the centralizer decomposition of Jacowski and McClure [9] for the classifying space at a given prime. We prove

Theorem D [Theorem 3.4]. *For a group G in the class $\mathcal{K}_1\mathcal{X}$ assume that there exists a G-p-acyclic complex all of whose isotropy groups contain non-trivial p-torsion, then there exists a natural p-local homology equivalence*

$$\pi\colon \operatorname*{hocolim}_{\mathcal{A}_*(G)^{op}} EG \times_G G/C_G(E) \longrightarrow BG$$

where $\mathcal{A}_*(G)$ denotes the Quillen category of (nontrivial) elementary abelian subgroups of G (refer Sect. 3.3).

In Sect. 5 we will show that the adjoint forms of Kač-Moody groups and their parabolic subgroups, as well as the adjoint forms of the normalizers of maximal tori admit acyclic complexes all of whose isotropy groups are of maximal rank (c.f. Remark 5.4). In particular, the above theorem applies for such groups.

It is worth pointing out that the construction of the class $\mathcal{K}_1\mathcal{X}$ resembles the construction of P. Kropholler and G. Mislin in [16]. Also related to the groups in $\mathcal{K}_1\mathcal{X}$ are the (discrete) groups studied by W. Luck in [18]. Consequently, all the results in this article are true for the groups considered in [18].

We would like to point out to the reader that the groups in the class $\mathcal{K}_1\mathcal{X}$ do not admit an a priori notion of a maximal torus. In this sense, they differ from compact Lie groups. Kač-Moody groups, however, retain this notion. One has a well defined maximal torus for a Kač-Moody group. Moreover, any two maximal tori within a Kač-Moody group are conjugate (c.f. [12]), and the normalizer of a maximal torus fits into an extension of a discrete (Weyl) group by the maximal torus. The Weyl group for a Kač-Moody group is a Coxeter group which is infinite in general.

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2. Orbit decompositions

Let H be a subgroup of G. For a given non trivial finite subgroup π of G one may ask how many conjugates of π lie in H and among those which are conjugated within H. Generally one finds a collection of elements $g_\lambda \in G$, such that $g_\lambda^{-1}\pi g_\lambda < H$ and any two $g_\lambda^{-1}\pi g_\lambda$ and $g_{\lambda'}^{-1}\pi g_{\lambda'}$ are not conjugated within H. This might be formalized in the following way.

Fix a finite group Π. Induced by the inclusion of H in G we have

$$i\colon \operatorname{Rep}(\Pi, H) \longrightarrow \operatorname{Rep}(\Pi, G)\,.$$

Let $\rho\colon \Pi \longrightarrow G$ be a representation for G and let $\pi = \rho(\Pi)$ be a finite subgroup of G. Now the set $\{g_\lambda\}$ indexes the counterimage $i^{-1}(\rho)$. Each g_λ providing the representation $c_{g_\lambda^{-1}} \circ \rho\colon \Pi \longrightarrow H$.

Next, we let π act on the orbit G/H. The centralizer $C_G(\pi)$ acts on the fix point space $(G/H)^\pi$. Notice that gH is a fix point in $(G/H)^\pi$ provided $egH = gH$ for every $e \in \pi$; that is, if and only if $g^{-1}\pi g$ is a subgroup of H.

On the other hand, given two fix points, gH and $g'H$, they are in the same orbit by the action of $C_G(\pi)$ if and only if there exists a fixed $h \in H$ that conjugates $g^{-1}eg$ to $g'^{-1}eg'$ for all $e \in \pi$. In fact, if $g'H = xgH$, for some $x \in C_G(\pi)$, then $h = g^{-1}x^{-1}g'$ is an element of H that conjugates $g'^{-1}eg'$ to $g^{-1}eg$ for all $e \in \pi$. Reciprocally, if $h \in H$ gives $hg'^{-1}eg'h^{-1} = g^{-1}eg$ for all $e \in \pi$, then $x = ghg'^{-1} \in C_G(\pi)$ and $xg'H = gH$.

We obtain therefore that the set $\{g_\lambda\}$ indexes the $C_G(\pi)$-orbits of $(G/H)^\pi$. The isotropy group of an orbit represented by $g_\lambda H$ is $C_G(\pi) \cap g_\lambda H g_\lambda^{-1}$, and then we obtain an orbit decomposition

$$\coprod_{g_\lambda} C_G(\pi)/C_G(\pi) \cap g_\lambda H g_\lambda^{-1} \cong (G/H)^\pi \tag{1}$$

of the fix point set $(G/H)^\pi$. We are interested in cases where Π is a finite p-group.

Theorem 2.1. *Let G be a group in the class $\mathcal{K}_1\mathcal{X}$ and let Π be a finite p-group. Then there are finitely many representations $\rho\colon \Pi \longrightarrow G$. Moreover,* *given any such representation with $\pi = \rho(\Pi)$ and a compact subgroup $H < G$, the decomposition (1) is a homeomorphism of $C_G(\pi)$-spaces.*

Proof. We pick some choice X of a G-p-acyclic complex. Since X^π is nonempty, we notice that the representation ρ factors through some isotropy group but since there are finitely many isotropy groups and the result is known for compact Lie groups, there are finitely many choices for ρ. It remains to show that for any compact subgroup H, the decomposition (1) is a homeomorphism of $C_G(\pi)$-spaces. The left hand side of (1) consist of a finite number of terms: $\coprod_{i=1}^n C_G(\pi)/C_G(\pi) \cap g_i H g_i^{-1}$, hence it will be enough to show that $C_G(\pi)/C_G(\pi) \cap g_i H g_i^{-1}$ is closed in $(G/H)^\pi$ in order to see that it is both closed and open.

First, $C_G(\pi)$ is closed in G because its closure still centralizes π. Also, by assumption, H is a compact subgroup of G. Hence it easily follows that $C_G(\pi) \cdot g_i H$ is a closed subspace of G. The result follows on projecting to G/H. □

Remark 2.2. Notice that the above proof strongly uses the compactness of the isotropy subgroups of the G-action on X. The above theorem is the only technical obstruction to extending the results of this paper to the class $\mathcal{K}_2\mathcal{X} = \mathcal{K}_1(\mathcal{K}_1\mathcal{X})$.

Corollary 2.3. *Let G be a group belonging to $\mathcal{K}_1\mathcal{X}$. Then for any finite p-group $\pi < G$, the group $C_G(\pi)$ also belongs to the class $\mathcal{K}_1\mathcal{X}$.*

Proof. Let X be a G-p-acyclic complex. The group $C_G(\pi)$ acts on the p-acyclic space X^π. Also, by 2.1, the space X^π is a finite $C_G(\pi)$-CW-complex. X^π will serve as our $C_G(\pi)$-acyclic complex. It is trivial to verify the required properties. □

3. Centralizers of finite p-subgroups

The geometric results in the previous section are used here in order to extend arguments due to Henn [8, 7] that will lead to some important homotopy theoretic structural theorems for groups in the class $\mathcal{K}_1\mathcal{X}$.

3.1. The Lannes' **T** *functor*

Let G be a group, and let V be a finite p-group. Let X be any G-space. For every representation $\rho\colon V \longrightarrow G$, let $C_G(\rho)$ denote the centralizer in G of the subgroup $\rho(V)$ and let X^ρ denote the fixed point space $X^{\rho(V)}$. Notice that the space X^ρ has a natural action of the group $C_G(\rho) \times V$ where V

acts trivially. The group homomorphism $C_G(\rho) \times V \longrightarrow G$ yields a map of the homotopy orbit spaces

$$\begin{aligned}&(EC_G(\rho) \times_{C_G(\rho)} X^\rho) \times BV \\ &\cong (EC_G(\rho) \times EV) \times_{C_G(\rho)\times V} X^\rho \longrightarrow EG \times_G X.\end{aligned}$$

On evaluating the above map in cohomology and looking at its adjoint we get the following map for every representation ρ

$$T_V(H^*_G(X)) \longrightarrow H^*_{C_G(\rho)}(X^\rho)\,.$$

Theorem 3.1. *Let G be an element of $\mathcal{K}_1\mathcal{X}$ and V an elementary abelian p-group.*

1. *Let X be any G-CW-complex of finite orbit type whose isotropy groups are all elements of $\mathcal{X}$. Then the following natural map is an isomorphism.*

$$T_V(H^*_G(X)) \xrightarrow{\simeq} \prod_{\rho\in \mathrm{Rep}(V,G)} H^*_{C_G(\rho)}(X^\rho)\,.$$

2. *The following natural map is an isomorphism.*

$$T_V(H^*(BG)) \xrightarrow{\simeq} \prod_{\rho\in \mathrm{Rep}(V,G)} H^*(BC_G(\rho))\,.$$

Proof. Proof will proceed by induction on the equivariant skeletons of X. Notice that the theorem is true for compact Lie groups [8]. An easy induction argument on the equivariant cells, reduces the proof of part 1 to the case of a single orbit $X = G/H$, where $H < G$ is an element of $\mathcal{X}$. Let $\rho \in \mathrm{Rep}(V,G)$ be a representation in the image of $\mathrm{Rep}(V,H)$. Fix elements $g_1,\dots,g_r \in G$ such that $c_{g_i^{-1}} \circ \rho$ are the possible factorizations of the representation ρ through H. By Theorem 2.1 one gets homeomorphisms

$$\begin{aligned}\coprod_{i=1}^r EG \times_{C_H(c_{g_i^{-1}}\circ\rho)} \{pt\} &\cong \coprod_{i=1}^r EG \times_{C_G(\rho)} C_G(E)/C_G(\rho)\cap g_iHg_i^{-1} \\ &\cong EG \times_{C_G(\rho)} (G/H)^\rho\end{aligned}$$

that provide the isomorphism

$$\prod_{i=1}^r H^*(BC_H(c_{g_i^{-1}\circ\rho})) \cong H^*_{C_G(\rho)}((G/H)^\rho)\,.$$

If we now bring into account all representations $\rho \in \mathrm{Rep}(V,G)$ that are in the image of $\mathrm{Rep}(V,H)$, the collections $c_{g_i^{-1}} \circ \rho$ will exhaust all representations $\sigma \in \mathrm{Rep}(V,H)$. Notice that for representations $\rho \in \mathrm{Rep}(V,G)$ that are not in the image of $\mathrm{Rep}(V,H)$, the space $(G/H)^\rho$ is empty. Hence we have an isomorphism

$$(2) \qquad \prod_{\sigma\in\mathrm{Rep}(V,H)} H^*(BC_H(\sigma)) \cong \prod_{\rho\in\mathrm{Rep}(V,G)} H^*_{C_G(\rho)}((G/H)^\rho)\,.$$

Now, $T_V(H^*_G(G/H)) \cong T_V(H^*(BH))$. Since H is a compact Lie group, it satisfies the theorem. In particular

$$(3) \qquad T_V(H^*(BH)) \cong \prod_{\sigma\in\mathrm{Rep}(V,H)} H^*(BC_H(\sigma))\,.$$

Combining the isomorphisms (2) and (3), we obtain that G satisfies part 1 of the theorem. The proof is complete once we observe that part 2 of the theorem is a special case of part 1 when we take X to be any G-p-acyclic complex. □

3.2. Mapping spaces

Let G be a group, and let π be a finite p-group. Let X be any G-space. For every representation $\rho\colon \pi \longrightarrow G$, let $C_G(\rho)$ denote the centralizer in G of the subgroup $\rho(\pi)$ and let X^ρ denote the fixed point space $X^{\rho(\pi)}$. Notice that the space X^ρ has a natural action of the group $C_G(\rho)\times\pi$ where π acts trivially. The group homomorphism $C_G(\rho)\times\pi \longrightarrow G$ yields a map of the homotopy orbit spaces

$$\begin{aligned}&(EC_G(\rho)\times_{C_G(\rho)} X^\rho)\times B\pi\\ &\cong (EC_G(\rho)\times E\pi)\times_{C_G(\rho)\times\pi} X^\rho \longrightarrow EG\times_G X.\end{aligned}$$

On completing this map and then taking the adjoint, we get a map for every ρ

$$((X^\rho)_{hC_G(\rho)})^\wedge_p \longrightarrow \mathrm{Map}(B\pi, (X_{hG})^\wedge_p)\,.$$

Theorem 3.2. *Let G be a group in the class $\mathcal{K}_1\mathcal{X}$ and π a finite p-group. Let X be any G-CW-complex of finite orbit type with isotropy subgroups in $\mathcal{X}$, then the following natural map is a homotopy equivalence.*

$$(4) \qquad \coprod_{\rho\in\mathrm{Rep}(\pi,G)} ((X^\rho)_{hC_G(\rho)})^\wedge_p \xrightarrow{\simeq} \mathrm{Map}(B\pi, (X_{hG})^\wedge_p)\,.$$

Taking X to be any G-p-acyclic complex, we obtain

Corollary 3.3. *If G is a group in the class $\mathcal{K}_1\mathcal{X}$ and π a finite p-group, then the following natural map is a homotopy equivalence*

$$(5) \qquad \coprod_{\rho\in\mathrm{Rep}(\pi,G)} BC_G(\rho)^\wedge_p \xrightarrow{\simeq} \mathrm{Map}(B\pi, BG^\wedge_p)\,.$$

□

We begin the proof of Theorem 3.2.

Proof. The above result is known in the case when G is a compact Lie group [5]. Now consider a G-CW-complex of finite orbit type. It is easy to verify that the space X_{hG} satisfies the conditions of Definition 6.2 in the appendix. Using the terminology of the appendix, X_{hG} is a finite $\mathfrak{X}$-CW-complex, where $\mathfrak{X}$ is the class consisting of spaces Y that are equivalent to finite disjoint unions of the classifying spaces of compact Lie groups.

The proof of the theorem for the case when X is a single orbit G/H is an easy consequence of Theorem 2.1 and previously known results about compact Lie groups [5]. It now follows by induction on the equivariant cells of X that the natural map

$$\coprod_{\rho\in\mathrm{Rep}(\pi,G)} ((X^\rho)_{hC_G(\rho)})^\wedge_p \longrightarrow \mathcal{M}(B\pi, X_{hG})^\wedge_p$$

is an equivalence, where $\mathcal{M}(B\pi, X_{hG})$ was defined in the appendix (Definition 6.6) to be the finite $\mathfrak{X}^1$-CW-complex with cells $\mathrm{Map}(B\pi, X_\alpha)$, where X_α are the cells of X_{hG}. By the main Theorem 6.11 in the appendix, the natural map

$$\mathcal{M}(B\pi, X_{hG})^\wedge_p \longrightarrow \mathrm{Map}(B\pi, (X_{hG})^\wedge_p)$$

is an equivalence. On composing the two equivalences, the proof is complete. □

3.3. The centralizers decomposition

For a group G, the Quillen category for G, $\mathcal{A}(G)$, has all elementary abelian p-subgroups of G as objects. Morphisms between two objects E and E' are group monomorphisms $\alpha\colon E \rightarrowtail E'$ for which there exists $g \in G$ such that $\alpha(e) = geg^{-1}$ for all $e \in E$. $\mathcal{A}_*(G)$ is the full subcategory of $\mathcal{A}(G)$ consisting of all objects except for the trivial subgroup.

Now let G be an element of $\mathcal{K}_1\mathcal{X}$ and X any G-space. If E is an elementary abelian subgroup of G, $C_G(E)$ denotes the centralizer in G of E and X^E the fixed point space by the action of E restricted from G. Notice that the space X^E has a natural action of the group $C_G(E)$ and we can define a functor

$$F_X\colon \mathcal{A}(G)^{op} \longrightarrow G\text{-}\mathcal{S}paces$$
$$F_X(E) = G \times_{C_G(E)} X^E \,.$$

Here the orbits are taken for the left action of $C_G(E)$ on $G \times X^E$ defined as $h \cdot (g, x) = (xh^{-1}, hx)$, for $h \in C_G(E)$ and $(g, x) \in G \times X^E$. The left action of G on $G \times_{C_G(E)} X^E$ is induced by the regular action of G on itself.

It is clear that the action $G \times X \longrightarrow X$ induces maps $G \times_{C_G(E)} X^E \longrightarrow X$, that form a natural transformation from F_X to the constant functor X hence a natural map

$$\mu_X\colon \operatorname*{hocolim}_{\mathcal{A}_*(G)^{op}} EG \times_G F_X \longrightarrow EG \times_G X\,.$$

Notice that if X and X^E are p-acyclic, μ_X is equivalent to the usual Jackowski-McClure map [9],

$$\pi\colon \operatorname*{hocolim}_{\mathcal{A}_*(G)^{op}} EG \times_G (G/C_G(E)) \longrightarrow EG \times_G \{pt\} = BG\,.$$

Theorem 3.4. *Let G be a group in the family $\mathcal{K}_1\mathcal{X}$.*

1. *Let X be any G-CW-complex with a finite number of equivariant orbits and such that all isotropy groups are elements of $\mathcal{X}$. Then the following natural map is a p-local homology equivalence.*

$$\mu_X\colon \operatorname*{hocolim}_{\mathcal{A}_*(G)^{op}} EG \times_G F_{X_s} \longrightarrow EG \times_G X_s$$

   where X_s denotes the p-singular locus of X, i.e. the set of all points in X which are fixed by some element of order p.
2. *Assume that there exists a G-p-acyclic complex X such that $X = X_s$, then the following natural map is a p-local homology equivalence.*

$$\pi\colon \operatorname*{hocolim}_{\mathcal{A}_*(G)^{op}} EG \times_G G/C_G(E) \longrightarrow BG\,.$$

Proof. It is easy to see that $X_s \subseteq X$ is a sub G-CW-complex. The proof now proceeds by induction on the equivariant skeletons of X_s. Notice again that the Theorem is true in the case of compact Lie groups [8, 0.1].

The induction step follows the arguments of [8, 2.6] that we just sketch. Firstly the proof is reduced to the case of a single orbit $X_s = G/H$, where $H < G$ is an element of $\mathcal{X}$ that contains nontrivial p-torsion. Then we use Theorem 2.1 in order to obtain

$$\begin{aligned}\operatorname*{hocolim}_{E\in\mathcal{A}_*(G)^{op}} EG\times_G (G \times_{C_G(E)} (G/H)^E) &\simeq \operatorname*{hocolim}_{E\in\mathcal{A}_*(G)^{op}} EG\times_{C_G(E)} (G/H)^E \\ &\simeq \operatorname*{hocolim}_{D\in\mathcal{A}_*(H)^{op}} EG \times_{C_H(D)} \{pt\}\end{aligned}$$

the last equality following from Theorem 2.1. Now the final space in the sequence of equivalences above is p-equivalent to $BH \simeq EG \times_G G/H$ since H is a compact Lie group.

It remains to show that (1) implies (2). This is achieved by taking X to be any G-p-acyclic complex with the property that $X_s = X$. □

4. Cohomology rings

4.1.

Once we have established Theorem 3.1 and the fact that the Quillen category of a group G in the class $\mathcal{K}_1\mathcal{X}$ is equivalent to a finite category, we can use arguments due to Henn, Lannes and Schwartz [6,7] in order to proof our structure theorems for mod p cohomology rings of groups in the class $\mathcal{K}_1\mathcal{X}$.

Theorem 4.1. *For a group G in the class $\mathcal{K}_1\mathcal{X}$ the Quillen map*

(6)
$$q_G\colon H^*(BG) \longrightarrow \varprojlim_{E\in\mathcal{A}(G)} H^*(BE)$$

is an F-isomorphism.

Proof. Classically an F-isomorphism is a homomorphism of algebras for which both the kernel and the cokernel are nilpotent algebras. In our context of unstable algebras over the Steenrod algebra, an algebra is nilpotent if it is nilpotent as an object in the category $\mathcal{U}$ of unstable modules over the Steenrod algebra. Recall that an unstable $\mathcal{A}$-module N is nilpotent if and only if $\mathrm{Hom}_{\mathcal{U}}(M, H^*V) = 0$ for any elementary abelian p-group V (c.f. [23]). It turns out that we only need to show that the induced map

(7)
$$q_G^\sharp\colon \mathrm{Hom}_{\mathcal{U}}\left(\varprojlim_{E\in\mathcal{A}(G)} H^*(BE), H^*V\right) \longrightarrow \mathrm{Hom}_{\mathcal{U}}(H^*(BG), H^*V)$$

is an isomorphism for any elementary abelian p-group V. And according to the linearization principle (c.f. [23, 3.8.6]) it will be enough to check $q_G^\sharp$ for Hom functors in the category $\mathcal{K}$ of unstable $\mathcal{A}$-algebras.

Now, according to Theorem 3.1 we can write the isomorphisms

(8)
$$\begin{aligned}\mathrm{Hom}_{\mathcal{K}}(H^*(BG), H^*V) &\cong \mathrm{Hom}_{\mathcal{K}}(T_V H^*(BG), \mathbb{F}_p)\\ &\cong \mathrm{Hom}_{\mathcal{K}}\left(\prod_{\rho\in\mathrm{Rep}(V,G)} H^*(BC_G(\rho), \mathbb{F}_p)\right) \cong \mathrm{Rep}(V,G)\,.\end{aligned}$$

On the other hand, $\mathrm{Rep}(V,G) \cong \mathrm{colim}_{E\in\mathcal{A}(G)}\,\mathcal{L}(V,E)$ is the colimit over the Quillen category of all linear maps from V to elementary abelian subgroups E of G. Furthermore, the functor $K \longmapsto \mathrm{Hom}_{\mathcal{K}}(K, H^*V)$ transforms finite limits into colimits, hence using the fact that the Quillen category $\mathcal{A}(G)$ for the group G is equivalent to a finite category, we can state isomorphisms

$$\begin{aligned}\operatorname{Rep}(V,G) &\cong \operatorname*{colim}_{E\in\mathcal{A}(G)} \mathcal{L}(V,E)\\ &\cong \operatorname*{colim}_{E\in\mathcal{A}(G)} \operatorname{Hom}_{\mathcal{K}}(H^*(BE),H^*V)\\ &\cong \operatorname{Hom}_{\mathcal{K}}(\varprojlim_{E\in\mathcal{A}(G)} H^*(BE),H^*V)\,. \end{aligned} \tag{9}$$

with which we finish the proof. □

Corollary 4.2. *For a group G in $\mathcal{K}_1\mathcal{X}$, the transcendence degree of $H^*(BG)$ coincides with the maximal rank of its elementary abelian p-subgroups.*

We have written the proof of Theorem 4.1 in a way that emphasizes the relevance of the functor $K \longmapsto \operatorname{Hom}_{\mathcal{K}}(K,H^*V)$, K an object of $\mathcal{K}$, in the investigation of the cohomology rings $H^*(BG)$.

For a general K of $\mathcal{K}$, $\operatorname{Hom}_{\mathcal{K}}(K,H^*V)$ has a natural structure of profinite set, induced by the collection of finitely generated subobjects of K. It also inherits from H^*V an action of the monoid $\operatorname{End} V$, so it becomes an object in the category $\mathcal{PS}\text{-}\operatorname{End} V$ of profinite $\operatorname{End} V$-sets. If V_d is an elementary abelian p-group of rank d the above functor is written

$$s_d\colon \mathcal{K} \longrightarrow (\mathcal{PS}\text{-}\operatorname{End} V_d)^{op}$$

with $s_d(K) = \operatorname{Hom}_{\mathcal{K}}(K,H^*V_d)$.

This structure is exploited in [6], where it is defined a functor

$$b_d\colon (\mathcal{PS}\text{-}\operatorname{End} V_d)^{op} \longrightarrow \mathcal{K}$$

as by the formula $b_d(S) = \operatorname{Hom}_{\mathcal{PS}\text{-}\operatorname{End} V_d}(S,H^*V_d)$. That is $b_d(S)^n$ is defined as $\operatorname{Hom}_{\mathcal{PS}\text{-}\operatorname{End} V_d}(S,H^nV_d)$ and the unstable $\mathcal{A}$-algebra structure is induced by that of H^*V_d. $b_d(K)$ is always a $\mathcal{N}il$-closed object of $\mathcal{K}$ with transcendence degree smaller that or equal to d and the pair s_d, b_d are inverse functors that define an equivalence of the categories $\mathcal{PS}\text{-}\operatorname{End} V_d$ and $\mathcal{K}_d/\mathcal{N}il$, the quotient category of unstable $\mathcal{A}$-algebras of transcendence degree smaller than or equal to d by the full subcategory of nilpotent $\mathcal{A}$-algebras.

With this notation, equations (8) and (9) providing the proof of Theorem 4.1 are written

$$s_d(H^*(BG)) \cong \operatorname{Rep}(V_d,G) \cong s_d(\varprojlim_{E\in\mathcal{A}(G)} H^*(BE))$$

in the category $\mathcal{PS}\text{-}\operatorname{End} V$, provided d is larger than the rang of any elementary abelian subgroup E of G. Since $\varprojlim_{E\in\mathcal{A}(G)} H^*(BE)$ is $\mathcal{N}il$-closed one identifies

$$b_d \circ s_d(H^*(BG)) \cong b_d \operatorname{Rep}(V_d,G) \cong \varprojlim_{E\in\mathcal{A}(G)} H^*(BE)$$

hence the Quillen map with the natural map $H^*(BG) \longrightarrow b_d \circ s_d(H^*(BG))$.

This pair of functors allows to interpret properties on objects in one of the categories $\mathcal{PS}\text{-}\operatorname{End} V_d$ or $\mathcal{K}_d/\mathcal{N}il$ in terms of properties of the corresponding objects in the other category. The concept of noetherian $\operatorname{End} V$-set is of particular interest to us. It requires the notion on kernel of an element in an $\operatorname{End} V$-set. We review these concepts and the results that are more relevant to us in the next paragraph.

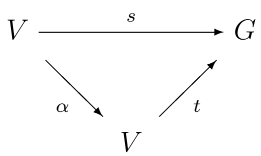
4.2. The notion of kernel for an element of an $\operatorname{End} V$*-set*

The results under review are all due to Henn, Lannes and Schwartz [6]. Fix an elementary abelian p-group V. An $\operatorname{End} V$-set is a set S together with a right action of $\operatorname{End} V$, considered as a monoid under composition. Our motivating examples are

1. $\operatorname{Rep}(V, G)$, for any group G and
2. $\operatorname{Hom}_{\mathcal{K}}(K, H^*V)$, for any unstable $\mathcal{A}$-algebra K.

In both cases, the action of $\operatorname{End} V$ is inherited from the natural action on V.

The notion of kernel is easily motivated by the first example. In fact if $s \in \operatorname{Rep}(V, G)$, the kernel of s is just defined as its group theoretic kernel. If we look at all possible factorizations of s as $t\alpha$, $\alpha \in \operatorname{End} V$



we can recover $\ker s$ as the maximal $\ker \alpha$ among all endomorphisms α that factor s. This is what we can generalize to any $\operatorname{End} V$-set.

Proposition-Definition 4.3. *If S is an $\operatorname{End} V$-set and $s \in S$, the kernel $\ker s$ is uniquely characterized by the properties:*

1. *For any $t \in S$ and $\alpha \in \operatorname{End} V$ such that $s = t\alpha$, $\ker \alpha \subset \ker s$.*
2. *There are elements $t_0 \in S$ and $\alpha_0 \in \operatorname{End} V$ with $s = t_0\alpha_0$ and $\ker \alpha_0 = \ker s$.*

We are now interested in the effect on a kernel of an element of $\operatorname{End} V$ and of a morphism of $\operatorname{End} V$-sets. In general, we obtain inclusions $\alpha^{-1}(\ker s) \subset \ker(s\alpha)$ for any element s of an $\operatorname{End} V$-set S and $\alpha \in \operatorname{End} V$, and $\ker s \subset \ker(\varphi s)$ for any morphism of $\operatorname{End} V$-sets $\varphi \colon S \longrightarrow S'$ and $s \in S$. The cases in which the equality holds are linked to finiteness properties.

Definition 4.4. An $\operatorname{End} V$-set S is called noetherian if

1. S is finite, and
2. $\alpha^{-1}(\ker s) = \ker(s\alpha)$ for any element $s \in S$ and $\alpha \in \operatorname{End} V$.

Definition 4.5. We say that a morphism of $\operatorname{End} V$-sets $\varphi\colon S \longrightarrow S'$ preserves kernels if $\ker s = \ker(\varphi s)$ for any $s \in S$.

As immediate examples we have that $\operatorname{Rep}(V, G)$ is a noetherian $\operatorname{End} V$-set as soon as $\operatorname{Rep}(V, G)$ is finite. Thus in particular, for any group G in the class $\mathcal{K}_1\mathcal{X}$. Furthermore, any injection of groups $\rho\colon G \rightarrowtail H$ induces a morphism of $\operatorname{End} V$-sets $\rho_\sharp\colon \operatorname{Rep}(V, G) \longrightarrow \operatorname{Rep}(V, H)$ that preserves kernels.

The importance of these concepts in our context comes mainly from the following results.

Theorem 4.6 **(**[6, 7.1]).

1. *If K is a noetherian unstable $\mathcal{A}$-algebra, then $s_d(K)$ is a noetherian $\operatorname{End} V$-set.*
2. *If S is a noetherian $\operatorname{End} V$-set, then $b_d(S)$ is a noetherian unstable $\mathcal{A}$-algebra.*

Theorem 4.7 **(**[6, 7.8]). *Let $\varphi\colon K \longrightarrow L$ be a map of unstable $\mathcal{A}$-algebras and $s_d(\varphi)\colon s_d(L) \longrightarrow s_d(K)$ the induced morphism of $\operatorname{End} V$-sets. Then,*

1. *If L is finitely generated as K-module via φ, $s_d(\varphi)$ preserves kernels.*
2. *If K and L are noetherian, the transcendence degree of L is d and $s_d(\varphi)$ preserves kernels, then L is a finitely generated K-module via φ.*

4.3.

We have now the necessary material in order to establish the main result of this section. The first immediate consequence is that for a group G in the class $\mathcal{K}_1\mathcal{X}$, the $\mathcal{N}il$-localization of $H^*(BG)$, $b_d \circ s_d(H^*(BG)) \cong \varprojlim_{E\in\mathcal{A}(G)} H^*(BE)$ is a noetherian ring, for $s_d(H^*(BG)) = \operatorname{Hom}_{\mathcal{K}}(H^*(BG), H^*(V_d)) \cong \operatorname{Rep}(V_d, G)$ is a noetherian $\operatorname{End} V$-set.

Theorem 4.8. *For any group G in the class $\mathcal{K}_1\mathcal{X}$, the mod p cohomology ring $H^*(BG)$ is noetherian. If a subgroup F belongs to $\mathcal{K}_1\mathcal{X}$, then $H^*(BF)$ is a finitely generated $H^*(BG)$-module via the restriction $H^*(BG) \longrightarrow H^*(BF)$.*

Proof. Pick a G-p-acyclic complex X. The acyclicity of X provides a p-equivalence

$$BG \simeq EG \times_G X\,.$$

Hence we have the associated isotropy spectral sequence of $H^*(BG)$-modules corresponding to the G-space X

$$E_2^{i,j} \cong H^i_G(X, H^j(BH_*)) \Longrightarrow H^{i+j}(BG) \tag{10}$$

that computes the cohomology of BG in terms of cohomology rings of the classifying spaces of its isotropy subgroups. Furthermore the E_2-term has a finite number of non-trivial columns hence in collapses at a finite stage (actually $E_1 \cong \prod_\Lambda H^*(BH)$ where Λ denotes the equivariant cells of X).

Next we claim the existence of a sub-$\mathcal{A}$-algebra K of $H^*(BG)$ which is noetherian and F-isomorphic to $H^*(BG)$. We have already seen that the $\mathcal{N}il$-localization of $H^*(BG)$, $L = \varprojlim_{E\in\mathcal{A}(G)} H^*(BE)$, is noetherian. If $\{x_1, \dots, x_s\}$ is a finite system of generators for L, we can find an integer k for which $L^{p^k} = \langle x_1^{p^k}, \dots, x_s^{p^k}\rangle$ is contained in the image of q_G. We can choose elements $y_1, \dots, y_s \in H^*(BG)$, with $q_G(y_j) = x_j^{p^k}$. Of course, the subalgebra generated by those elements in $H^*(BG)$ need not be closed under the Steenrod operations, however in each case the defect will be an element in the kernel of q_G and therefore we will find a large enough integer l such that the subalgebra K generated by $y_1^{p^l}, \dots, y_s^{p^l}$ is closed under the Steenrod operations and is then a sub-$\mathcal{A}$-algebra of $H^*(BG)$. Moreover, $q_G(K)$ is the subalgebra of L generated by $x_1^{p^{k+l}}, \dots, x_s^{p^{k+l}}$ and then K is F-isomorphic to L, hence also to $H^*(BG)$.

Now if $H < G$ is one of the isotropy groups for the space X, then the $\mathcal{A}$-map $K \longrightarrow H^*(BG) \longrightarrow H^*(BH)$ induces, for any elementary abelian p-group V, a diagram of $\mathrm{End}\, V$-sets

$$\begin{array}{ccccc}
\mathrm{Rep}(V,H) & \longrightarrow & \mathrm{Rep}(V,G) & & \\
\cong\downarrow & & \downarrow\cong & & \\
\mathrm{Hom}_{\mathcal{K}}(H^*(BH), H^*V) & \longrightarrow & \mathrm{Hom}_{\mathcal{K}}(H^*(BG), H^*V) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{K}}(K, H^*V)
\end{array}$$

Since $\mathrm{Rep}(V,H) \longrightarrow \mathrm{Rep}(V,G)$ preserves kernels and $H^*(BH)$ is noetherian, we notice that $H^*(BH)$ is finitely generated as a K-module.

Now the above spectral sequence might be considered a spectral sequence of K-modules by restriction via $K \longrightarrow H^*(BG)$, an as a such is has finitely generated E_1-term; that is, E_1 is a noetherian K-module hence each page E_i is a noetherian K-module, an therefore $H^*(BG)$ is itself a noetherian K-module, in particular a noetherian algebra.

In order to finish the proof we only need to observe that same argument as with the isotropy groups H will show that for any subgroup F of G, that belongs to the class $\mathcal{K}_1\mathcal{X}$, $H^*(BF)$ becomes a finitely generated K-module, or just a finitely generated $H^*(BG)$-module. $\square$

5. Groups related to Kač-Moody groups

As indicated in the introduction, we will show that Kač-Moody groups yield nontrivial examples of groups in the class $\mathcal{K}_1\mathcal{X}$. Indeed, it was our desire to understand the homotopic nature of Kač-Moody groups in the first place, that let to this paper. By a Kač-Moody group, we shall mean the unitary form of a complex Kač-Moody group. Similarly, a parabolic subgroup of a Kač-Moody group will mean the unitary form of the complex parabolic. It is our goal to show that groups of the form P/C belong to the class $\mathcal{K}_1\mathcal{X}$, where P is the parabolic subgroup of some Kač-Moody group and $C < P$ is a compact central subgroup. Let K_P be the semisimple factor in the Levi decomposition of P, so that P is a split extension of a torus with K_P. Notice that K_P is equivalent to a Kač-Moody group. To each group of the form $G = P/C$ as above, we associate a G-CW-complex X_G. We begin by describing X_G whenever $G = K$ is a Kač-Moody group of rank n.

Let $\mathcal{C}$ be the category of subsets of the n-element set $\{1, 2, \ldots, n\}$ such that the corresponding standard parabolic subgroup is compact. We define the morphisms to be usual inclusions of subsets. Consider the functor from $\mathcal{C}$ to topological spaces, which sends the subset I to the homogeneous space K/P_I, where P_I denotes the standard parabolic subgroup corresponding to the subset I. The homotopy colimit of this functor is our choice for the K-CW-complex X_K. Notice that whenever K is of finite type, the category $\mathcal{C}$ has a terminal object and the value of the functor on this object is a point, hence it is clear that X_K is contractible. In this special case one can use Smith theory to show that the space $(X_K)^\pi$ is acyclic for a finite p-subgroup $\pi < K$. For a noncompact Kač-Moody group, notice that the category $\mathcal{C}$ is a full subcategory of the category of all proper subsets of the set $\{1, 2, \ldots, n\}$. Consequently, we can identify $B\mathcal{C}$ with a subspace of the barycentric subdivision of the $(n-1)$ simplex (the latter being canonically identified with the classifying space of the poset of all proper subsets of the set $\{1, 2, \ldots, n\}$.)

$$X_K = (K/T \times B\mathcal{C})/\sim, \ (gT, x) \sim (hT, y) \Leftrightarrow x = y \in \Delta_I^\circ, \ sP_I = uP_I$$

where T is the maximal torus of K and Δ_I° denotes the interior of the face of the $(n-1)$-simplex Δ that corresponds to the subset I. The space X is acyclic and the proof of its acyclicity is similar to the one given in [21]. We recall the proof briefly.

Let B be the positive Borel subgroup of a complex Kač-Moody group that contains K as the unitary form. Consider the skeletal filtration of K/T:

$$F_k(K/T) = \coprod_{l(w) \leq k} BwB/B\,.$$

Here w denotes the elements of the Weyl group, W, of K. One uses this filtration to define a filtration of X_K by subcomplexes as follows:

$$F_k(X_K) = (F_k(K/T) \times B\mathcal{C})/\sim \ .$$

In [21, Thm. 2,16] it is shown that the obvious map

$$\zeta\colon F_k(K/T) \times B\mathcal{C} \longrightarrow F_k(X_K)/F_{k-1}(X_K)$$

is given by collapsing a subspace to a point and that the associated quotients are

$$F_k(X_K)/F_{k-1}(X_K) = \bigvee_{l(w)=k} \zeta(\overline{BwB/B} \times B\mathcal{C})\,.$$

We now proceed to show that the spaces $\zeta(\overline{BwB/B} \times B\mathcal{C})$ are acyclic. This shows that each filtration $F_k(X_K)$ is acyclic and hence since X_K is a colimit of such, it is itself acyclic.

It is straightforward to see that for a given $w \in W$, the space $\zeta(\overline{BwB/B}\times B\mathcal{C})$ is a suitable suspension of the quotient space $B\mathcal{C}/B\mathcal{D}_w$ where $\mathcal{D}_w \subset \mathcal{C}$ is the full subcategory of all subsets I such that w is identified with an element of lesser length within W/W_I. Now notice that $\mathcal{C}$ has an initial object given by the empty set. Consequently $B\mathcal{C}$ is contractible. Hence to show that $\zeta(\overline{BwB/B} \times B\mathcal{C})$ is acyclic, it is sufficient to show that $B\mathcal{D}_w$ is also contractible. Define I_w to be the set $\{i \in \{1, 2, \dots, n\} \mid l(wr_i) < l(w)\}$. It follows from the general theory of Coxeter groups that W_{I_w} is a finite group and hence that $I_w \in \mathcal{D}_w$. Consider the functor $\mathcal{F}\colon \mathcal{D}_w \longrightarrow \mathcal{D}_w$ which maps a subset I to the subset $I \cap I_w$. Notice that this functor provides us with a zig-zag between the identity functor and the constant functor with value I_w. It follows that $B\mathcal{D}_w$ is contractible. For future use, we now establish the acyclicity of various other subcomplexes of X_K.

Let S be any subcomplex of K/T. Let us define a filtration of S via

$$F_k(S) = F_k(K/T) \cap S = \coprod_{l(\bar{w})\leq k} B\bar{w}B/B$$

where $\bar{w}$ are elements of W that index the cells in S. One can define a subcomplex X_S of X_K as

$$X_S = (S \times B\mathcal{C})/\sim,\ (sT, x) \sim (uT, y) \Leftrightarrow x = y \in \Delta_I^\circ\,,\ sP_I = uP_I.$$

Lemma 5.1. *Let S be a subcomplex of K/T, then the subcomplex X_S of X_K is acyclic whenever X_K is acyclic.*

Proof. Define a filtration of X_S by subcomplexes

$$F_k(X_S) = F_k(X_K) \cap X_S.$$

It is straightforward to see that

$$F_k(X_S)/F_{k-1}(X_S) = \bigvee_{l(\bar{w})=k} \zeta(\overline{B\bar{w}B/B} \times B\mathcal{C}).$$

Since the spaces $\zeta(\overline{B\bar{w}B/B} \times B\mathcal{C})$ are acyclic whenever X_K is contractible, it follows that the spaces $F_k(X_S)$ are acyclic and consequently that X_S is acyclic. □

We now generalize the above construction to the groups G of the form P/C where P is a parabolic subgroup of some Kač-Moody group and C is a compact subgroup of the center in P. Let K_P be the semisimple factor of P so that P is a split extension of a torus with K_P. Since K_P can be identified with a Kač-Moody group, we define the space X_G to be X_{K_P}. To see that X_G is a G-CW-complex , one only needs to observe that each homogeneous spaces K_P/P_I can be written as P/Q, where P_I is a standard parabolic in K_P and $Q < P$ is a suitable subparabolic that contains C. If H is an isotropy subgroup of the G action on X, then it is easy to see that $H < G$ is a compact Lie group of maximal rank. In particular, if G is nontrivial, then every isotropy subgroup of the G action on X_G contains nontrivial p-torsion.

We record the following fact about Kač-Moody groups to be used in the following theorem.

Theorem 5.2. *Let K be a Kač-Moody group of rank n. Given $J \subseteq \{1,2,\ldots,n\}$, let P_J denote the corresponding standard parabolic subgroup. Let Π be a finite subgroup of P_J, then up to conjugacy within P_J, there exists a standard parabolic subgroup of finite type $P_I < P_J$ that contains Π.*

Proof. This is essentially a slight generalization of a theorem of Kač-Peterson. In [12] it is shown that any finite subgroup of a Kač-Moody group can be conjugated into a standard parabolic subgroup of finite type. Their proof can be easily extended to show that any finite subgroup of a standard parabolic subgroup P_J can be conjugated within P_J into a standard parabolic subgroup of finite type $P_I < P_J$. □

Theorem 5.3. *Let G be any group of the form P/C as described above. Let Π be a finite p-group, and let $\rho\colon \Pi \longrightarrow G$ be a representation. Then the spaces $(X_G)^\rho$ are p-acyclic.*

Proof. One may assume that P is a standard parabolic subgroup in some Kač-Moody group. By Theorem 5.2 one may further assume that there exists a standard parabolic of finite type $Q < P$ such that ρ factors through the compact Lie group $Q/C < P/C$. The semisimple factor K_P inside P can be identified with a Kač-Moody group and we need only consider the case when K_P is a noncompact Kač-Moody group. By definition, we have

$X_G = X_{K_P}$. Let $P_I = K_P \cap Q$ be the standard compact parabolic subgroup $P_I < K_P$ and let $T < K_P$ be the standard maximal torus. Define a filtration of the space K_P/T by finite subcomplexes:

$$S_k = P_I \cdot F_k(K_P/T)$$

where $F_k(K_P/T)$ is the standard skeletal filtration of K_P/T. Notice that by assumption the action of Π via ρ on the space K_P/T preserves the finite subcomplex S_k. Since these subcomplexes provide a filtration of K_P/T, the finite complexes $(X_G)_{S_k}$ of Lemma 5.1 provide a filtration of X_G by finite subcomplexes which are preserved under the action of Π via ρ. From Lemma 5.1, each space $(X_G)_{S_k}$ is acyclic. So by Smith theory, the fixed point sets $(X_G)^{\rho}_{S_k}$ are p-acyclic. Now the result follows by taking direct limits. □

We will now show that the Weyl group, W, of a Kač-Moody group K belongs to the class $\mathcal{K}_1\mathcal{X}$. As before let $\mathcal{C}$ be the category of subsets of the n-element set $\{1, 2, \ldots, n\}$ such that the corresponding standard parabolic subgroup is compact. We define the morphisms to be usual inclusions of subsets. Consider the functor from $\mathcal{C}$ to topological spaces, which sends the subset I to the homogeneous space W/W_I, where W_I denotes the Weyl group corresponding to the subset I. The homotopy colimit of this functor is our choice for the W-space X_W. An essential difference from the space X_K is that X_W is a finite dimensional CW-complex. This is to be expected since W is a discrete group. As before, to show the required properties for X_W, we may assume that W is infinite. The proof that shows X_W is acyclic is similar to the nondiscrete situation. One identifies X_W with the space

$$X_W = (W \times B\mathcal{C})/\sim\,,\ (u,x) \sim (v,y) \Leftrightarrow x = y \in \Delta_I^\circ\,,\ uW_I = vW_I\,.$$

Next, one filters the space X_W using the length of elements in the Weyl group. The associated quotients are shown to be contractible for exactly the same reasons as before. Notice that X_W can be considered as a finite $N_K(T)$-CW-complex , where $N_K(T)$ is the normalizer of the maximal torus in K, and is seen to act on X_W via the projection $N_K(T) \longrightarrow W$. This shows that $N_K(T)$ is an element of $\mathcal{K}_1\mathcal{X}$. Notice also that the finite dimensionality of X_W forces the acyclicity of $(X_W)^\pi$ for finite p-groups $\pi < W$.

We can extend this construction to normalizers of maximal tori within groups of the form $G = P/C$, where P is a parabolic subgroup within a Kač-Moody group and $C < P$ is a compact central subgroup. Let $N < G$ be the normalizer of the maximal torus T/C. Let W_G be the Weyl group of P. It is easy to see that W_G is canonically isomorphic to the Weyl group W_P of the Kač-Moody group given by the semisimple factor $K_P < P$. We define $X_N = X_{W_G}$ to be the space X_{W_P}. It is straightforward to check the details and we leave them to the interested reader.

Remark 5.4. If G is a group of the form P/C considered above, then the isotropy groups of the G-action on X_G contain a conjugate of the torus T/C. The same is true for groups G that are normalizers of T/C within P/C. It follows that, in these cases, all the isotropy subgroups contain nontrivial p-torsion for any prime p. Note that this condition is not true in general for groups G that are the Weyl groups of T/C within P/C.

6. Appendix : $\mathfrak{X}$-CW-complexes

Spaces that can be obtained by successively "attaching" a finite number of reasonable spaces are indeed special. In this section we analyze the behavior of such spaces. Before we begin, we remind the reader that all cohomology is to be understood with p-primary coefficients.

Let $\mathfrak{X}$ be any class of spaces such that given Y in $\mathfrak{X}$, and a finite p-group π

1. $H^*(Y)$ and $H^*(\mathrm{Map}(B\pi, Y))$ are of finite type.
2. The natural map $\mathrm{Map}(B\pi, Y)^\wedge_p \longrightarrow \mathrm{Map}(B\pi, Y^\wedge_p)$ is an equivalence.

Remark 6.1. An example of such a class which will be of interest is the class $\mathfrak{X}$ of spaces that are equivalent to a finite disjoint union of the classifying spaces of compact Lie groups. The justification of this observation is the content of Sect. 7.

Another example of such a class is the class of spaces equivalent to finite disjoint unions of the classifying spaces of p-compact groups.

From now on, fix a class $\mathfrak{X}$ that satisfies the above two conditions.

Definition 6.2. A space Y is a $\mathfrak{X}$-CW-complex if there exist a space $\overline{Y}$, and a ladder of inclusions

$$\begin{array}{ccccccccc}
\emptyset = Y_{-1} & \subset & Y_0 & \subset & Y_1 & \subset & \dots & Y = \operatorname*{colim}_{n\in\mathbb{Z}} Y_n \\
 & & \downarrow g_0 & & \downarrow g_1 & & & \downarrow g \\
\emptyset = \overline{Y}_{-1} & \subset & \overline{Y}_0 & \subset & \overline{Y}_1 & \subset & \dots & \overline{Y} = \operatorname*{colim}_{n\in\mathbb{Z}} \overline{Y}_n
\end{array}$$

where the spaces Y_n and $\overline{Y}_n$ are obtained inductively as homotopy pushouts:

$$\begin{array}{ccc} \amalg_{\Lambda_n} Y_\alpha \times S^{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \amalg_{\Lambda_n} Y_\alpha \times * & \longrightarrow & Y_n \end{array} \qquad \begin{array}{ccc} \amalg_{\Lambda_n} S^{n-1} & \xrightarrow{\overline{f}_{n-1}} & \overline{Y}_{n-1} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \amalg_{\Lambda_n} * & \longrightarrow & \overline{Y}_n \, , \end{array}$$

where π denotes the obvious projection map and the maps f_{n-1} and $\overline{f}_{n-1}$ are any maps so that the following diagram commutes:

$$\begin{array}{ccc} \amalg_{\Lambda_n} Y_\alpha \times S^{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle g_{n-1}} \\ \amalg_{\Lambda_n} S^{n-1} & \xrightarrow{\overline{f}_{n-1}} & \overline{Y}_{n-1} \, . \end{array}$$

We define the map g_n as the canonical map induced by the commutativity of the above diagram. The spaces $\{Y_\alpha; \alpha \in \Lambda_n\}$ are called the n-cells of Y and belong to $\mathfrak{X}$. If the total number of cells of Y are finite, we shall call Y a finite $\mathfrak{X}$-CW-complex. The spaces Y_n are called the n-skeleta of Y.

Remark 6.3. Roughly speaking, a $\mathfrak{X}$-CW-complex Y is a space over a regular CW-complex $\overline{Y}$, so that the fiber over every point in $\overline{Y}$ belongs to $\mathfrak{X}$.

Definition 6.4. Define a new class $\mathfrak{X}^1$ to contain all spaces that are finite disjoint unions of spaces of the form $\{\mathrm{Map}(B\kappa, Y)_{h\rho} \mid Y \in \mathfrak{X}\}$ where κ, ρ are finite p-groups with the action of ρ on $\mathrm{Map}(B\kappa, Y)$ induced by an action on $B\kappa$.

Theorem 6.5. *The class $\mathfrak{X}^1$ also satisfies the two conditions stated earlier.*

Proof. Given $Y \in \mathfrak{X}$ and finite p-groups π and κ, with π acting on $B\kappa$, consider the fibration

$$\mathrm{Map}(B\kappa, Y) \longrightarrow \mathrm{Map}(B\kappa, Y)_{h\rho} \longrightarrow B\rho \, . \tag{11}$$

Since $H^*(\mathrm{Map}(B\kappa, Y))$ and $H^*(B\rho)$ are both of finite type, it follows that so is $H^*(\mathrm{Map}(B\kappa, Y)_{h\rho})$. Next consider the fibration of mapping spaces of 11 over a general component:

(12)
$$\mathrm{Map}(B\kappa, Y)^{h\pi} \longrightarrow \mathrm{Map}(B\pi, \mathrm{Map}(B\kappa, Y)_{h\rho})_{(f)} \longrightarrow \mathrm{Map}(B\pi, B\rho)_f \, .$$

Since $\mathrm{Map}(B\kappa, Y)^{h\pi} = \mathrm{Map}((B\kappa)_{h\pi}, Y)$, we know that $H^*(\mathrm{Map}(B\kappa, Y)^{h\pi})$ is of finite type. Clearly $H^*(\mathrm{Map}(B\pi, B\rho)_f)$ is also of finite type, hence it follows that $H^*(\mathrm{Map}(B\pi, \mathrm{Map}(B\kappa, Y)_{h\rho})_{(f)})$ is of finite type verifying the first condition. To verify the second condition, notice that the completion of the fibration 12 still remains a fibration. Alternatively, we can complete the fibration 11 and then consider the mapping spaces. We have a diagram of fibrations over a general component:

$$\begin{array}{ccccc} (\mathrm{Map}(B\kappa,Y)^{h\pi})^\wedge_p & \longrightarrow & (\mathrm{Map}(B\pi,\mathrm{Map}(B\kappa,Y)_{h\rho})_{(f)})^\wedge_p & \longrightarrow & (\mathrm{Map}(B\pi,B\rho)_f)^\wedge_p \\ \downarrow & & \downarrow & & \downarrow \\ (\mathrm{Map}(B\kappa,Y)^\wedge_p)^{h\pi} & \longrightarrow & \mathrm{Map}(B\pi,(\mathrm{Map}(B\kappa,Y)_{h\rho})^\wedge_p)_{(f)} & \longrightarrow & \mathrm{Map}(B\pi,(B\rho)^\wedge_p)_f \end{array}$$

Since $\mathrm{Map}(B\kappa,Y)^{h\pi} = \mathrm{Map}((B\kappa)_{h\pi},Y)$, our assumptions on the class $\mathfrak{X}$ imply that the induced map on the fiber is an equivalence. The induced map on the base is clearly an equivalence. It follows that the map on the total spaces is also an equivalence, completing the proof. □

Definition 6.6. Given a $\mathfrak{X}$-CW-complex Y, and a finite p-group π, consider the map

$$\mathrm{Map}(B\pi,Y) \xrightarrow{g_*} \mathrm{Map}(B\pi,\overline{Y}).$$

Let $\mathcal{M}(B\pi,Y) = g_*^{-1}(\overline{Y})$ be the subspace of $\mathrm{Map}(B\pi,Y)$ given by the inverse image of the constant maps $\overline{Y} \subseteq \mathrm{Map}(B\pi,\overline{Y})$.

It is left to the reader to verify that $\mathcal{M}(B\pi,Y)$ is a $\mathfrak{X}^1$-CW-complex with n-cells $\{\mathrm{Map}(B\pi,Y_\alpha), \alpha \in \Lambda_n\}$, and attaching maps $\mathrm{Map}(B\pi,f_n)$.

Proposition 6.7. *Given a finite $\mathfrak{X}$-CW-complex Y, let $Z = Y^\wedge_p$. Then for an elementary abelian p-group V, the natural map*

$$\mathcal{M}(BV,Y)^\wedge_p \xrightarrow{F_{BV}} \mathrm{Map}(BV,Z)$$

is an equivalence.

Proof. The skeletal filtration for $\mathfrak{X}$-CW-complex es gives rise to a spectral sequence in cohomology. Let $M_*^{*,*}$ denote the spectral sequence converging to the mod p cohomology of $\mathcal{M}(BV,Y)$ and $E_*^{*,*}$ the one converging to $H^*(Y)$. These might be considered as spectral sequences of unstable modules over the Steenrod algebra (the action is vertical and is induced from the geometric filtration).

Evaluation provides a filtration preserving map

$$ev\colon BV \times \mathcal{M}(BV,Y) \longrightarrow Y.$$

Since both tensoring with $H^*(BV)$, and T_V are exact functors, we get a maps of spectral sequences $E_*^{*,*} \longrightarrow H^*(BV) \otimes M_*^{*,*}$ and its adjoint $T_V E_*^{*,*} \longrightarrow M_*^{*,*}$. By [17, 3.4.1], our assumptions on $\mathfrak{X}$ imply that the evaluation map induces an isomorphism on each cell Y_α of Y

$$T_V H^*(Y_\alpha) \cong H^*(\mathrm{Map}(BV,Y_\alpha))$$

Furthermore, T_V commutes with finite products, hence the map of spectral sequences is an isomorphism at the first page, hence also $T_V E_\infty^{*,*} \cong M_\infty^{*,*}$. Since the filtration is finite, we have an isomorphism

$$T_V H^*(Y) \cong H^* \mathcal{M}(BV, Y) \tag{13}$$

induced by the evaluation map. Since all spaces involved are of finite type, by [17, 3.3.1] we obtain that the map F_{BV} is an equivalence completing the proof. □

We now recall two results that will be important in the sequel.

Proposition 6.8. [17, 4.2] *Let V be an elementary abelian p-group, and let X be a V-space. Then the BV-equivariant map*

$$BV \times X^{hV} \longrightarrow \mathrm{Map}(BV, X_{hV})_{(1)} .$$

induced by the inclusion $X^{hV} \longrightarrow \mathrm{Map}(BV, X_{hV})_{(1)}$ and the action of BV on the space $\mathrm{Map}(BV, X_{hV})_{(1)}$, is a homeomorphism.

Proposition 6.9. [17, 4.3.1] *The following natural map is a homotopy equivalence:*

$$(X_p^\wedge)_{hV} \longrightarrow (X_{hV})_p^\wedge .$$

As a consequence of these results we get

Corollary 6.10. *Given a finite $\mathfrak{X}$-CW-complex Y, and an extension of finite p-groups*

$$\kappa \longrightarrow \pi \longrightarrow \mathbb{Z}/p .$$

Let $\mathcal{B}\kappa$ denote the $\mathbb{Z}/p$-space $E\pi/\kappa$. Then the following natural map is an equivalence

$$\mathcal{M}(B\pi, Y)_p^\wedge \xrightarrow{\simeq} (\mathcal{M}(\mathcal{B}\kappa, Y)_p^\wedge)^{h\mathbb{Z}/p} .$$

Proof. Notice that the space $\mathcal{M}(\mathcal{B}\kappa, Y)_{h\mathbb{Z}/p}$ is a $\mathfrak{X}^1$-CW-complex. Applying Proposition 6.7 to the class $\mathfrak{X}^1$, we get the equivalence

$$\mathcal{M}(B\mathbb{Z}/p, \mathcal{M}(\mathcal{B}\kappa, Y)_{h\mathbb{Z}/p})_p^\wedge \xrightarrow{F_{B\mathbb{Z}/p}} \mathrm{Map}(B\mathbb{Z}/p, (\mathcal{M}(\mathcal{B}\kappa, Y)_{h\mathbb{Z}/p})_p^\wedge) .$$

Define

$$\beta \colon \mathcal{M}(B\pi, Y)_p^\wedge \times B\mathbb{Z}/p \longrightarrow \mathrm{Map}(B\mathbb{Z}/p, (\mathcal{M}(\mathcal{B}\kappa, Y)_p^\wedge)_{h\mathbb{Z}/p})_{(1)}$$

to be the map induced by

$$\mathcal{M}(B\pi, Y)_p^\wedge \longrightarrow \mathrm{Map}(B\mathbb{Z}/p, (\mathcal{M}(\mathcal{B}\kappa, Y)_p^\wedge)_{h\mathbb{Z}/p})_{(1)}$$

and the $B\mathbb{Z}/p$-action on $\mathrm{Map}(B\mathbb{Z}/p, (\mathcal{M}(\mathcal{B}\kappa, Y)^\wedge_p)_{h\mathbb{Z}/p})_{(1)}$. One has a commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(B\pi, Y)^\wedge_p \times B\mathbb{Z}/p & \xrightarrow{\beta} & \mathrm{Map}(B\mathbb{Z}/p, (\mathcal{M}(\mathcal{B}\kappa, Y)^\wedge_p)_{h\mathbb{Z}/p})_{(1)} \\ \downarrow \alpha & & \downarrow \gamma \\ (\mathcal{M}(B\mathbb{Z}/p, \mathcal{M}(\mathcal{B}\kappa, Y)_{h\mathbb{Z}/p})^\wedge_p)_{(1)} & \xrightarrow{F_{B\mathbb{Z}/p}} & \mathrm{Map}(B\mathbb{Z}/p, (\mathcal{M}(\mathcal{B}\kappa, Y)_{h\mathbb{Z}/p})^\wedge_p)_{(1)} . \end{array}$$

where α is the (cellular) equivalence defined using Proposition 6.8 and γ is the equivalence induced by the map in Proposition 6.9. It follows that β is an equivalence. The restriction of β to the fiber over the identity element in $B\mathbb{Z}/p$ is the equivalence we require. This completes the proof. □

We can now use Proposition 6.7 as the induction step in proving the main theorem of this section.

Theorem 6.11. *Given a finite $\mathfrak{X}$-CW-complex Y, let $Z = Y^\wedge_p$. Let π be a finite p-group, then the natural map*

$$\mathcal{M}(B\pi, Y)^\wedge_p \xrightarrow{F_{B\pi}} \mathrm{Map}(B\pi, Z)$$

is an equivalence.

Proof. We will argue by induction on the order of π. If π is elementary abelian we are in the situation of Proposition 6.7, hence induction starts.

Assume now π is a finite p-group and the result is true for p-groups of smaller order. We can find an extension

$$\kappa \longrightarrow \pi \longrightarrow \mathbb{Z}/p .$$

Our induction hypothesis apply to κ and then, if $\mathcal{B}\kappa = E\pi/\kappa$, we have an equivalence

$$\mathcal{M}(\mathcal{B}\kappa, Y)^\wedge_p \xrightarrow{F_{\mathcal{B}\kappa}} \mathrm{Map}(\mathcal{B}\kappa, Z)$$

which is $\mathbb{Z}/p$-equivariant for the induced action of $\mathbb{Z}/p$ on $\mathcal{B}\kappa$. Hence we obtain an equivalence of homotopy fixed points

$$(\mathcal{M}(\mathcal{B}\kappa, Y)^\wedge_p)^{h\mathbb{Z}/p} \xrightarrow{F^{h\mathbb{Z}/p}_{\mathcal{B}\kappa}} \mathrm{Map}(\mathcal{B}\kappa, Z)^{h\mathbb{Z}/p} . \tag{14}$$

Now consider the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(B\pi, Y)^\wedge_p & \xrightarrow{F_{B\pi}} & \mathrm{Map}(B\pi, Z) \\ \downarrow & & \downarrow \iota \\ (\mathcal{M}(\mathcal{B}\kappa, Y)^\wedge_p)^{h\mathbb{Z}/p} & \xrightarrow{F^{h\mathbb{Z}/p}_{\mathcal{B}\kappa}} & \mathrm{Map}(\mathcal{B}\kappa, Z)^{h\mathbb{Z}/p} \end{array}$$

where ι is the obvious equivalence and the vertical map on the left is the equivalence from 6.10. It follows that $F_{B\pi}$ is also an equivalence. □

7. Appendix: Maps from BP to a completed classifying space

The results of this section have been borrowed from a joint work between C. Broto, R. Levi and B. Oliver.

The main result in this section is a description of $\mathrm{Map}(BP, BG^\wedge_p)$ when P is a finite p-group and G is a compact Lie group. This result is well known, and proofs of parts of it have been published (c.f. [20]); but a complete proof doesn't seem to be written up anywhere.

We first note the known result about maps between (uncompleted) classifying spaces. For any pair G, G' of topological groups, $\mathrm{Rep}(G, G') \overset{\text{def}}{=} \mathrm{Hom}(G, G')/\mathrm{Inn}(G')$ denotes the set of (continuous) homomorphisms $G \to G'$ modulo conjugation by elements of G'. And for any homomorphism $\rho\colon G \to G'$, we write $C_{G'}(\rho) \overset{\text{def}}{=} C_{G'}(\rho(G)) \subseteq G'$.

Proposition 7.1. *If G and G' are discrete groups, then the map*

$$B\colon \mathrm{Rep}(G, G') \longrightarrow [BG, BG']$$

that sends a homomorphism ρ to $B\rho$ is a bijection. And for any homomorphism $\rho\colon G \longrightarrow G'$, the product map $(\mathrm{incl}, \rho)\colon C_{G'}(\rho) \times G \longrightarrow G'$ induces a homotopy equivalence

$$BC_{G'}(\rho) \longrightarrow \mathrm{Map}(BG, BG')_{B\rho}\,.$$

Proof. The proof is elementary. Classical results about maps to an Eilenberg-MacLane space imply that the space of pointed maps $\mathrm{Map}_*(BG, BG')$ is homotopically discrete and the group of components is naturally isomorphic to $\mathrm{Hom}(G, G')$. It then follows from the fibre sequence

$$\mathrm{Map}_*(BG, BG') \longrightarrow \mathrm{Map}(BG, BG') \xrightarrow{ev} BG$$

that the set of components of the space of unpointed maps $\mathrm{Map}(BG, BG')$ is naturally isomorphic to $\mathrm{Rep}(G, G')$, and that for each homomorphism $\rho\colon G \longrightarrow G'$, $\mathrm{Map}(BG, BG')_{B\rho}$ is aspherical with $C_{G'}(\rho) \cong \pi_1 \mathrm{Map}(BG, BG')_{B\rho}$. □

This result was generalised to the case where the target is the classifying space of a compact Lie group by Dwyer and Zabrodsky.

Proposition 7.2. [5] *Let P be a finite p-group and G a compact Lie group. Then the map*

$$B\colon \mathrm{Rep}(P, G) \xrightarrow{\cong} [BP, BG]\,,$$

which sends a homomorphism $\rho\colon P \to G$ to $B\rho$, is a bijection. And for each homomorphism $\rho\colon P \longrightarrow G$, the product map $(\mathrm{incl}, \rho)\colon C_G(\rho) \times P \longrightarrow G$ induces

$$BC_G(\rho) \longrightarrow \mathrm{Map}(BP, BG)_{B\rho}$$

which in turn induces an isomorphism of fundamental groups and of $\mathbb{F}_p$-homology groups. □

The description of mapping spaces $\mathrm{Map}(BP, BG_p^\wedge)$ will be shown by comparing it with $\mathrm{Map}(BP, BSU(n)_p^\wedge)$ for some embedding $G \leq SU(n)$. The following elementary lemma will be needed.

Lemma 7.3. *Fix $n > 1$, and let $\varphi\colon U(n-1) \to SU(n)$ be the homomorphism which sends a matrix A to $\left(\begin{smallmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{smallmatrix}\right)$. Then for any $H \leq \mathrm{Im}(\varphi)$, the centralizer of H in $SU(n)$ is connected.*

Proof. For any $H \leq \mathrm{Im}(\varphi)$, the space $C_{SU(n)}(H)$ is a retract of $C_{U(n)}(H)$ by the retraction which sends a matrix A to its product with $\mathrm{diag}(1, \ldots, 1, \det(A)^{-1})$. (Any such matrix centralizes H by the assumption $H \leq \mathrm{Im}(\varphi)$.) Since $C_{U(n)}(H)$ is connected (a product of unitary groups), $C_{SU(n)}(H)$ is also connected. □

The next lemma is a special case of [4, Lemma 10.6]. Recall first that for a given topological group G and G-space X, the space of homotopy fixed points is defined as

$$X^{hG} = \mathrm{Map}_G(EG, X)\,,$$

the space of G-equivariant maps from the total space of the universal G-bundle EG to X. It can also be described as the space of sections of the projection map $EG \times_G X \longrightarrow BG$.

Lemma 7.4. *Fix a fibration $p\colon E \to X$ with fibre F, a finite group G, and a map $f\colon BG \to X$. Assume there is an action of G on F, and a map $\varphi\colon EG \times_G F \to E$, such that the following is a homotopy pullback square*

$$\begin{array}{ccc} EG \times_G F & \xrightarrow{\varphi} & E \\ {\scriptstyle \mathrm{proj}}\downarrow & & \downarrow{\scriptstyle p} \\ BG & \xrightarrow{f} & X\,. \end{array}$$

Let $\mathrm{Map}(BG, E)_f$ be the space of those maps $f'\colon BG \to E$ such that $p \circ f' \simeq f$. Then the fibration

$$\mathrm{Map}(BG, p)_f\colon \mathrm{Map}(BG, E)_f \longrightarrow \mathrm{Map}(BG, X)_f$$

has fibre homotopy equivalent to F^{hG}.

Proof. Since p is a fibration, so is $\mathrm{Map}(BG, p)_f$. The fibre over the element $f \in \mathrm{Map}(BG, X)_f$ is homeomorphic to the space of sections of the pullback, hence the space of sections of the Borel map $EG \times_G F \to BG$; and this is the homotopy fixed point set F^{hG}. □

We are now ready to prove Proposition 7.5: to show for any p-group P and any compact Lie group G, that $\mathrm{Map}(BP, BG^\wedge_p)$ is weakly homotopy equivalent to the p-completion of $\mathrm{Map}(BP, BG)$.

Proposition 7.5. *Let G be a compact Lie group. Then for any finite p-group P, the p-completion map $BG \longrightarrow BG^\wedge_p$ induces a homotopy equivalence*

$$\mathrm{Map}(BP, BG)^\wedge_p \xrightarrow{\simeq} \mathrm{Map}(BP, BG^\wedge_p).$$

In other words, the map

$$B\colon \mathrm{Rep}(P, G) \xrightarrow{\cong} [BP, BG^\wedge_p],$$

which sends a homomorphism $\rho\colon P \to G$ to $B\rho$, is a bijection; and for each $\rho\colon P \to G$ the product map $(\mathrm{incl}, \rho)\colon C_G(\rho) \times P \longrightarrow G$ induces a homotopy equivalence

$$BC_G(\rho)^\wedge_p \longrightarrow \mathrm{Map}(BP, BG^\wedge_p)_{B\rho}\,.$$

Proof of Proposition 7.5. **Case 1: G simply connected and $\mathbf{C_G}(\rho)$ connected.** Assume first that G is connected and simply connected. Let F denote the homotopy fibre of the completion map $BG \to BG^\wedge_p$. The obstructions to lifting a map $BP \to BG^\wedge_p$ to BG lie in $H^n(BP; \pi_{n-1}(F))$ (all $n \geq 1$), and the obstructions to the uniqueness of such liftings (up to homotopy) lie in $H^n(BP; \pi_n(F))$. Since F is simply connected with uniquely p-divisible homotopy groups, these obstruction groups all vanish, and hence $[BP, BG] \cong [BP, BG^\wedge_p]$.

Similarly, for any $i \geq 0$, we obtain isomorphisms of sets of homotopy classes of pointed maps

$$\mathrm{Map}_*(BP, BG) \xrightarrow{\simeq} \mathrm{Map}_*(BP, BG^\wedge_p)$$

is a homotopy equivalence. This is a particular case of [19, Theorem 1.5].

The corresponding map between unpointed mapping spaces fits in the diagram of fibrations

$$\begin{array}{ccccc}
\mathrm{Map}_*(BP, BG) & \longrightarrow & \mathrm{Map}(BP, BG) & \xrightarrow{ev} & BG \\
\simeq\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_*(BP, BG^\wedge_p) & \longrightarrow & \mathrm{Map}(BP, BG^\wedge_p) & \xrightarrow{ev} & BG^\wedge_p\,.
\end{array} \tag{15}$$

Since BG, and therefore $BG^\wedge_p$, are simply connected, by [1, Lemma II.5.2(i)], these fibrations remain fibrations after p-completion, and therefore we will have finish if we prove that given a homomorphism $\rho\colon P \longrightarrow G$ for which $C_G(\rho)$ is connected, the mapping space $\mathrm{Map}(BP, BG^\wedge_p)_{B\rho}$ is p-complete.

Now, we fix a homomorphism $\rho\colon P \longrightarrow G$ and assume furthermore that $C_G(\rho)$ is connected. The first observation is that in this case $\mathrm{Map}(BP, BG)_{B\rho}$ is simply connected by [5] (c.f. Proposition 7.2), and from diagram (15) we obtain that $\mathrm{Map}(BP, BG^\wedge_p)_{B\rho}$ is also simply connected. Hence, again by [1, Lemma II.5.2(i)], $\mathrm{Map}(BP, BG^\wedge_p)_{B\rho}$ is p-complete if and only if $\Omega\,\mathrm{Map}(BP, BG^\wedge_p)_{B\rho}$ is p-complete.

Let $\Lambda(BG^\wedge_p)$ denote the free loop space, and let $e\colon \Lambda(BG^\wedge_p) \to BG^\wedge_p$ be evaluation at the basepoint of S^1. Let P act on $G^\wedge_p$ by conjugation induced via ρ. Then Lemma 7.4 applies to show that there is a homotopy equivalence

$$(G^\wedge_p)^{hP} \xrightarrow{\simeq} \mathrm{fiber}_{B\rho}(\mathrm{Map}(BP, \Lambda(BG^\wedge_p)) \xrightarrow{e\circ -} \mathrm{Map}(BP, BG^\wedge_p))\,.$$

Notice also that $(G^\wedge_p)^{hP} \xrightarrow{\simeq} \Omega\,\mathrm{Map}(BP, BG^\wedge_p)_{B\rho}$. By the generalized Sullivan conjecture (c.f. [2], [3], or [17]), $(G^\wedge_p)^{hP} \simeq (G^P)^\wedge_p$. In particular $(G^\wedge_p)^{hP}$ is p-complete, hence so is $\Omega\,\mathrm{Map}(BP, BG^\wedge_p)_{B\rho}$.

Case 2: Now let G be arbitrary, and fix an embedding $G \le U(n-1) \le SU(n)$ as in Lemma 7.3. Fix $\rho \in \mathrm{Hom}(P, SU(n))$, and let $\rho_1, \ldots, \rho_k$ be G-conjugacy class representatives of homomorphisms $P \to G$ which are $SU(n)$-conjugate to ρ. Consider the following diagram:

$$\begin{array}{ccccc}
\coprod_{i=1}^k C_{SU(n)}(\rho_i)/C_G(\rho_i) & \longrightarrow & \coprod_{i=1}^k BC_G(\rho_i) & \longrightarrow & BC_{SU(n)}(\rho) \\
\downarrow & & \downarrow & & \downarrow \\
(SU(n)/G)^{hP} & \longrightarrow & \coprod_{i=1}^k \mathrm{Map}(BP, B'G)_{B\rho_i} & \longrightarrow & \mathrm{Map}(BP, BSU(n))_{B\rho}\,,
\end{array}$$

Here, $B'G = ESU(n)/G \simeq BG$, and the action of P on $SU(n)/G$ is that induced by ρ. Each row is a fibration sequence (Lemma 7.4), and the fibre in the top row describes the $C_{SU(n)}(\rho)$ orbit decomposition of $(SU(n)/G)^P$. Furthermore, the maps between the total spaces and base spaces are homotopy equivalences after p-completion by [5] (c.f. Proposition 7.1). Furthermore, both fibrations remain fibrations after p-completion because the base spaces satisfy $\pi_1\,\mathrm{Map}(BP, BSU(n))_{B\rho} \cong \pi_1 BC_{SU(n)}(P) = 0$ by [5] (c.f. Proposition 7.1) and Lemma 7.3, and then [1, II.5.2(i)] applies.

It then follows that the map between fibres also induces a homotopy equivalence

$$((SU(n)/G)^P)^\wedge_p \xrightarrow{\simeq} ((SU(n)/G)^{hP})^\wedge_p$$

and this combines with the generalized Sullivan conjecture to give a commutative triangle of homotopy equivalences

$$\begin{array}{ccc} & \big((SU(n)/G)^P\big)^\wedge_p & \\ \swarrow \simeq & & \simeq \searrow \\ \big((SU(n)/G)^{hP}\big)^\wedge_p & \xrightarrow{\ \simeq\ } & \big((SU(n)/G)^\wedge_p\big)^{hP} . \end{array}$$

Now set $\mathrm{Map}(BP, BG)_{\overline{B\rho}} = \amalg_{i=1}^{k} \mathrm{Map}(BP, BG)_{B\rho_i}$, and consider the following diagram:

$$\begin{array}{ccccc} \big((SU(n)/G)^{hP}\big)^\wedge_p & \longrightarrow & \big(\mathrm{Map}(BP, BG)_{\overline{B\rho}}\big)^\wedge_p & \longrightarrow & \big(\mathrm{Map}(BP, BSU(n))_{B\rho}\big)^\wedge_p \\ \downarrow & & \downarrow & & \downarrow \\ ((SU(n)/G)^\wedge_p)^{hP} & \longrightarrow & \mathrm{Map}(BP, BG^\wedge_p)_{\overline{B\rho}} & \longrightarrow & \mathrm{Map}(BP, BSU(n)^\wedge_p)_{B\rho} . \end{array}$$

The top row is a fibration sequence as we have argued above. The bottom row is a fibration sequence by Lemma 7.4, since

$$(SU(n)/G)^\wedge_p \longrightarrow BG^\wedge_p \longrightarrow BSU(n)^\wedge_p$$

is a fibration sequence by [1, II.5.2(i)] again. We have shown that the map between fibres is a homotopy equivalence. The map between base spaces is a homotopy equivalence by Case 1; and hence the map between total spaces is a homotopy equivalence. Upon taking the union over all $\rho \in \mathrm{Rep}(P, SU(n))$ that factors through $G \leq SU(n)$, this shows that $\mathrm{Map}(BP, BG^\wedge_p) \simeq \mathrm{Map}(BP, BG)^\wedge_p$. □

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