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Real Structures and Morava K-Theories

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Abstract. We show that real k-structures coincide for $k = 1, 2$ on all formal groups for which multiplication by 2 is an epimorphism. This enables us to give explicit polynomial generators for the Morava $K(n)$-homology of *BSpin* and *BSO* for $n = 1, 2$.

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1. Introduction and Statement of Results

Let F be a formal group law over a ring R and let S be a R-algebra. A power series f over S in k variables is called a k-structure on F if it satisfies the relations:

(i) $f(0, x_2, x_3, \ldots, x_k) = 1$,
(ii) $f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = f(x_1, x_2, \ldots, x_k)$ for all $\sigma \in \Sigma_k$,
(iii) $f(x_1, \ldots, x_k) f(x_0, x_1 +_F x_2, x_3, \ldots, x_k)$
$= f(x_0 +_F x_1, x_2, \ldots, x_k) f(x_0, x_1, x_3, \ldots, x_k)$.

Important examples for k-structures come from topology as follows: for any complex oriented cohomology theory we are given an isomorphism

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \cong \pi^* E[\![x_1, x_2, \ldots, x_k]\!].$$

Here, the class x_i for $i = 1 \ldots k$ is the first Chern class of the complex line bundle L_i which sits over the ith factor as the tautological bundle. An elementary calculation using the algebraic independence of the elementary symmetric polynomials shows that the Chern classes c_j $(0 < j < k)$ for the virtual bundle $(L_1 - 1)(L_2 - 1) \cdots (L_k - 1)$ are trivial. Hence, this stable bundle is classified by a map to the $2k$th connected cover $BU\langle 2k\rangle$ of BU. The induced stable map

$$\begin{aligned} f\colon \mathbb{C}P^\infty_+ \wedge \ldots \wedge \mathbb{C}P^\infty_+ &\longrightarrow BU\langle 2k\rangle_+ \cong BU\langle 2k\rangle_+ \wedge S^0 \\ &\longrightarrow BU\langle 2k\rangle_+ \wedge E \end{aligned}$$

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gives a power series over $S = E_*BU\langle 2k\rangle$, which satisfies the relations above with F being the formal group law associated to the oriented cohomology theory E (cf. [AHS99].)

Ando, Hopkins and Strickland showed that for $k \leqslant 3$ all k-structures are represented in topology this way: let F be the formal group law associated to E over $R = \pi_*E$. Then given any such k-structure g on F over a R-algebra S there is a unique algebra homomorphism

$$\varphi\colon E_*BU\langle 2k\rangle \longrightarrow S$$

which satisfies $\varphi f = g$. Explicitly, let C_kF be the R-algebra freely generated by symbols $a_{i_1,\dots i_k}$ for $i_j \geqslant 0$ subject to the relations (i)–(iii) above for the power series

$$f(x_1, \dots, x_k) = \sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k} x_1^{i_1} \cdots x_k^{i_k}.$$

Then this generating function f is the universal k-structure on the group law F. The results of Hopkins, Ando and Strickland show that the map

$$\alpha\colon C_kF \longrightarrow E_*BU\langle 2k\rangle$$

that classifies the k-structure associated to $E_*BU\langle 2k\rangle$ is an isomorphism. Moreover, the canonical map $\iota\colon BU\langle 2k\rangle \to BU\langle 2k-2\rangle$ is given in E-homology in terms of the generating functions by

$$\iota_* f(x_1, x_2, \dots, x_k) = \frac{f(x_1 +_F x_2, x_3, \dots x_k)}{f(x_2, \dots, x_k) f(x_1, x_3, \dots, x_k)}. \tag{1}$$

The classifying spaces of the unitary groups have real counter parts which for $k \leqslant 2$ fit into a diagram

$$\begin{array}{ccc} BU\langle 4\rangle = BSU & \longrightarrow & BSpin = BO\langle 4\rangle \\ \downarrow & & \downarrow \\ BU\langle 2\rangle = BU & \longrightarrow & BSO = BO\langle 2\rangle. \end{array}$$

The involution τ induced by complex conjugation is reflected by the equalities

$$\tau f(x) = f(-_F x) \text{ for 1-structures,} \tag{2}$$

$$\tau f(x, y) = f(-_F x, -_F y) \text{ for 2-structures.} \tag{3}$$

For $k = 1, 2$ we let $C_k^r F$ be the universal ring of real k-structures on the formal group law F. That is, the R-algebra $C_k^r F$ is the quotient of C_kF subject to the real relations given by demanding that the universal k-structure be invariant under τ. It was shown in [AHS99] that the canonical map

$$\alpha_r\colon C_k^r F \longrightarrow E_*BO\langle 2k\rangle$$

is an isomorphism for Morava's theory $K(1)$ if $k=1$ and for $K(1)$ and $K(2)$ if $k=2$. The prime under consideration is understood to be 2 throughout this paper unless stated otherwise.

THEOREM 1.1. *Let F be the Honda formal group law of height n over $\mathbb{F}_2[v_n^{\pm}]$ for some n. Then the canonical map*

$$\iota_*: C_2^r F \longrightarrow C_1^r F$$

is an isomorphism.

We proceed to show that for the Honda formal group law the ring $C_1^r F$ is a polynomial algebra in generators $a_2, a_4, \ldots$. These results enable us to show the following theorem:

THEOREM 1.2. *Let $b_i \in K(n)_{2i} BSpin$ be the image of the class $c_1^{i\,*}$ under the map induced by the inclusion of the maximal torus*

$$BS^1 \longrightarrow BSpin(3) \longrightarrow BSpin.$$

(i) *For $n = 1, 2$ we have $K(n)_* BSpin \cong \pi_* K(n)\,[b_{2^n\cdot 2}, b_{2^n\cdot 4}, b_{2^n\cdot 6}, \ldots]$,*
(ii) *$K(1)_* BSpin \cong K(1)_* BSO$,*
(iii) *The following is an exact sequence of algebras*

$$1 \longrightarrow K(2)_* BSpin \longrightarrow K(2)_* BSO \xrightarrow{w_2} K(2)_* K(\mathbb{Z}/2, 2) \longrightarrow 1.$$

Moreover, the algebra $K(2)_* K(\mathbb{Z}/2, 2)$ is generated by one generator b in degree 6 which satisfies the relation $b^2 = v_2 b$.

Remark 1.3. It is known that $K(2)_* BSO$ is a polynomial algebra. Consequently, the above short exact sequence does not split.

Theorem 1.1 allows a generalization to any formal group G for which multiplication by 2 is an epimorphism. (A convenient source for the language of formal schemes and formal groups is provided by [NS].)

THEOREM 1.4. *If $2: G \longrightarrow G$ is an epimorphism, then the map*

$$\iota: \operatorname{spec} C_1^r G \longrightarrow \operatorname{spec} C_2^r G$$

is an isomorphism. In particular this applies to any formal group of finite height over a field of any positive characteristic.

Although Theorem 1.1 is an obvious consequence of 1.4, we postpone the proof of 1.4 to the very end. Instead, we start with the proof of 1.1 since it only uses the language of power series rather than schemes and helps in the understanding of the proof of the general result.

2. Real Structures on the Honda Formal Group Law

In this section we will prove Theorem 1.1. The Honda formal group law F of height n describes the tensor product of complex line bundles in $K(n)$-theory. It is characterized by the fact that it is 2-typical and its 2-series satisfies

$$[2](x) = x +_F x = x^{2^n}.$$

It is a well-known difficulty that for $n \geqslant 2$ the formula of its formal sum becomes very hard to deal with. Hence, we cannot prove Theorem 1.1 by staring at coefficients but need a general argument which only uses 2-typicality and the [2]-series.

We will construct an explicit inverse to the canonical map

$$\iota_* \colon C_2^r = C_2^r F \longrightarrow C_1^r = C_1^r F.$$

For that first observe the following lemma:

LEMMA 2.1. *The power series* $S(x) = f(x, x) f(x, -_F x)^{-1}$ *with coefficients in* C_2^r *satisfies the relation*

$$f(x^{2^n}, y^{2^n}) = \frac{S(x +_F y)}{S(x)S(y)}.$$

Proof. The cocylce relation (iii) gives

$$\begin{aligned}\frac{S(x +_F y)}{S(x)S(y)} &= \frac{f(x +_F y, x +_F y) f(x, -_F x) f(y, -_F y) f(-_F x, -_F y)}{f(-_F x, x +_F y) f(y, -_F y) f(x, x) f(y, y)} \\ &= \frac{f(x +_F y, x +_F y) f(x, y) f(-_F x, -_F y)}{f(x, x) f(y, y)}\end{aligned}$$

and

$$\begin{aligned}f(x^{2^n}, y^{2^n}) &= \frac{f(x^{2^n} +_F y, y) f(x^{2^n}, y)}{f(y, y)} \\ &= \frac{f(x, y) f(x +_F y, x +_F y) f(x, y) f(x, x +_F y)}{f(y, y) f(x +_F y, x) f(x, x)} \\ &= \frac{f(x, y) f(x +_F y, x +_F y) f(x, y)}{f(y, y) f(x, x)}.\end{aligned}$$

Hence, the claim follows from the real relation $f(x, y) = f(-_F x, -_F y)$. □

LEMMA 2.2. *For any* E *the map*

$$(\iota + f)_* \colon E_* BSU \otimes_{\pi_* E} E_* \mathbb{C}P^\infty \longrightarrow E_* BSU \times \mathbb{C}P^\infty \longrightarrow E_* BU$$

is an isomorphism of $E_* BSU$*-modules.*

Proof. The diagram

$$\begin{array}{ccccc} BSU \times BSU \times \mathbb{C}P^\infty & \xrightarrow{\iota\times\iota\times f} & BU \times BU \times BU & \xrightarrow{1\times\mu} & BU \times BU \\ \downarrow{\scriptstyle \mu\times 1} & & \downarrow{\scriptstyle \mu\times 1} & & \downarrow{\scriptstyle \mu} \\ BSU \times \mathbb{C}P^\infty & \xrightarrow{\iota\times f} & BU \times BU & \xrightarrow{\mu} & BU \end{array}$$

commutes. □

There is a ring in between C_2^r and C_1^r which will be useful in the sequel: let $T(x)$ be the power series $f(x)f(-_F x)^{-1}$ and let $C_1^{r'}$ be the quotient ring of C_1 subject to the relation generated by the set I' consisting of the nonconstant coefficients of

$$T(x +_F y)/T(x)T(y).$$

Then we have:

LEMMA 2.3. (i) *The canonical map* ι': $C_2^r \longrightarrow C_1^{r'}$ *is an injection.*
(ii) T *seen as a power series with coefficients in* $C_1^{r'}$ *is a power series in* x^{2^n}.

Proof. Let I be the subset of C_2 consisting of the nonconstant coefficients of $f(x, y)f(-_F x, -_F y)^{-1}$. For (i) first observe that we have

$$\iota\frac{f(x, y)}{f(-_F x, -_F y)} = \frac{f(x +_F y)f(x)f(y)}{f(-_F x -_F y)f(-_F x)f(-_F y)} = \frac{T(x +_F y)}{T(x)T(y)}.$$

It follows that I' is the image of I and that the map ι' is well defined. Hence, it suffices to check that the ideal IC_2 generated by $f(x, y)f(-_F x, -_F y)^{-1}$ is the intersection of the ideal $I'C_1$ with C_2. By 2.2 there exists a retraction homomorphism

$$\rho\colon C_1 \cong K(n)_*BU \longrightarrow K(n)_*BSU \cong C_2$$

of C_2-modules. Hence, any $a = \sum_k i_k s_k \in C_2$ with $s_k \in C_1$ and $i_k \in I'$ satisfies

$$a = \rho(a) = \sum_k i_k\,\rho(s_k) \in IC_2$$

and the first part of the lemma follows.

For the second part, let $T(x) = f(x)f(-_F x)^{-1}$ be considered as a power series with coefficients in $C_1^{r'}$. Since $-_F x = x + \mathrm{O}(x^{2^n})$, it follows that $T(x) = 1 + \mathrm{O}(x^{2^n})$. By extending coefficients, we may assume that $\pi_*(K(n))$ contains all roots of unity. Let $p \neq 2$ be any prime. Let ζ be a primitive pth root of unity. On repeatedly applying the equality $T(x +_F y) = T(x)T(y)$ and using 2-typicality, we get the equation

$$1 = T(0) = T\left(\sum_0^{p-1} \zeta^i x\right) = \prod_0^{p-1} T(\zeta^i x).$$

On taking logarithmic derivatives, we get the following identity:

$$0 = \sum_{0}^{p-1} \zeta^i x T'(\zeta^i x) T(\zeta^i x)^{-1}.$$

Since p is any odd prime, it follows that the power series $xT'(x)T(x)^{-1}$ may be expressed as

$$xT'(x)T(x)^{-1} = \sum_{i \geqslant 1} c_i x^{2^i}.$$

An easy induction argument now shows that $T(x)$ is an even power series. Let $T(x) = T_1(x^2)$, where $T_1(x)$ is a power series with coefficients in $C_1^{r'}$. Notice that $T_1(x) = 1 + \mathrm{O}(x^{2^{n-1}})$.

Let F_1 denote the Frobenius of the formal group law F. We have the equalities

$$\frac{T_1(x^2 +_{F_1} y^2)}{T_1(x^2)T_1(y^2)} = \frac{T_1((x +_F y)^2)}{T_1(x^2)T_1(y^2)} = \frac{T(x +_F y)}{T(x)T(y)} = 1.$$

Hence, we also have the equality

$$T_1(x +_{F_1} y) = T_1(x)T_1(y).$$

Since F_1 is also 2-typical, we may repeat the above procedure n times to establish the existence of a power series $T_n(x)$ such that $T(x) = T_n(x^{2^n})$. □

Proof of 1.1. First we claim that the power series $S(x)$ of Lemma 2.1 is a power series in x^{2^n}. Using the injection ι' of Lemma 2.3 we get

$$\iota' S(x) = \iota' \frac{f(x, x)}{f(x, -_F x)} = \frac{f(x^{2^n})}{T(x)}.$$

Hence, the claim follows from 2.3. Next define the ring homomorphism κ from C_1^r to C_2^r by demanding

$$\kappa f(x^{2^n}) = S(x).$$

Then we see with 2.1

$$\kappa \iota f(x^{2^n}, y^{2^n}) = \frac{\kappa f(x^{2^n} +_F y^{2^n})}{\kappa(f(x^{2^n}))\kappa(f(y^{2^n}))} = \frac{S(x +_F y)}{S(x)S(y)} = f(x^{2^n}, y^{2^n}).$$

Thus the universal property of C_2^r shows that we have constructed a left inverse to the map ι. Similarly, we see with

$$\iota \kappa f(x^{2^n}) = \iota S(x) = f(x^{2^n})$$

that ι is indeed an isomorphism. □

Now recall that by definition the algebra C_1^r is generated by the coefficients of the universal 1-structure

$$f(x) = \sum_{i \geqslant 0} a_i x^i$$

which satisfies the real relation $f(x) = f(-_F x)$.

LEMMA 2.4. *We have*

$$-_F x = x + x^{2^n} + \mathrm{O}(x^{2^n+1}).$$

Proof. This is easily verified by writing the right-hand side as the formal sum $x +_F x^{2^n}$ up to terms of higher order. □

PROPOSITION 2.5. *The algebra C_1^r is already generated by $a_2, a_4, a_6, \ldots$.*

Proof. Decompose $-_F x$ as a sum of power series $-_F x = g_-(x) + g_+(x)$, where $g_-(x)$ is an odd power series and $g_+(x)$ is an even power series with the property that

$$g_+(x) = x^{2^n} \epsilon(x)$$

for some unit power series $\epsilon(x)$. Now consider $f(x)$ as a power series with coefficients in C_1^r. We have the equality $f(-_F x) = f(x)$. On comparing the even powers on both sides of this equality, we get

$$\sum_i a_{2i}(-_F x)^{2i} + \sum_i a_{2i+1}(-_F x)^{2i} g_+(x) = \sum_i a_{2i} x^{2i}.$$

This equality gives us the identity

$$g_+(x) \sum_i a_{2i+1}(-_F x)^{2i} = \sum_i a_{2i}(x^{2i} + (-_F x)^{2i}).$$

Let $z(x)$ be the right-hand side. Then the last equality tells us that there is a power seires $w(x)$ such that

$$z(x) = x^{2^n} w(x).$$

Thus we have

$$\sum_i a_{2i+1}(-_F x)^{2i} = \epsilon(x)^{-1} w(x).$$

Notice that the right-hand side in this identity is a well-defined power series with coefficients that can be written only in terms of a_{2i}. Replacing x by $-_F x$ in the above identity shows that each odd generator a_{2j+1} can be written in terms of the even generators a_{2i}. This proves the proposition. □

LEMMA 2.6. *Let $\tau : BS^1 \longrightarrow BS^1$ be the map which classifies the complex conjugate $\bar{L}$ and let $a_i \in K(n)_{2i}BS^1$ be the dual of x^i. Then there are λ_{ij} such that for all j $\tau_* a_j = a_j + \sum_{i=1}^{j-1} \lambda_{ij} a_i$.*

Proof. Compute

$$\langle \tau_* a_j, x^i \rangle = \langle a_j, (-_F x)^i \rangle = \langle a_j, (x + \mathrm{O}(x^{2^n}))^i \rangle . \qquad \square$$

Now the first part of the Theorem 1.2 will follow from the following proposition:

PROPOSITION 2.7. *The classes $a_2, a_4, a_6, \ldots$ are algebraically independent in the algebra C_1^r.*

Proof. The strategy will be similar to the proof of 8.5 in [Sna75] where the special case $n = 1$ was shown. It is enough to check the independence of the images of the generators under the algebra homomorphism

$$C_1^r \xrightarrow{\alpha_r} K(n)_* BSO \xrightarrow{c} K(n)_* BU \cong C_1.$$

When composed with the projection map $C_1 \to C_1^r$ it coincides with the induced map in $K(n)$-homology of

$$BU \xrightarrow{\Delta} BU \times BU \xrightarrow{1\times\tau} BU \times BU \xrightarrow{\mu} BU.$$

Hence, we see with the lemma that the generator a_{2i} is mapped to the sum $\sum_{p+q=2i} a_p \tau_*(a_q) = \sum_{p+q=2i} a_p a_q$ + terms of lower degree $= a_i^2$ + terms of lower degree and the claim follows. $\square$

3. The Homology Rings of *BSpin* and *BSO*

We first consider the homology ring of $BSpin$ for $K(n)$-theory with $n \leqslant 2$. In view of the result of Hopkins, Ando and Strickland and Theorem 1.1 we have isomorphisms

$$K(2)_* BSpin \cong C_2^r \cong C_1^r \cong \pi_* K(n)[a_2, a_4, \ldots].$$

To prove Theorem 1.2: consider the inclusion of tori

$$\begin{array}{ccccc}
S^1 & \xrightarrow{z^2} & S^1 & \searrow & \\
\downarrow & & \downarrow & & \\
Spin(3) & \longrightarrow & SO(3) & \longleftarrow & SO(2) = U(1) \\
\downarrow & & \downarrow & & \\
Spin & \longrightarrow & SO. & &
\end{array}$$

In $K(n)$-homology a generator $b_{2^n \cdot m} \in K(n)_* BS^1$ is sent to $v_n^m a_m$ as one easily checks. Since $v_n \in \pi_* K(n)$ is a unit, the structure of $K(n)_* BSpin$ follows.

For the structure of $K(2)_*BSO$, we will analyze the Rothenberg Steenrod spectral sequence for the fibration

$$K(\mathbb{Z}/2, 1) \longrightarrow BSpin \longrightarrow BSO. \tag{4}$$

As a prelude to this calculation, we consider the fibration

$$K(\mathbb{Z}/2, 1) \longrightarrow * \longrightarrow K(\mathbb{Z}/2, 2). \tag{5}$$

In the Rothenberg–Steenrod spectral sequence for the fibration (5), we have

$$E_2 = \mathrm{Tor}_{K(2)_*K(\mathbb{Z}/2,1)}(K(2)_*, K(2)_*) \Rightarrow K(2)_*K(\mathbb{Z}/2, 2).$$

Its calculation is due to Ravenel and Wilson [RW80] who cautiously assume that $p > 2$. However, as explained by Johnson and Wilson in [JW85] their arguments are valid at 2 as well. For the reader's convenience we go through the calculation below and only recall three well-known facts:

(i) The spectral sequence is a spectral sequence of Hopf-algebras,
(ii) $K(2)_*K(\mathbb{Z}/2, 2)$ is a two-dimensional vector space over $K(2)_*$,
(iii) $K(2)_*K(\mathbb{Z}/2, 1) = K(2)_*[b_2]/(b_2^4)$, where b_2 is the class that maps to the corresponding element in $K(2)_4BS^1$.

From these facts we get a description of the E_2 term of the spectral sequence as a Hopf algebra

$$E_2 = \Lambda(x) \otimes \Gamma[y],$$

where the bidegrees of the classes are given by $|x| = (1, 4)$, and $|y| = (2, 16)$. For $n \geqslant 0$, let y_n be the generator of $\Gamma[y]$ in bidegree $(2^{n+1}, 2^{n+4})$, so that $y = y_0$. We start with the following claim:

CLAIM 3.1. *In the above spectral sequence, $d_r = 0$ for all $r \neq 3$. Moreover, for $n \geqslant 1$, the differential d_3 is given by the formula*

$$d_3(y_n) = y_{n-1} \cdot y_{n-2} \dots y_1 \cdot x \cdot v_2^5.$$

Proof. Let d_k be the first nontrivial differential in the spectral sequence. Let m be the smallest integer so that $d_k(y_m) \neq 0$. Notice that $d_k(y_m)$ must be a primitive element in odd homogeneous degree. The only element with this property has the form $x \cdot v_2^s$ for some integer s. It follows that $k = 2^{m+1} - 1$ and that $s = 3 \cdot 2^m - 1$. From the value of k we notice that y_n is a permanent cycle for $n < m$. From the coalgebra structure of this spectral sequence, we deduce that $d_k(y_n) \neq 0$ for $n \geqslant m$. Since we know that the dimension of the E_∞ term over $K(2)_*$ is 2, we are forced to have $m = 1$. Finally, the explicit formula for d_3 given above is forced by trivial dimensional reasons. □

We now proceed to analyze the Rothenberg–Steenrod spectral sequence for the fibration (4). Notice that we have a diagram of fibrations

$$\begin{array}{ccccc} K(\mathbb{Z}/2,1) & \xrightarrow{i} & BSpin & \longrightarrow & BSO \\ \downarrow = & & \downarrow & & \downarrow w_2 \\ K(\mathbb{Z}/2,1) & \longrightarrow & * & \longrightarrow & K(\mathbb{Z}/2,2). \end{array} \tag{6}$$

As before, the Rothenberg–Steenrod spectral sequence for the fibration (4) is a spectral sequence of Hopf algebras. It is clear from our calculations that the inclusion of the fiber $i: K(\mathbb{Z}/2, 1) \to BSpin$ is trivial in $K(2)$-homology. Hence, we get an expression of the E_2 term as a Hopf algebra

$$E_2 = \Lambda(x) \otimes \Gamma[y] \otimes K(2)_*BSpin,$$

where the classes x and y are as before and $K(2)_*BSpin$ is in external degree zero. We have the following.

CLAIM 3.2. *In the Rothenberg–Steenrod spectral sequence for the fibration (4) we have $d_r = 0$ for all $r \neq 3$. Moreover, for $n \geqslant 1$, the differential d_3 is given by the formula*

$$d_3(y_n) = y_{n-1} \cdot y_{n-2} \dots y_1 \cdot x \cdot v_2^5.$$

Proof. The fact that $d_2 = 0$ follows for dimensional reasons. That $d_3(y_n) \neq 0$ follows by Claim 3.1 on comparing this spectral sequence with the spectral sequence for the fibration (5) using the diagram (6). The proof will be complete once we verify the formula for $d_3(y_n)$ as given above. For this we proceed by induction on n. As before, $d_3(y_1)$ must be primitive in external degree one. Hence, we get $d_3(y_1) = x \cdot v_2^5$. Here we are using the fact that $P(H_1 \otimes H_2) = P(H_1) \oplus P(H_2)$, where $P(H)$ denotes the primitive elements in the Hopf algebra H. Now assume we know that

$$d_3(y_n) = y_{n-1} \cdot y_{n-2} \dots y_1 \cdot x \cdot v_2^5.$$

Consider the general expression for $d_3(y_{n+1})$

$$d_3(y_{n+1}) = y_n \cdot y_{n-1} \dots y_1 \cdot x \cdot p,$$

where $p \in K(2)_*BSpin$ is some nonzero element that can be written as $p = v_2^5 + q$ for some $q \in \tilde{K}(2)_*BSpin$. Let τ denote the element $y_n \cdot y_{n-1} \dots y_1 \cdot x \cdot q$. Using the coalgebra structure and the induction hypothesis, we have the equality

$$d_3(\Delta y_{n+1}) = 1 \otimes \tau + \tau \otimes 1 + \Delta(y_n \cdot y_{n-1} \dots y_1 \cdot x \cdot v_2^5).$$

On the other hand,

$$d_3(\Delta y_{n+1}) = \Delta d_3(y_{n+1}) = \Delta(y_n \cdot y_{n-1} \dots y_1 \cdot x \cdot v_2^5) + \Delta\tau.$$

On comparing the two expressions, we notice that the element τ is primitive. Using the structure of primitives in a tensor product of Hopf algebras, we deduce that this can happen only if $q = 0$. This completes the induction step and we are done. □

Proof of 1.2 (iii). Using Claims 3.1 and 3.2 one notices that the Rothenberg–Steenrod spectral sequence for fibration (4) is a free $K(2)_*BSpin$ module on the spectral sequence for the fibration (5). This structure is induced by the diagram (6). It follows that $K(2)_*BSO$ is a free $K(2)_*BSpin$ module on $K(2)_*K(\mathbb{Z}/2, 2)$.

We finish the proof of 1.2 with the following fact which is the case $n = 2$ of Theorem 9.2(a) of [RW80]. This theorem was later used by Ando and Strickland in [AS98] to establish a relationship to Weil pairings. Using their point of view we give a short proof below. □

PROPOSITION 3.3. *Let* $b \in K(2)_6K(\mathbb{Z}/2, 2)$ *be the image of* $b_1 \otimes b_2$ *under the multiplication map*

$$\mu : K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 1) \longrightarrow K(\mathbb{Z}/2, 2).$$

Then we have

$$K(2)_*K(\mathbb{Z}/2, 2) \cong K(2)_*[b]/(b^2 - v_2 b).$$

Proof. The composite

$$f : K(\mathbb{Z}/2, 1)_+ \wedge K(\mathbb{Z}/2, 1)_+ \xrightarrow{\mu_+} K(\mathbb{Z}/2, 2)_+ \longrightarrow K(2) \wedge K(\mathbb{Z}/2, 2)_+$$

gives a polynomial

$$f(x, y) = 1 + \sum_{1 \leqslant i, j \leqslant 3} b_{ij} x^i y^j \in K(2)_*K(\mathbb{Z}/2, 2)[x, y]/(x^4, y^4)$$

which satisfies

(i) $f(x +_F y, z) = f(x, z) f(y, z)$,
(ii) $f(z, x +_F y) = f(z, x) f(z, y)$,
(iii) $f(x, x) = 1$.

This means that f defines a Weil pairing in the sense of Ando and Strickland [AS98]. Moreover, since f is concentrated in dimensions $6n$ the only non-trivial coefficients can be $b_{1,2}$ and $b_{3,3}$. One easily verifies that the relations are equivalent to

$$v_2 b_{1,2} = b_{1,2}^2 = b_{3,3}.$$

Finally, any of these must generate since f is universal among all Weil pairings ([AS98], [RW80]) and $K(2)_*K(\mathbb{Z}/2, 2)$ is 2-dimensional. □

4. The Proof of 1.4

Before proving the general result we should explain the meaning of the theorem in terms of the geometry of 1 and 2-structures. A 1-structure on G is just a pointed function f from G to the multiplicative formal group $\mathbb{G}_m$; while a 2-structure g defines a commutative central extension $\mathbb{G}_m \longrightarrow E \longrightarrow G$. Here E is the formal scheme $G \times \mathbb{G}_m$ and the group structure is given by the formula

$$(a, \lambda) \cdot (b, \mu) = (a + b, g(a, b)\lambda\mu).$$

The kernel of the map $\iota : \text{spec } C_1G \longrightarrow \text{spec } C_2G$ defined by (1) is $\hom(G, \mathbb{G}_m)$. In fact the sequence

$$\hom(G, \mathbb{G}_m) \longrightarrow \text{spec } C_1G \xrightarrow{\iota} \text{spec } C_2G$$

of group schemes is short exact, and there is a nonadditive splitting

$$\text{spec } C_1G \cong \hom(G, \mathbb{G}_m) \times \text{spec } C_2G.$$

(If G is the formal group associated to a complex orientable cohomology theory E, then the short exact sequence of group schemes is isomorphic to the sequence

$$\text{spec } E_*\mathbb{C}P^\infty \longrightarrow \text{spec } E_*BU \longrightarrow \text{spec } E_*BSU,$$

and the splitting reflects the homotopy equivalence $BU \cong \mathbb{C}P^\infty \times BSU$.)

For a 2-structure g the real condition $g(-a, -b) = g(a, b)$ has the following meaning in terms of central extensions: the homomorphism

$$\tau : G \longrightarrow G;\ a \mapsto -a$$

gives a central extension τ^*E. There is a natural isomorphism

$$E \longrightarrow \tau^*E;\ (a, \lambda) \mapsto (a, \lambda)$$

of formal schemes over G, and g is a real 2-structure if it is a homomorphism of groups.

Now consider the restriction of ι to spec C_1^rG. Its kernel consists of homomorphisms f from G to $\mathbb{G}_m$ such that $f(a) = f(-a)$ is satisfied. This implies that $1 = f(a)^2 = f(2a)$. If $2 : G \longrightarrow G$ is an epimorphism, then this implies that $f \equiv 1$, and so

$$\iota\colon \text{spec } C_1^rG \longrightarrow \text{spec } C_2^rG$$

is injective. Hence, it remains to show that the map ι is surjective. For that we construct a splitting in the same way as we did for the proof of 1.1. Given a real 2-structure g on G, let $sg\colon G \longrightarrow \mathbb{G}_m$ be given by the formula

$$sg(a) = \frac{g(a, a)}{g(a, -a)}.$$

LEMMA 4.1. *If* $2a = \pm 2b$*, then* $sg(a) = sg(b)$.

Proof. We treat the case $2a = 2b$; the other case is similar. Suppose that $a = b + u$ with $2u = 0$. The equation

$$((b, 1)(u, 1))((b, 1)(u, 1)) = ((b, 1)(b, 1))((u, 1)(u, 1))$$

in the central extension E implies that

$$g(b+u, b+u)g(b, u)^2 = g(2b, 2u)g(b, b)g(u, u) = g(b, b)g(u, u). \tag{7}$$

The equation

$$((b, 1)(u, 1))((-b, 1)(-u, 1)) = ((b, 1)(-b, 1))((u, 1)(-u, 1))$$

implies that

$$g(b+u, -b-u)g(b, u)g(-b, -u) = g(b, -b)g(u, -u). \tag{8}$$

If $g \in C_2^r$ then $g(-b, -u) = g(b, u)$. If $2u = 0$ then $g(u, -u) = g(u, u)$. Then (8) becomes

$$g(b+u, -b-u)g(b, u)^2 = g(b, -b)g(u, u), \tag{9}$$

and dividing (7) by (9) yields

$$sg(a) = \frac{g(b+u, b+u)}{g(b+u, -b-u)} = \frac{g(b, b)}{g(b, -b)} = sg(b),$$

as desired. □

If $2\colon G \longrightarrow G$ is an epimorphism, then by the lemma we may define a function $tg\colon G \longrightarrow \mathbb{G}_m$ by the formula $tg(a) = sg(c)$, where c is chosen so that $2c = a$. Moreover, the lemma implies that tg is a real 1-structure.

LEMMA 4.2. *Suppose that* $2\colon G \longrightarrow G$ *is an epimorphism, so that* tg *is defined. Then* $\iota tg = g$.

Proof. This is essentially as in Lemma 2.1 and hence left to the reader. □

We conclude that $t\colon \operatorname{spec} C_2^r \longrightarrow \operatorname{spec} C_1^r$ is the desired splitting.

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