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Universal moduli spaces of surfaces with flat bundles and cobordism theory

Ralph L. Cohen [a,*,1], Søren Galatius [a,2], Nitu Kitchloo [b,3]

[a] Dept. of Mathematics, Stanford University, Stanford, CA 94305, United States
[b] Dept. of Mathematics, UC San Diego, La Jolla, CA 92093, United States

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Abstract

For a compact, connected Lie group G, we study the moduli of pairs (Σ, E), where Σ is a genus g Riemann surface and $E \to \Sigma$ is a flat G-bundle. Varying both the Riemann surface Σ and the flat bundle leads to a moduli space $\mathcal{M}_g^G$, parametrizing families Riemann surfaces with flat G-bundles. We show that there is a stable range in which the homology of $\mathcal{M}_g^G$ is independent of g. The stable range depends on the genus of the surface. We then identify the homology of this moduli space in the stable range, in terms of the homology of an explicit infinite loop space. Rationally, the stable cohomology of this moduli space is generated by the Mumford–Morita–Miller κ-classes, and the ring of characteristic classes of principal G-bundles, $H^*(BG)$. Equivalently, our theorem calculates the homology of the moduli space of semi-stable holomorphic bundles on Riemann surfaces.

We then identify the homotopy type of the category of one-manifolds and surface cobordisms, each equipped with a flat G-bundle. Our methods combine the classical techniques of Atiyah and Bott, with the new techniques coming out of Madsen and Weiss's proof of Mumford's conjecture on the stable cohomology of the moduli space of Riemann surfaces.

Keywords: Universal moduli spaces; Surfaces; Connections; Cobordism theory

* Corresponding author.
E-mail address: ralph@math.stanford.edu (R.L. Cohen).
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Contents

1. Introduction and statement of results

Let G be a connected compact Lie group. Our goal is to study a certain moduli space of flat G-bundles on Riemann surfaces. In the first part of our paper, we calculate the homology of this moduli space in a "stable range." In the second part we construct such moduli spaces in the case where the Riemann surface has boundary, and study a cobordism category built out of such moduli spaces.

The study of moduli spaces of flat bundles on Riemann surfaces, and its connection with semistable holomorphic bundles, goes back to the seminal work of Atiyah–Bott [1]. Fix a smooth principal G-bundle $E \to \Sigma$ on an oriented surface Σ, and let $\mathcal{A}_F(E)$ denote the space of flat connections on E. Assume for now that $\mathcal{A}_F(E)$ is non-empty (in general we define $\mathcal{A}_F(E)$ to the space of *central* connections, cf. Definition 4 below). Let $\mathcal{G}$ denote the gauge group of $E \to \Sigma$, i.e. the group of G-equivariant maps $E \to E$ which live over the identity map of Σ. A main result of Atiyah-Bott is an inductive calculation of the $\mathcal{G}$-equivariant homology of $\mathcal{A}_F(E)$, or in other words, a calculation of the homology of the *homotopy orbit space*

$$\mathcal{A}_F(E)/\!\!/\mathcal{G}. \tag{1}$$

Here, we use the notation $X/\!\!/G$ when G acts on X, for the homotopy orbit space. Explicitly, $X/\!\!/G = (EG \times X)/G$, where EG is a contractible space with free G-action. There is a projection map $X/\!\!/G \to X/G$ which is a homotopy equivalence if the G-action on X is free. The quotient (1) parametrizes families of flat vector bundles over a fixed surface Σ, where the flat bundle is allowed to vary in the family. Thus it can be viewed as a moduli space of flat vector bundles on Σ.

Another moduli space, whose study goes back to Riemann, is the moduli space of Riemann surfaces; it parametrizes families of Riemann surfaces. We construct it as the homotopy quotient

$$\mathcal{M}_g = J(\Sigma)/\!\!/\operatorname{Diff}(\Sigma), \tag{2}$$

where $J(\Sigma)$ is the space of (almost) complex structures on a genus g Riemann surface Σ, and $\operatorname{Diff}(\Sigma)$ is the group of diffeomorphisms of Σ. In [7], Madsen and Weiss gave a complete calculation of the cohomology $H^*(\mathcal{M}_g)$, as long as $* < (g-1)/2$, using tools of stable homotopy theory.

The moduli space under study in this paper combines the moduli spaces (1) and (2). It parametrizes families of flat vector bundles $E \to \Sigma$, where we allow *both* the flat bundle *and* the Riemann surface Σ to vary. Explicitly our moduli space can be constructed as

$$\mathcal{M}^G_{g,\gamma} = \big(\mathcal{A}_F(E) \times J(\Sigma)\big) /\!\!/ \operatorname{Aut}(E), \tag{3}$$

where $\operatorname{Aut}(E)$ is the group of G-equivariant maps $E \to E$ which are over some diffeomorphism $\Sigma \to \Sigma$. The topological type of the bundle $E \to \Sigma$ is encoded by the pair (g, γ). Here $g \in \mathbb{N}$ is the genus of Σ, and $\gamma \in \pi_1(G)$ labels the isomorphism type of the bundle. If the bundle is classified by a map $f : \Sigma \to BG$, the corresponding element is

$$\gamma = f_*\big([\Sigma]\big) \in H_2(BG) = H_1(G) = \pi_1(G).$$

In the case where no flat connection on $E \to \Sigma$ exists, the definition (3) has to be modified slightly, cf. Definition 6 below.

Forgetting the bundle $E \to \Sigma$ altogether gives a map to the moduli space $\mathcal{M}_g$, and there is a fibration sequence

$$\mathcal{A}_F(E)/\!\!/\mathcal{G} \to \mathcal{M}^G_{g,\gamma} \to \mathcal{M}_g.$$

Our main theorem, Theorem 1 below, amounts to a calculation of $H_q(\mathcal{M}^G_{g,\gamma})$ when $g > 2q + 4$. The result turns out to be independent of g and γ as long as $\mathcal{A}_F(E)$ is non-empty, i.e. as long as the bundle $E \to \Sigma$ admits a flat connection. Actually our theorem holds for any bundle, if we define $\mathcal{A}_F(E)$ and $\mathcal{M}^G_{g,\gamma}$ using central connections, cf. Definitions 4 and 6 below.

To state our main theorem, let us introduce some notation. Let L denote the canonical line bundle over $\mathbb{CP}^\infty$, and let $\mathbb{CP}^\infty_{-1} = (\mathbb{CP}^\infty)^{-L}$ be the Thom spectrum of the virtual inverse $-L$ (graded so that the Thom class is in H^{-2}). Let BG_+ denote the classifying space BG, with a disjoint basepoint added. We get a spectrum $\mathbb{CP}^\infty_{-1} \wedge BG_+$, and let $\Omega^\infty(\mathbb{CP}^\infty_{-1} \wedge BG_+)$ be the corresponding infinite loop space. Our main theorem on the homology of $\mathcal{M}^G_{g,\gamma}$ is the following.

Theorem 1. *Let G be a connected, compact, semisimple Lie group. There is a map $\mathcal{M}^G_{g,\gamma} \to \Omega^\infty(\mathbb{CP}^\infty_{-1} \wedge BG_+)$, inducing an isomorphism*

$$H_q\big(\mathcal{M}^G_{g,\gamma}\big) \to H_q\big(\Omega^\infty_\bullet\big(\mathbb{CP}^\infty_{-1} \wedge BG_+\big)\big)$$

when $g \geqslant 2q+4$. Here $\Omega^\infty_\bullet(\mathbb{CP}^\infty_{-1} \wedge BG_+) \subseteq \Omega^\infty(\mathbb{CP}^\infty_{-1} \wedge BG_+)$ denotes the connected component of the basepoint. In particular, in this range, the homology groups of $\mathcal{M}^G_{g,\gamma}$ are independent of g and $\gamma \in \pi_1(G)$.

See Section 2.2 below for an explicit description of the rational cohomology of $\Omega^\infty_\bullet(\mathbb{CP}^\infty_{-1} \wedge BG_+)$, and hence of $\mathcal{M}^G_{g,\gamma}$ in the stable range.

Remarks. (i) One may canonically identify $\pi_0(\Omega^\infty(\mathbb{CP}^\infty_{-1} \wedge BG_+))$ with $\mathbb{Z} \times \pi_1(G)$. The isomorphism in the theorem is induced by an explicit map $\mathcal{M}^G_{g,\gamma} \to \Omega^\infty(\mathbb{CP}^\infty_{-1} \wedge BG_+)$, defined by a Pontryagin–Thom construction, cf. [2,4,6,7]. The direct definition gives a map into the component labelled $(g-1,\gamma) \in \mathbb{Z} \times \pi_1(G)$, rather than the (homotopy equivalent) base point component.

(2) We have chosen to work with *homotopy quotients* in the definitions of the moduli spaces (1), (2), and (3). One could alternatively consider the *quotient stacks*. Theorem 1 would also hold in that case since, by definition, the homology of the stacks would be homology of the homotopy quotients.

Following Atiyah–Bott [1], we can reinterpret our results in the setting of semi-stable holomorphic structures. In this introduction we will consider only the case of vector bundles, corresponding to the case $G = U(n)$, see Section 2.3 for the general case. The *slope* of a complex vector bundle $V \to \Sigma$ is the number $\mu(V) = c_1(V)/\dim(V)$. A holomorphic vector bundle $V \to \Sigma$ is semi-stable if each holomorphic sub-bundle $V' \subseteq V$ satisfies $\mu(V') \leqslant \mu(V)$. Let $\mathcal{C}(V)$ denote the space of all holomorphic structures on $V \to \Sigma$ and let $\mathcal{C}_{\mathrm{ss}}(V)$ denote the subspace of semi-stable ones. This has an action of the group $\mathrm{Aut}(V)$ of fiberwise linear maps $V \to V$ which are over some diffeomorphism $\Sigma \to \Sigma$. We consider the moduli space

$$\mathcal{M}_{\mathrm{ss}}(V) = \mathcal{C}_{\mathrm{ss}}(V) /\!\!/ \mathrm{Aut}(V),$$

which parametrizes families of semi-stable holomorphic vector bundles $V \to \Sigma$, where we allow both the Riemann surface Σ and the vector bundle to vary in the family. Other authors, e.g. [5], have studied moduli spaces of semi-stable vector bundles over a fixed Riemann surface, allowing only the bundle to vary.

We prove the following corollary to our main theorem.

Corollary 2. *There is a map $\mathcal{M}_{\mathrm{ss}}(V) \to \Omega^\infty(\mathbb{CP}^\infty_{-1} \wedge BU(n)_+)$, inducing an isomorphism*

$$H_q\big(\mathcal{M}_{\mathrm{ss}}(V)\big) \to H_q\big(\Omega^\infty_\bullet\big(\mathbb{CP}^\infty_{-1} \wedge BU(n)_+\big)\big)$$

for $g \geqslant 2q+4$.

Finally, we give an application to the $\mathrm{Out}(\pi_1(\Sigma_g))$-equivariant homology of the representation variety, $\mathrm{Rep}(\pi_1(\Sigma_g), G)$. Here $\pi_1(\Sigma_g)$ is the fundamental group of a closed, connected, oriented surface Σ_g of genus g, and $\mathrm{Out}(\pi_1(\Sigma_g))$ is the outer automorphism group. See Theorems 13 and 14 below.

The second main result of the paper regards a cobordism category of surfaces with flat connections. We call this category $\mathcal{C}^F_G$ whose objects are closed, oriented one-manifolds S equipped with connections on the trivial principal bundle $S \times G$, and whose morphisms are oriented cobordisms Σ between the one-manifold boundary components, equipped with flat G bundles $E \to \Sigma$ that restrict on the boundaries in the obvious way. (See Section 3.1 for a careful definition.) Our result is the identification of the homotopy type of the geometric realization of this category.

Theorem 3. *There is a homotopy equivalence,*

$$B\mathcal{C}^F_G \simeq \Omega^\infty\big(\Sigma\big(\mathbb{CP}^\infty_{-1} \wedge BG_+\big)\big).$$

In order to prove this theorem, we will compare the category $\mathcal{C}_G^F$ of surfaces with flat connections to the category of surfaces with *any* connection, $\mathcal{C}_G$. That is, this category is defined exactly as was the category $\mathcal{C}_G^F$, except that we omit the requirement that the connection ω on the principal G-bundle $E \to \Sigma$ be flat. We will prove that the inclusion of cobordism categories $\mathcal{C}_G^F \hookrightarrow \mathcal{C}_G$ induces a homotopy equivalence on their geometric realizations, and then use the results of [4] to identify the resulting homotopy type.

2. A stability theorem for the universal moduli space of flat connections

2.1. Review of results of Atiyah and Bott

Let Σ be a Riemann surface, G a compact Lie group, and $E \to \Sigma$ a smooth principal G-bundle. Let $\mathcal{A}(E)$ be the space of connections on E. Using a Riemannian metric on Σ and an invariant metric on the Lie algebra $\mathfrak{g}$ of G, we get an L^2 metric on $\Omega^2(\Sigma; \mathrm{ad}(E))$, and we consider the Yang–Mills functional

$$L : A \mapsto \left\| F(A) \right\|^2.$$

The critical points are precisely the Yang–Mills connections, i.e. connections satisfying

$$d * F(A) = 0.$$

The underlying philosophy of Atiyah–Bott [1] is to study $L : \mathcal{A}(E) \to \mathbb{R}$ from a Morse theoretic point of view, equivariantly with respect to the gauge group $\mathcal{G}$, the group of sections of $\mathrm{Ad}(E)$. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{z}$ be its center. Then we have a subset

$$\Omega^2(\Sigma) \otimes \mathfrak{z} \subseteq \Omega^2\big(\Sigma; \mathrm{ad}(E)\big),$$

and Atiyah–Bott ([1, Proposition 6.16] and the following lines) proves that a connection $A \in \mathcal{A}(E)$ is a minimum of L if and only if its curvature is of the form $\mathrm{vol} \otimes \eta$ for some $\eta \in \mathfrak{z}$, where vol is the volume form on Σ. Furthermore these local minima are also global, and the element $\eta \in \mathfrak{z}$ is uniquely determined by the topology of the bundle $E \to \Sigma$. Bundles $E \to \Sigma$, equipped with a connection $A \in \mathcal{A}(E)$ which is a minimum for $L : \mathcal{A}(E) \to \mathbb{R}$, are the central object of study in this paper, so we give a them a name.

Definition 4. A connection $A \in \mathcal{A}(E)$ is *central* if its curvature is of the form $\mathrm{vol} \otimes \eta$ for some $\eta \in \mathfrak{z}$. Let $\mathcal{A}_F(E) \subseteq \mathcal{A}(E)$ denote the space of central connections.

Thus, a connection is central if it minimizes $L : \mathcal{A}(E) \to \mathbb{R}$. In many cases a central connection is the same as a flat connection. Obviously, a flat connection is always central. If E admits a flat connection, then all central connections are flat and hence $\mathcal{A}_F(E)$ is the space of flat connections. In general there can be topological obstructions to flatness. In the case $\mathfrak{z} = 0$, central is always the same as flat. In particular, this is the case when G is semisimple. For completeness, we note the following criterion for flatness.

Lemma 5. *Let $E \to \Sigma$ be a principal G-bundle whose topological type is classified by the element $\gamma \in \pi_1(G)$. Then the bundle admits a flat connection if and only if γ is a torsion element in the abelian group $\pi_1(G)$.*

Proof. Suppose first that $E \to \Sigma$ is flat, and hence induced by a homomorphism $\pi_1(\Sigma) \to G$. If γ were not torsion, there would be a homomorphism $\rho : G \to U(1)$ which detects γ. Then ρ gives a flat $U(1)$-bundle with non-trivial first Chern class, a contradiction.

Conversely, suppose $\gamma \in \pi_1(G)$ is torsion. Then there is a finite covering space $\tilde{\Sigma} \to \Sigma$ such that the pullback of $E \to \Sigma$ becomes trivial. The curvature of $E \to \Sigma$ gives a class in $H^2(\Sigma; \mathfrak{z})$ which vanishes when pulled back to $\tilde{\Sigma}$. But the projection $\tilde{\Sigma} \to \Sigma$ is injective in H^2, so the curvature class of $E \to \Sigma$ must also vanish. If we pick a central connection, its curvature is of the form $\mathrm{vol} \otimes \eta$, and hence must vanish. □

Unlike the notion of flatness, the notion of centrality depends on the metric on Σ. Hence a little explanation is needed when defining $\mathcal{M}^G_{g,\gamma}$ in the general case. Here is the correct definition. Recall that the uniformization theorem gives a canonical (spherical, Euclidean or hyperbolic) metric on any Riemann surface.

Definition 6. Define $\mathcal{M}^G_{g,\gamma}$ as in (3) above, with $\mathcal{A}_F(E) \times J(\Sigma)$ interpreted as the set of pairs (A, j) with $j \in J(\Sigma)$ and A a connection on E which is central with respect to the metric induced by j.

Probably the most interesting case of the general (central, non-flat) case of our theorem the case $G = U(n)$, which can be re-interpreted in terms of semi-stable holomorphic structures on vector bundles, as in Corollary 2 in Section 1.

From a Morse theoretic point of view, $\mathcal{A}(E)$ is built starting with the space of minima $\mathcal{A}_F(E)$. The "stable manifold" of $\mathcal{A}_F(E)$ is the space of connections which flow to $\mathcal{A}_F(E)$ under the gradient flow of $L : \mathcal{A}(E) \to \mathbb{R}$. Using analytic results, which were later established by Råde [9] and Daskalopoulus [3], the gradient flow deformation retracts this "stable manifold" to $\mathcal{A}_F(E)$. Then build $\mathcal{A}(E)$ from there by successively attaching critical points for L, i.e. Yang–Mills connections, of higher codimension. In the end this gives a $\mathcal{G}$-equivariant stratification of $\mathcal{A}(E)$. Using Riemann–Roch, Atiyah–Bott [1, formulas 5.10, 7.15 and 10.7] calculates the codimension of the strata. This formula lets us calculate the connectivity of the inclusion $\mathcal{A}_F(E) \to \mathcal{A}(E)$. It turns out that the codimension of all higher strata is bounded below by a function which grows linearly with genus.

2.2. *The main theorem*

The goal of this section is to prove Theorem 1, as stated in Section 1. Recall from Eq. (3) in Section 1 that $\mathcal{M}^G_{g,\gamma}$ is defined as the homotopy quotient

$$\mathcal{M}^G_{g,\gamma} = \big(\mathcal{A}_F(E) \times J(\Sigma)\big) /\!\!/ \operatorname{Aut}(E),$$

where $\mathcal{A}_F(E)$ is the space of flat (or central) connections on $E \to \Sigma$.

Similarly, let $\mathcal{A}(E)$ be the affine space of *all* connections on the bundle E (no flatness required). Define the configuration space, $\mathcal{B}^G_{g,\gamma}$ to be the homotopy orbit space,

$$\mathcal{B}^G_{g,\gamma} = \big(J(\Sigma_g) \times \mathcal{A}(E)\big) /\!\!/ \operatorname{Aut}(E). \tag{4}$$

Including $\mathcal{A}_F(E) \hookrightarrow \mathcal{A}(E)$ defines a natural map $j : \mathcal{M}^G_{g,\gamma} \hookrightarrow \mathcal{B}^G_{g,\gamma}$. The following is a straightforward consequence of the works [1,3,9].

Theorem 7. *The map* $j : \mathcal{M}^G_{g,\gamma} \hookrightarrow \mathcal{B}^G_{g,\gamma}$ *is* $2(g-1)r$*-connected. Here* g *is the genus of* Σ*, and* r *denotes the smallest number of the form* $\frac{1}{2}\dim(G/Q)$*, where* $Q \subset G$ *is any proper compact subgroup of maximal rank (see Remark* 8 *below for more details). In particular* j *induces an isomorphism in homotopy groups and in homology groups in dimensions less than* $2(g-1)r$.

Proof. As done by Atiyah–Bott [1], the space of G-connections on E can be identified with the space of holomorphic structures on the induced complexified bundle, $E^c = E \times_G G^c$. Moreover, in the Atiyah–Bott stratification of the space of holomorphic bundles one has that the space $\mathcal{A}_F(E)$ of central connections is homotopy equivalent to the stratum of semistable holomorphic bundles [3,9]. By considering the codimension of the next smallest stratum (in the partial order described in [1]) one knows that the inclusion of the semistable stratum into the entire space of holomorphic bundles is $2(g-1)r$-connected. Translating to the setting of connections, this says that the inclusion

$$j : \mathcal{A}_F(E) \hookrightarrow \mathcal{A}(E)$$

is $2(g-1)r$-connected.

Now consider the following diagram of principal $\mathrm{Aut}(E)$-fibrations,

$$\begin{array}{ccc}
J(\Sigma) \times \mathcal{A}_F(E) & \xrightarrow{\hookrightarrow} & J(\Sigma) \times \mathcal{A}(E) \\
\downarrow & & \downarrow \\
\mathcal{M}^G_{g,\gamma} & \xrightarrow{j} & \mathcal{B}^G_{g,\gamma} \\
\downarrow & & \downarrow \\
B\mathrm{Aut}(E) & \xrightarrow{=} & B\mathrm{Aut}(E).
\end{array}$$

The above discussion implies that the top horizontal arrow induces an isomorphism of homotopy groups through dimension $2(g-1)r$. Applying the five-lemma to the long exact sequences in homotopy groups induced by the two bundles, we get that the middle horizontal arrow, $j_g : \mathcal{M}^G_{g,\gamma} \to \mathcal{B}^G_{g,\gamma}$ also induces an isomorphism of homotopy groups in this range. □

Remark 8. The formula for r may be derived from [1, Eq. (10.7)]. By the formula given there, the connectivity of the map j is at least $2(g-1)r$, where r denotes the minimum of the numbers $r(P)$, where P runs over the set of all proper parabolic subgroups of G^c, and $r(P)$ denotes the number of positive roots of G, which are not roots of P. Since any proper, compact, connected, maximal-rank subgroup $Q \subset G$ belongs to a proper parabolic subgroup P of G^c, and $\dim(G/Q) \geqslant \dim(G/G \cap P) = 2r(P)$, we may write r as the minimum value of the numbers of the form $\{\frac{1}{2}\dim(G/Q)\}$, with Q running through all compact proper maximal-rank subgroups of G. We note that if G is the special unitary group $SU(n)$, then the largest parabolic in $SL_n(\mathbb{C})$ is isomorphic to $GL_{n-1}(\mathbb{C})$. Hence the number r is given by $n-1$ in this case.

Since $J(\Sigma)$ and $\mathcal{A}(E)$ are both contractible, Theorem 7 states that through a range of dimensions, the universal moduli space $\mathcal{M}^G_{g,\gamma}$ has the homotopy type of the classifying space of the automorphism group, $B\mathrm{Aut}(E)$. Now observe that this classifying space has the following description.

Let $EG \to BG$ be a smooth, universal principal G-bundle, so that EG is contractible with a free G-action. The mapping space of smooth equivariant maps, $C^\infty_G(E, EG)$ is also contractible and has a free action of the group $\mathcal{G}(E)$. The action is pointwise and clearly has slices. Thus one has a model for the classifying space of this gauge group,

$$B\big(\mathcal{G}(E)\big) \simeq C^\infty_G(E, EG)/\mathcal{G}(E) \cong C^\infty(\Sigma, BG)_E,$$

where $C^\infty(\Sigma, BG)_E$ denotes the component of the mapping space classifying the isomorphism class of the bundle E.

Let $E\mathrm{Diff}(\Sigma) \to B\mathrm{Diff}(\Sigma)$ be a smooth, universal principal $\mathrm{Diff}(\Sigma)$-bundle. A nice model for $E\mathrm{Diff}(\Sigma)$ is the space of smooth embeddings, $E\mathrm{Diff}(\Sigma) = \mathrm{Emb}(\Sigma, \mathbb{R}^\infty)$. The product of the action of $\mathrm{Aut}(E)$ on $C^\infty_G(E, EG)$ and the action (through $\mathrm{Diff}(\Sigma)$) on $E\mathrm{Diff}(\Sigma)$ gives a free action on the product $E\mathrm{Diff}(\Sigma) \times C^\infty_G(E, EG)$. The quotient of this action is a model of the classifying space $B\mathrm{Aut}(E)$. This quotient is homeomorphic to the homotopy orbit space,

$$B\mathrm{Aut}(E) \simeq E\mathrm{Diff}(\Sigma) \times_{\mathrm{Diff}(\Sigma)} C^\infty(\Sigma, BG)_E \tag{5}$$

where $\mathrm{Diff}(\Sigma)$ acts on $C^\infty(\Sigma, BG)_E$ by precomposition.

Corollary 9. *There is a* $2(g-1)r$*-connected map*

$$\tilde{j} : \mathcal{M}^G_{g,\gamma} \longrightarrow E\mathrm{Diff}(\Sigma) \times_{\mathrm{Diff}(\Sigma)} C^\infty(\Sigma, BG)_E.$$

We recall from [2] that the space $E\mathrm{Diff}(\Sigma) \times_{\mathrm{Diff}(\Sigma)} C^\infty(\Sigma, X)$ can be viewed as the space of smooth surfaces in the background space X in the following sense. As in [2], define

$$\mathcal{S}_g(X) = \big\{(M, f)\text{: where } M \subset \mathbb{R}^\infty \text{ is a smooth oriented surface of genus } g \\ \text{and } f : M \to X \text{ is a smooth map}\big\}.$$

The topology was described carefully in [2], which used the embedding space $\mathrm{Emb}(\Sigma, \mathbb{R}^\infty)$ for $E\mathrm{Diff}(\Sigma)$. In particular, $\mathcal{S}_g(BG)$ is a model for $E\mathrm{Diff}(\Sigma) \times_{\mathrm{Diff}(\Sigma)} C^\infty(\Sigma, BG)$, and therefore Corollary 9 defines a $2(g-1)r$-connected map

$$\coprod_{\gamma \in \pi_1(G)} \mathcal{M}^G_{g,\gamma} \xrightarrow{j} \mathcal{S}_g(BG).$$

Again, $S_g(X)$ need not be connected. In the case where X is simply connected, the correspondence $(M, f) \mapsto f_*[M]$ defines an isomorphism $\pi_0 S_g(X) \cong H_2(X) = \pi_2(X)$. For $\gamma \in H_2(X)$, we let

$$S_{g,\gamma}(X) \subseteq S_g(X)$$

be the corresponding connected component.

Now in [2] the stable topology of $\mathcal{S}_{g,\gamma}(X)$ was studied for a simply connected space X. The following is the main result of [2].

Theorem 10. *For X simply connected, the homology group $H_q(\mathcal{S}_{g,\gamma}(X))$ is independent of g and γ, as long as $2q+4 \leqslant g$. For q in this range,*

$$H_q\big(\mathcal{S}_{g,\gamma}(X)\big) \cong H_q\big(\Omega^\infty_\bullet\big(\mathbb{CP}^\infty_{-1} \wedge X_+\big)\big).$$

Remark. In [2] the spaces $\mathcal{S}_{g,\gamma}(X)$ were actually defined using the continuous mapping spaces $\mathrm{Map}(\Sigma, X)$ in the compact-open topology, rather than the smooth mapping spaces $C^\infty(\Sigma_g, X)$ used here. However since the inclusion $C^\infty(\Sigma, X) \hookrightarrow \mathrm{Map}(\Sigma, X)$ is a homotopy equivalence which is equivariant with respect to the $\mathrm{Diff}(\Sigma)$-action, the results of [2] apply to the smooth mapping spaces as well.

Proof of Theorem 1. Since G is assumed to be a compact, connected Lie group, BG is simply connected, so we can apply Theorem 10. If we let $X = BG$ and put Corollary 9 and Theorem 10 together, Theorem 1 follows. □

For completeness we conclude this section with an explicit description of $H^*(\mathcal{M}^G_{g,\gamma}; \mathbb{Q})$ in the stable range, following the description of stable rational cohomology of $\mathcal{S}_{g,\gamma}(X)$ given in [2]. This stable cohomology is generated by the Miller–Morita–Mumford κ-classes, and the rational cohomology of BG. We also give a geometric description of how these generating classes arise.

For a graded vector space V over the rationals, let V_+ be positive part of V, i.e.

$$V_+ = \bigoplus_{n=1}^{\infty} V_n.$$

Let $A(V_+)$ be the free graded-commutative $\mathbb{Q}$-algebra generated by V_+. Given a homogeneous basis of V_+, $A(V_+)$ is the polynomial algebra generated by the even-dimensional basis elements, tensor the exterior algebra generated by the odd-dimensional basis elements. Let $\mathcal{K}$ be the graded vector space $H^*(\mathbb{CP}^\infty_{-1}; \mathbb{Q})$. It is generated by one basis element, κ_i, of dimension $2i$ for each $i \geqslant -1$. Explicitly, κ_{-1} is the Thom class, and $\kappa_i = c_1^{i+1}\kappa_{-1}$, for $c_1 = c_1(L) \in H^2(\mathbb{CP}^\infty)$. Consider the graded vector space

$$V = H^*\big(\mathbb{CP}^\infty_{-1} \wedge BG_+; \mathbb{Q}\big) = \mathcal{K} \otimes H^*(BG; \mathbb{Q}).$$

Then $H^*(\Omega^\infty_\bullet(\mathbb{CP}^\infty_{-1} \wedge BG_+); \mathbb{Q})$ is canonically isomorphic to $A(V_+)$, and we get the following corollary of the stable rational cohomology $H^*(\mathcal{S}_{g,\gamma}(X))$ given in [2] and Corollary 9 above.

Corollary 11. *There is a homomorphism of algebras,*

$$\Theta : A\big(\big(\mathcal{K} \otimes H^*(BG; \mathbb{Q})\big)_+\big) \longrightarrow H^*\big(\mathcal{M}^G_{g,\gamma}; \mathbb{Q}\big)$$

which is an isomorphism in dimensions less than or equal to $(g-4)/2$.

Given an element $\alpha \in H^*(BG; \mathbb{Q})$, we describe the image

$$\Theta(\kappa_i \otimes \alpha) \in H^*\big(\mathcal{M}^G_{g,\gamma}; \mathbb{Q}\big).$$

The topological type of $E \to \Sigma$ is immaterial here, so in order not to complicate notation we take the disjoint union

$$\mathcal{M}^G = \coprod_{g,\gamma} \mathcal{M}^G_{g,\gamma}.$$

This is the moduli space of *all* connected Riemann surfaces Σ equipped with a flat principal G-bundle $E \to \Sigma$. It comes equipped with a universal surface bundle

$$\Sigma \to \mathcal{M}^G_1 \xrightarrow{p} \mathcal{M}^G.$$

$\mathcal{M}^G_1$ is the moduli space of pairs $(E \to \Sigma, x \in \Sigma)$, where $(E \to \Sigma) \in \mathcal{M}^G$, and x is a marked point. Explicitly, we could let $\mathcal{M}^G_{g,\gamma,1} = (J(\Sigma) \times \mathcal{A}_F(E) \times \Sigma) /\!/ \operatorname{Aut}(E)$ in analogy with (3), and take the disjoint union over all (g, γ).

The space $\mathcal{M}^G_1$ has two canonical bundles over it. The first is the "vertical tangent bundle," $T_{\mathrm{vert}}\mathcal{M}^G_1$. This is a complex line bundle whose fiber over $(E \to \Sigma, x \in \Sigma)$ is the tangent space $T_x\Sigma$. The second canonical bundle is a principal G-bundle, $E^G_1 \to \mathcal{M}^G_1$, whose fiber over $(E \to \Sigma, x \in \Sigma)$ is E_x, the fiber of $E \to \Sigma$ at $x \in \Sigma$.

View a class $\alpha \in H^*(BG; \mathbb{Q})$ as a characteristic class for G-bundles. Then $\alpha(E^G_1) \in H^*(\mathcal{M}^G_1; \mathbb{Q})$ is a well defined cohomology class. Similarly, since $T_{\mathrm{vert}}\mathcal{M}^G_1$ is an complex line bundle, it has a well defined Chern class $c_1 \in H^2(\mathcal{M}^G_1; \mathbb{Q})$. One then defines $\Theta(\kappa_i \otimes \alpha) \in H^*(\mathcal{M}^G; \mathbb{Q})$ to be the image under integrating along the fiber,

$$\Theta(\kappa_i \otimes \alpha) = \int_{\mathrm{fib}} c_1^{i+1} \cup \alpha\big(E^G_1\big).$$

Remarks. 1. The smoothness of the moduli spaces, $\mathcal{M}^G$ and $\mathcal{M}^G_1$ has not been discussed, so that fiberwise integration has not been justified. However, as described in [2] and [6], the Pontrjagin–Thom construction, which realizes fiberwise integration in the smooth setting, is well defined, and gives the definition of the map Θ we are using.

2. When $\alpha = 1$, $\Theta(\kappa_i)$ is exactly the Miller–Morita–Mumford class coming from $H^*(B\operatorname{Diff}(\Sigma); \mathbb{Q}) = H^*(\mathcal{M}_g; \mathbb{Q})$, the cohomology of the moduli space of Riemann surfaces.

3. Notice that the above formula makes good sense, even when $i = -1$, in that $\Theta(\kappa_{-1} \otimes \alpha) = \int_{\mathrm{fib}} \alpha(E^G_1)$.

2.3. Applications to semistable bundles and surface group representations

We now deduce two direct corollaries of Theorem 1 that stem from the close relationship between the space of flat connections, the space of semistable holomorphic bundles on a Riemann surface, and the space of representations of the fundamental group of the surface.

As above, let Σ be a fixed closed, oriented, smooth surface of genus g, and let $J(\Sigma)$ be the space of (almost) complex structures on Σ. Let E be a principal G bundle over Σ where as before, G is a compact, connected Lie group. Given $J \in J(\Sigma)$, a semistable structure on $E^c = E \times_G G^c$ is a holomorphic structure so that the induced holomorphic adjoint bundle $E \times_G \mathfrak{g}^c$ is a semistable vector bundle in the sense explained in Section 1 (i.e. no sub-bundles has higher slope). For a fixed $J \in J(\Sigma)$, let $\mathcal{C}^J_{ss}(E^c) \subset \mathcal{C}^J(E^c)$ be the space of semistable holomorphic structures inside the full affine space of all holomorphic structures on $E^c \to \Sigma$. As explained in [1], the full space $\mathcal{C}^J(E^c)$ is homeomorphic to the space $\mathcal{A}(E)$ of all connections on $E \to \Sigma$, and the open subspace $\mathcal{C}^J_{ss}(E^c)$ is identified with the open stratum for the Yang–Mills flow [1]. In particular there is a homotopy equivalence $\mathcal{A}_F(E) \simeq \mathcal{C}^J_{ss}(E^c)$ which is equivariant with respect to $\mathrm{Aut}(E)$.

Define the space

$$\mathcal{C}_{ss}\big(E^c\big) = \big\{(J, B)\colon\ J \in J(\Sigma),\ \text{and } B \in \mathcal{C}^J_{ss}\big(E^c\big)\big\}. \tag{6}$$

Let $\mathcal{C}(E^c)$ be the full space of all holomorphic structures on E^c, i.e. $\mathcal{C}(E) = \{(J, B)\colon\ J \in J(\Sigma)$, and $B \in \mathcal{C}^J(E)\}$. As before $\mathrm{Aut}(E)$ acts on $\mathcal{C}(E)$ with the semistable bundles $\mathcal{C}_{ss}(E)$ as an invariant subspace. Namely, an automorphism $(\tilde{\phi}, \phi)$ of the principal bundle,

$$\begin{array}{ccc} E & \xrightarrow{\ \tilde{\phi}\ } & E \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\ \phi\ } & \Sigma \end{array}$$

pulls back a holomorphic structure on $(E, J) \times_G G^c$ to a holomorphic structure on the bundle $(E, g^*(J)) \times_G G^c$. Actually, the larger group $\mathrm{Aut}(E^c)$, consisting of G^c-equivariant maps $E^c \to E^c$ which are over some diffeomorphism $\Sigma \to \Sigma$, acts on $\mathcal{C}_{ss}(E)$ in the same way.

We define the universal moduli space of semistable holomorphic bundles, and the universal moduli space of all holomorphic bundles,

$$\mathcal{M}_{ss}(E) = \mathcal{C}_{ss}\big(E^c\big) /\!\!/ \mathrm{Aut}\big(E^c\big), \qquad \mathcal{D}(E) = \mathcal{C}(E) /\!\!/ \mathrm{Aut}\big(E^c\big)$$

to be the homotopy orbit spaces. Notice that the space of all holomorphic structures $\mathcal{C}(E)$ is contractible, so the homotopy orbit space $\mathcal{D}(E)$ is a model of the classifying space, $B\,\mathrm{Aut}(E^c) \simeq E\mathrm{Diff}(\Sigma) \times_{\mathrm{Diff}(\Sigma)} C^\infty(\Sigma, BG)_E$.

We now have the following:

Theorem 12. *The inclusion of the universal moduli space of semistable holomorphic bundles into all holomorphic bundles,*

$$\mathcal{M}_{ss}(E) \hookrightarrow \mathcal{D}(E) \simeq E\big(\mathrm{Diff}(\Sigma)\big) \times_{\mathrm{Diff}(\Sigma)} C^\infty(\Sigma, BG)_E$$

is a $2(g-1)r$*-connected map.*

Proof. This follows from Theorem 7 together with the equivariant homotopy equivalence between the space of flat connections and the space of semistable holomorphic bundles [1,3,9], and the homotopy equivalence between $\mathrm{Aut}(E)$ and $\mathrm{Aut}(E^c)$. □

We therefore have the following stability theorem for the universal moduli space of semistable holomorphic bundles as a corollary to Theorem 10. Corollary 2 in Section 1 is the special case where $G = U(n)$. Indeed, a principal $U(n)$ bundle $E \to \Sigma$ is the same thing as a vector bundle $V \to \Sigma$, and the two notions of semistability coincide (cf. [1, Lemma 10.9]).

Theorem 13. *Let G be a connected, compact Lie group. Then the homology group of the universal moduli space of semistable holomorphic bundles, $H_q(\mathcal{M}_{\mathrm{ss}}(E))$ is independent of the genus g of Σ and the topological type of the bundle E, so long as $2q+4 \leqslant g$. For q in this range,*

$$H_q\big(\mathcal{M}_{\mathrm{ss}}(E)\big) \cong H_q\big(\Omega^\infty_\bullet\big(\mathbb{CP}^\infty_{-1} \wedge BG_+\big)\big).$$

We conclude with an application to the space of representations of the fundamental group. Choose a basepoint $x_0 \in \Sigma_g$, and let $\pi = \pi_1(\Sigma_g, x_0)$ be the fundamental group based at that point. Let $\mathrm{Hom}(\pi, G)$ denote the space of homomorphisms from π to G . We topologize this space as a subspace of the mapping space, $\mathrm{Map}(\pi, G)$.

Let $\mathrm{Aut}(\pi)$ denote the group of homotopy classes of basepoint preserving, orientation preserving, homotopy equivalences of Σ_g. As suggested by the notation, we may identify $\mathrm{Aut}(\pi)$ with the subgroup of automorphisms of the fundamental group that acts by the identity on $H^2(\pi, \mathbb{Z}) = \mathbb{Z}$. The group $\mathrm{Aut}(\pi)$ acts on the space of homomorphisms $\mathrm{Hom}(\pi, G)$, by precomposition. This action descends to an action of the *outer* automorphism group, $\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ on the strict quotient variety, $\mathrm{Hom}(\pi, G)/G$, where G acts by conjugation. Here $\mathrm{Inn}(\pi)$ is the normal subgroup of *inner* automorphisms. We now study how this action of $\mathrm{Out}(\pi)$ lifts to an action of the orientation preserving diffeomorphism group, $\mathrm{Diff}(\Sigma_g)$ on the *homotopy quotient* space,

$$\mathrm{Rep}(\pi, G) = \mathrm{Hom}(\pi, G)/\!\!/G.$$

Now let $E \to \Sigma$ be a principal bundle which admits a flat connection. Then $\mathcal{A}_F(E)$ is the space of flat connections on E. Recall that holonomy defines a homeomorphism from the space of based gauge equivalence classes of flat connections to the corresponding component of the space of homomorphisms:

$$h : \mathcal{A}_F(E)/\mathcal{G}_0(E) \xrightarrow{\cong} \mathrm{Hom}(\pi, G)_E$$

where $\mathcal{G}_0(E)$ is the based gauge group (which fixes a fiber pointwise), and $\mathrm{Hom}(\pi, G)_E$ denotes the connected component corresponding to the bundle E. This holonomy map is G-equivariant, where G acts as usual on $\mathrm{Hom}(\pi, G)$ by conjugation, and on the space of flat connections $\mathcal{A}_F(E)/\mathcal{G}_0(E)$, it acts by identifying G as the quotient group $G = \mathcal{G}(E)/\mathcal{G}_0(E)$, and by using the action of the full gauge group $\mathcal{G}(E)$ on $\mathcal{A}_F(E)$. We therefore have a homeomorphism,

$$\mathcal{A}_F(E)/\!\!/\mathcal{G}(E) = EG \times_G \big(\mathcal{A}_F(E)/\mathcal{G}_0(E)\big) \xrightarrow{\cong} EG \times_G \mathrm{Hom}(\pi, G)_E = \mathrm{Rep}(\pi, G)_E,$$

where the subscript denotes the component of $\mathrm{Rep}(\pi, G)$ that correspond to the bundle $E \to \Sigma$. Notice that we can take an alternative model of $\mathcal{A}_F(E)/\!\!/\mathcal{G}(E)$ as

$$\mathcal{A}_F(E)/\!\!/\mathcal{G}(E) \simeq \big(\mathcal{A}_F(E) \times E\big(\mathrm{Aut}(E)\big)/\mathcal{G}(E)$$

which has a residual action of $\mathrm{Aut}(E)/\mathcal{G}(E) \cong \mathrm{Diff}(\Sigma)$. Thus the representation space, $\mathrm{Rep}(\pi, G)_E$, is homotopy equivalent to a space with a $\mathrm{Diff}(\Sigma)$ action, and this action clearly lifts the action of $\mathrm{Out}(\pi)$ on the honest quotient space, $\mathcal{A}_F(E)/\mathcal{G}(E) \cong \mathrm{Hom}(\pi, G)/G$. Furthermore, for genus $g \geqslant 2$, this diffeomorphism group has contractible path components, and so is homotopy equivalent to its discrete group of path components, the mapping class group, $\mathrm{Diff}(\Sigma_g) \simeq \Gamma_g = \mathrm{Out}(\pi)$. We therefore define the $\mathrm{Out}(\pi)$-equivariant homology

$$H_q^{\mathrm{Out}(\pi)}\big(\mathrm{Rep}(\pi, G)_E\big) = H_q^{\mathrm{Diff}(\Sigma)}\big(\mathcal{A}_F(E) \times E\big(\mathrm{Aut}(E)\big)/\mathcal{G}(E)\big).$$

Theorem 14. *Let $g \geqslant 2$. Then the $\mathrm{Out}(\pi)$-equivariant homology of the representation variety is independent of the genus g, so long as $2q + 4 \leqslant g$. For q in this range,*

$$H_q^{\mathrm{Out}(\pi)}\big(\mathrm{Rep}(\pi, G)_E\big) \cong H_q\big(\Omega_\bullet^\infty\big(\mathbb{CP}_{-1}^\infty \wedge BG_+\big)\big).$$

Proof. $H_q^{\mathrm{Out}(\pi)}(\mathrm{Rep}(\pi, G)_E) = H_q^{\mathrm{Diff}(\Sigma)}(\mathcal{A}_F(E)/\!\!/\mathcal{G}(E))$, but the latter group is equal to

$$H_q\big(E\mathrm{Diff}(\Sigma) \times_{\mathrm{Diff}(\Sigma)} \mathcal{A}_F(E)/\!\!/\mathcal{G}(E)\big) = H_q\big(\mathcal{A}_F(E)/\!\!/\,\mathrm{Aut}(E)\big) = H_q\big(\mathcal{M}_{g,\gamma}^G\big).$$

The result follows by Theorem 10. □

3. The cobordism category of surfaces with flat connections

In this section we study the cobordism category $\mathcal{C}_G^F$ of surfaces equipped with flat G-bundles. In Section 1 we defined moduli spaces $\mathcal{M}_g^G$ of pairs (Σ, E) consisting of a closed Riemann surface Σ and a flat G-bundle E over Σ. In this section we generalize to Riemann surfaces with boundary. In this section we always consider flat connections rather than central, even in the case where the Lie algebra $\mathfrak{g}$ has non-trivial center. As we shall see, all principal bundles over a surface with boundary possess flat connections (cf. the proof of Proposition 17 below).

The moduli spaces of flat bundles on Riemann surfaces with boundary form morphisms in $\mathcal{C}_G^F$, and gluing along common boundaries define composition of morphisms. We then identify the homotopy type of its classifying space, proving Theorem 3 in Section 1.

Some care is needed to define a category whose morphism spaces are moduli spaces of surfaces with flat bundles and where composition is defined by gluing. We need to assure that composition is associative (not just associative up to homotopy). We will actually present two different ways of achieving this, leading to two different definitions of the cobordism category. Although the categories are not (quite) homotopy equivalent, their classifying spaces are, and Theorem 3 will be proved for both versions. The first approach follows the spirit of the first part of the paper, and defines the morphism spaces as a homotopy quotient in analogy with our previous definition of $\mathcal{M}_{g,\gamma}^G$, cf. (3) above. The second is in some sense simpler and more geometric, but includes only morphisms with non-empty outgoing boundary. When all surfaces have non-empty boundary, the homotopy quotients in the definition of the moduli spaces will be homotopy equivalent to the actual quotients, i.e. the space of orbits. This is because the relevant group actions are free.

3.1. Moduli of surfaces with boundary

We first define the relevant moduli spaces. Let Σ be a compact oriented 2-manifold (not necessarily connected, possibly with boundary). Let $J(\Sigma)$ be the space of (almost) complex structures on Σ. A principal G-bundle $E \to \Sigma$ restricts to a principal G-bundle $\partial E \to \partial\Sigma$. For a flat connection ω on ∂E, let $\mathcal{A}_F(E, \omega)$ denote the space of flat connections on E which restrict to ω on ∂E. Let $\mathrm{Aut}(E;\partial)$ denote the group of automorphisms of E, which restrict to the identity on a neighborhood of ∂E. Thus, $\mathrm{Aut}(E;\partial)$ fits into an exact sequence

$$1 \to \mathcal{G}(E;\partial) \to \mathrm{Aut}(E,\partial) \to \mathrm{Diff}(\Sigma;\partial) \to 1,$$

where $\mathcal{G}(E;\partial)$ is the group of gauge transformations of E which restrict to the identity near the boundary, and $\mathrm{Diff}(\Sigma;\partial)$ is the groups of diffeomorphisms of Σ which restrict to the identity near the boundary. Let $\mathcal{M}(E,\omega)$ be the homotopy orbit space

$$\mathcal{M}(E,\omega) = \bigl(J(\Sigma) \times \mathcal{A}_F(E,\omega)\bigr) /\!\!/ \mathrm{Aut}(E;\partial). \tag{7}$$

An imprecise definition of $\mathcal{C}_G^F$ goes as follows.

Definition 15. An object is a triple $x = (S, E, \omega)$, where S is a closed 1-manifold, $E \to S$ is a principal G-bundle, and ω is a connection on E. The space of morphisms from $x_0 = (S_0, E_0, \omega_0)$ to $x_1 = (S_1, E_1, \omega_1)$ is the disjoint union

$$\mathcal{C}_G^F(x_0, x_1) = \coprod_E \mathcal{M}(E,\omega),$$

where the disjoint union is over all $E \to \Sigma$ with $\partial E = E_0 \amalg E_1$, one E in each diffeomorphism class, and $\omega = \omega_0 \amalg \omega_1$.

This definition is imprecise because it only defines the homotopy type of the space of morphisms, not the underlying set (the homotopy quotient involved in defining $\mathcal{M}(E,\omega)$ involves a choice). We must give a precise, set-level description of the homotopy quotient, and define an associative composition on the point set level. We present a way of doing this. Recall that the definition of homotopy quotient involves the choice of a free, contractible $\mathrm{Aut}(E;\partial)$-space $E\mathrm{Aut}(E;\partial)$. As constructed in Eq. (5), a convenient choice of this space is given by

$$E\mathrm{Aut}(E;\partial) = \mathbb{R}_+ \times \mathrm{Emb}\bigl(\Sigma, [0,1] \times \mathbb{R}^\infty\bigr) \times C_G^\infty(E, EG), \tag{8}$$

where $\mathbb{R}_+$ denotes the positive real numbers, $\mathrm{Emb}(\Sigma, [0,1] \times \mathbb{R}^\infty)$ denotes the space of embeddings which restrict to embeddings of incoming and outgoing boundaries $S_\nu \to \{\nu\} \times \mathbb{R}^\infty$, $\nu = 0, 1$, and $C_G^\infty(E, EG)$ denotes the space of G-equivariant smooth maps. Using this space in the definition of the homotopy quotient, we get the following definition of the set of objects and the set of morphisms.

Definition 16. A point in the space of objects $\mathrm{Ob}(\mathcal{C}_G^F)$, is given by a triple (S, c, ω), where $\mathrm{S} \subset \mathbb{R}^\infty$ is an embedded, closed, oriented one-manifold, $c : S \to BG$ is a smooth map, and ω is a principal connection on the pullback along c of $EG \to BG$.

A point in the space of morphisms $\mathrm{Mor}(\mathcal{C}_G^F)$, is given by the data: (t, M, i, c, σ), where t is a positive real number, $M \subset [0,t] \times \mathbb{R}^\infty$ is a 2-dimensional cobordism, $i \in J(M)$ is a complex structure, and $c : M \to BG$ is a smooth map. Let $E \to M$ be the pullback along c of the universal smooth G-bundle $EG \to BG$. Finally, σ is a flat connection on E.

For an explicit comparison of the two definitions, consider the $\mathrm{Aut}(E;\partial)$-invariant map

$$E\mathrm{Aut}(E;\partial)\times\big(J(\Sigma)\times\mathcal{A}_F(E;\omega)\big)\longrightarrow \mathrm{Mor}\big(\mathcal{C}_G^F\big)$$

defined in the following way. Given elements $(t,\phi,b)\in E\mathrm{Aut}(E;\partial)$ as in (8) and $(j,\tau)\in J(\Sigma)\times\mathcal{A}_F(E;\omega)$, let $M\subseteq[0,t]\times\mathbb{R}^\infty$ be obtained by stretching the first coordinate of the image $\phi(\Sigma)\subseteq[0,1]\times\mathbb{R}^\infty$, and letting i, c, and σ be induced from j, b, and τ by the identification $M\cong\Sigma$. This map factors through an injection of $\mathrm{Aut}(E;\partial)$-orbits

$$\mathcal{M}(E,\omega)\longrightarrow \mathrm{Mor}\big(\mathcal{C}_G^F\big). \tag{9}$$

Taking disjoint union over Σ's and E's, we get an identification of the morphism spaces in Definition 15 and those in Definition 16. Moreover, it is now clear how to define an associative composition rule: take union of subsets $M_0\subseteq[0,t_1]\times\mathbb{R}^\infty$ and $M_1\subseteq[t_1,t_1+t_2]\times\mathbb{R}^\infty$. (For this to be a smooth submanifold, we should insist that all cobordism $M\subseteq[0,t]\times\mathbb{R}^\infty$ are "collared" as in [4]. Similarly, c and ω should be constant in the collar direction. We omit the details.)

The goal of this section is to identify the homotopy type of the classifying space of this category $B\mathcal{C}_G^F$. (The classifying space of a category is the geometric realization of the simplicial nerve of the category.) More specifically, our goal is to prove Theorem 3 as stated in Section 1. In order to prove this theorem, we will compare the category $\mathcal{C}_G^F$ of surfaces with flat connections to the category of surfaces with *any* connection. Namely, let $\mathcal{C}_G$ be the category defined exactly as was the category $\mathcal{C}_G^F$, except that we omit the requirement that the principal connection σ be flat.

The inclusion of flat connections into all connections defines a functor

$$\iota:\mathcal{C}_G^F\hookrightarrow\mathcal{C}_G.$$

We will observe that this inclusion restricts to the "positive boundary subcategories" defined as follows. Let $\mathcal{C}$ represent either of the cobordism categories, $\mathcal{C}_G^F$ or $\mathcal{C}_G$. Let $\mathcal{C}_\partial$ denote the subcategory that has the same objects as $\mathcal{C}$, but the morphisms of $\mathcal{C}_\partial$ are those morphisms of $\mathcal{C}$ that involve surfaces, each path component of which has a non-empty "outgoing" boundary. (The "outgoing boundary" of a surface $\Sigma\subset\mathbb{R}^\infty\times[0,t]$ is $\Sigma\cap(\mathbb{R}^\infty\times\{t\})$.) An important step in proving Theorem 3 is the following proposition, which we prove in Section 3.2.

Proposition 17. *The inclusion functor $\iota:\mathcal{C}_{G,\partial}^F\hookrightarrow\mathcal{C}_{G,\partial}$ induces a homotopy equivalence of classifying spaces*

$$\iota:B\mathcal{C}_{G,\partial}^F\xrightarrow{\simeq}B\mathcal{C}_{G,\partial}.$$

This proposition will allow us to identify the homotopy type of $B\mathcal{C}_G^F$, because we will be able to identify $B\mathcal{C}_{G,\partial}$ with the classifying space of a cobordism category studied in [4]. In that paper, the authors identified the homotopy type of a broad range of topological cobordism categories. We will be interested in a particular such category which we call $\mathcal{C}_2(BG)$, defined as follows.

Definition 18. Let X be a space. The space of objects $\mathrm{Ob}(\mathcal{C}_2(X))$ is given by pairs (S,c), where $\mathrm{S}\subset\mathbb{R}^\infty$ is an embedded, closed, oriented one-manifold, and $c:S\to X$ is a continuous map.

The space of morphisms $\mathrm{Mor}(\mathcal{C}_2(X))$, is given by triples, (t, Σ, c), where t is a nonnegative real number, $\Sigma \subset [0, t] \times \mathbb{R}^\infty$ is an oriented cobordism, and $c : \Sigma \to X$ is a continuous map. The embedded surface is collared at the boundaries as in [4]. In particular, $\Sigma_0 = \Sigma \cap (\mathbb{R}^\infty \times \{0\})$ and $\Sigma_t = \Sigma \cap (\mathbb{R}^\infty \times \{t\})$ are smoothly embedded, oriented one-manifolds. Again, morphisms are assumed to be collared, and composition is defined by gluing of cobordisms and maps. (See [4] for details.)

The homotopy type of the classifying space $B\mathcal{C}_2(X)$ was determined in [4]. Namely, the following was proved there.

Theorem 19. *(See [4].)*

(a) *Let $\mathcal{C}_{2,\partial}(X)$ denote the positive boundary subcategory. Then the inclusion functor, $\mathcal{C}_{2,\partial}(X) \hookrightarrow \mathcal{C}_2(X)$ induces a homotopy equivalence of classifying spaces,*

$$B\mathcal{C}_{2,\partial}(X) \xrightarrow{\simeq} B\mathcal{C}_2(X).$$

(b) *There is a homotopy equivalence,*

$$B\mathcal{C}_2(X) \simeq \Omega^\infty\big(\Sigma\big(\mathbb{CP}^\infty_{-1} \wedge X_+\big)\big).$$

Because of this theorem, applied to the space $X = BG$, Theorem 3 will follow from Proposition 17 and the following two results, which we prove in Section 3.2 below.

Proposition 20. *The functor $\mathcal{C}_G \to \mathcal{C}_2(BG)$ which on the level of morphisms is given by $(t, \Sigma, j, c, \omega) \mapsto (t, \Sigma, c)$ induces homotopy equivalences of classifying spaces*

$$B\mathcal{C}_G \simeq B\mathcal{C}_2(BG),$$
$$B\mathcal{C}_{G,\partial} \simeq B\mathcal{C}_{2,\partial}(BG).$$

Proposition 21. *The inclusion of the positive boundary subcategory, $\mathcal{C}^F_{G,\partial} \hookrightarrow \mathcal{C}^F_G$ induces a homotopy equivalence of classifying spaces*

$$B\mathcal{C}^F_{G,\partial} \simeq B\mathcal{C}^F_G.$$

3.2. The positive boundary subcategories

In this subsection we prove Propositions 17, 20 and 21. In view of Theorem 19, these imply the string of equivalences

$$B\mathcal{C}^F_G \simeq B\mathcal{C}^F_{G,\partial} \simeq B\mathcal{C}_{G,\partial} \simeq B\mathcal{C}_{2,\partial}(BG) \simeq B\mathcal{C}_2(BG) \simeq \Omega^\infty\big(\Sigma\big(\mathbb{CP}^\infty_{-1} \wedge BG_+\big)\big)$$

(the five homotopy equivalences follow from Propositions 21, 17, 20, and Theorem 19(a) and 19(b), respectively). Thus the proof of Theorem 3 will be completed.

Proof of Proposition 17. Morphisms in both categories are given by tuples $(t, \Sigma, j, c, \omega)$, where ω is a connection on a principal G bundle $E \to \Sigma$. The only difference between the two categories is that in one of them, ω is required to be flat. Let (t, Σ, j, c) be fixed. We prove that under the "positive boundary" assumption on Σ, the inclusion of flat connections into all connections,

$$\mathcal{A}_F(E) \to \mathcal{A}(E),$$

is a homotopy equivalence.

The "positive boundary" assumption implies that no connected component of Σ is a closed 2-manifold. Hence Σ deformation retracts onto its 1-skeleton $X \subseteq \Sigma$. Choose a 1-parameter family $\phi_t : \Sigma \to \Sigma$ of smooth maps which starts at the identity and ends at a map ϕ_1 which retracts Σ onto its 1-skeleton. We can lift this family to a 1-parameter family $\Phi_t : E \to E$ of maps of principal G-bundles with Φ_0 the identity. Then we can let ω_t be the connection on E obtained by pullback along Φ_t. The curvature of ω_t can be computed by naturality:

$$F_{\omega_t} = (\phi_t)^*(F_\omega),$$

and therefore ω_1 is flat, because ϕ_1 has one-dimensional image and the curvature is a two-form. Thus the identity map of $\mathcal{A}(E)$ is homotopic to a map into $\mathcal{A}_F(E)$, and since $\mathcal{A}(E)$ is contractible, $\mathcal{A}_F(E)$ is contractible too.

This implies that the functor $\mathcal{C}^F_{G,\partial} \to \mathcal{C}_{G,\partial}$ induces a homotopy equivalence on morphism spaces or, in other words,

$$N_1\mathcal{C}^F_{G,\partial} \to N_1\mathcal{C}_{G,\partial}.$$

For $k \geqslant 2$, the argument is similar: An element in $N_k\mathcal{C}_{G,\partial}$ is given by (Σ, j, c, ω) as before, together with a k-tuple $(t_1, \ldots, t_k)$ of positive real numbers. Here $\Sigma \subseteq [0, t] \times \mathbb{R}^\infty$ with $t = t_1 + \cdots + t_k$. The same procedure as for $k = 1$ gives a path from ω to a flat connection. Therefore the functor induces homotopy equivalences on k-nerves for all k, and hence on the geometric realization. □

Remark 22. Notice that the above proof holds for any Lie group. In particular, it shows that the space of flat connections is gauge equivariantly contractible, for any principal bundle over a connected Riemann surface with non-empty boundary.

We now go about proving Proposition 20.

Proof of Proposition 20. The map in the proposition is induced by the functor $(t, \Sigma, j, c, \omega) \mapsto (t, \Sigma, c)$ which forgets the complex structure j on the oriented surface Σ, and forgets the connection ω on the principal G-bundle $E \to \Sigma$. Thus the functor gives a fibration

$$N_1\mathcal{C}_G \to N_1\mathcal{C}_2(BG) \tag{10}$$

whose fiber over (t, Σ, c) is the space

$$\mathcal{N}(\Sigma, c) = J(\Sigma) \times \mathcal{A}(E),$$

where $J(\Sigma)$ is the space of (almost) complex structures on the oriented surface Σ and $\mathcal{A}(E)$ is the space of connections on E. But both spaces are contractible, so (10) is a homotopy equivalence. The higher levels of the simplicial nerve are completely similar, and we get a homotopy equivalence of geometric realizations. $\square$

Proof of Proposition 21. In the case where G is the trivial group (in other words, omit the flat G-bundle $E \to X$ from Definition 15), we recover the cobordism category $\mathcal{C}_d$ from [4], when $d = 2$. The positive boundary subcategory is denoted $\mathcal{C}_{d,\partial}$. In [4, Section 6], it is proved that the inclusion $B\mathcal{C}_{d,\partial} \to B\mathcal{C}_d$ is a weak homotopy equivalence when $d \geqslant 2$. The proof of Proposition 21 will follow [4, Section 6] very closely. We first recall an outline of that argument. Let $\mathcal{C} = \mathcal{C}_{\{e\}}$, the cobordism category in the case G is the trivial group.

In [4] a functor D from smooth manifolds to sets was defined, where $D(X)$ is the set of smooth manifolds $W \subseteq X \times \mathbb{R} \times \mathbb{R}^\infty$ such that the projection $(\pi, f) : W \to X \times \mathbb{R}$ is proper, and the projection $\pi : W \to X$ is a submersion with 2-dimensional fibers. A *concordance* is an element $W \in D(X \times \mathbb{R})$; in that case the restrictions to $X \times \{0\}$ and $X \times \{1\}$ are called *concordant*. This is an equivalence relation on $D(X)$, and the set of equivalence classes is denoted $D[X]$. The equivalence $B\mathcal{C} \to B\mathcal{C}_\partial$ was proved in [4] by proving two natural isomorphisms:

$$D[X] \cong [X, B\mathcal{C}], \tag{11}$$

$$D[X] \cong [X, B\mathcal{C}_\partial]. \tag{12}$$

The first is proved as follows (again, in outline). Given an element $W \in D(X)$ and a point $x \in X$, let $W_x \subseteq \mathbb{R} \times \mathbb{R}^\infty$ be the d-manifold $W_x = \pi^{-1}(x)$ and let $f_x : W_x \to \mathbb{R}$ denote the projection to the first factor. A choice of regular value $a \in \mathbb{R}$ for f_x defines an object $(f_x)^{-1}(a)$ of $\mathcal{C}$, and if $a_0 < a_1$ are both regular values, then $(f_x)^{-1}([a_0, a_1])$ is a morphism in $\mathcal{C}$ between the two corresponding objects. This is used to define a map from left to right in (11) which is an isomorphism.

To construct (12), we need to ensure that only morphisms satisfying the positive boundary condition arise as $f^{-1}([a_0, a_1])$. At the heart of this is [4, Lemma 6.2], which constructs a continuous family of pairs (K_t, f_t), $t \in \mathbb{R}$, consisting of a d-manifold K_t containing the open subset $U = \mathbb{R}^d - D^d \subseteq K_t$, and a smooth function $f_t : K_t \to \mathbb{R}$ which is constant on U and proper when restricted to $K_t - U$. Furthermore $K_0 = \mathbb{R}^d$ and f_0 is constant, and $K_1 = \mathbb{R}^d - \{0\}$ and $f_1(x)$ goes to infinity as $x \to 0$.

Now let $W \in D(X)$ and let $a_0 < a_1$ be regular values of $f_x : W_x \to \mathbb{R}$. If we are lucky, $f_x^{-1}([a_0, a_1])$ already satisfies the positive boundary condition. If not, let $Q \subseteq f_x^{-1}([a_0, a_1])$ be a connected component not touching $f_x^{-1}(a_1)$, and let $e : \mathbb{R}^d \to Q$ be an embedding. Gluing in the family (K_t, f_t), we get a one-parameter family of pairs (W_t, f_t). Repeat this procedure for each such component Q, and we get a one-parameter family (W_t, f_t) starting at (W_x, f_x) at time 0, and ending at some other pair (W_x', f_x') at time 1, for which $(f_x')^{-1}([a_0, a_1])$ satisfies the positive boundary condition. The rest of the proof in [4, Section 6] describes how this construction together with a (somewhat complicated) gluing procedure can be used to construct the map in (12).

For the purposes of proving Proposition 21, we need to construct a version of the "standard" one-parameter family (K_t, f_t) where K_t is equipped with a flat G-bundle $E_t \to K_t$, which is specified over K_0. Fortunately, this is not hard. Namely, the proof of [4, Lemma 6.2] also constructs a continuous family of immersions $j_t : K_t \to K_0$ which is the identity for $t = 0$. Then given any flat G-bundle $E_0 \to K_0$, it has a canonical extension to a flat G-bundle $E_t = (j_t)^*(E_0)$ over K_t. With this extension of [4, Lemma 6.2] in place, the rest of the proof in [4, Section 6] applies verbatim if we add flat G-bundles to every surface in sight. □

This completes the proof of Theorem 3.

Finally, let us briefly discuss an alternative definition of the cobordism category. It is (homotopy) equivalent to $\mathcal{C}^F_{G,\partial}$, but replaces the homotopy quotients by actual "strict" quotients (i.e. the space of orbits in the quotient topology). We first define the strict version of (7) above as

$$\mathcal{M}^{\text{strict}}(E,\omega) = \big(J(\Sigma) \times \mathcal{A}_F(E,\omega)\big)/\operatorname{Aut}(E;\partial),$$

and then define the strict cobordism category as in Definition 15 above, using $\mathcal{M}^{\text{strict}}(E,\omega)$ instead of $\mathcal{M}(E,\omega)$, and taking only disjoint union over $E \to \Sigma$ where Σ is in the positive boundary category. Actually, a concrete and very geometric description can be given of the morphism spaces $\mathcal{M}^{\text{strict}}(E,\omega)$ and also of the gluing maps, in the following way.

Let $\mathcal{M}^{\text{strict}}(\Sigma) = J(\Sigma)/\operatorname{Diff}(\Sigma,\partial\Sigma)$ be the moduli space of complex structures on Σ, and consider the forgetful map, $\mathcal{M}^{\text{strict}}(E,\omega) \to \mathcal{M}^{\text{strict}}(\Sigma)$. Then the preimage of any diffeomorphism class of complex structure on a genus g surface with b boundary components is given by [8] as

$$\Big\{(a,c,\omega) \in \mathrm{G}^{2g} \times \mathrm{G}^{b-1} \times \mathcal{A}(\partial E) \;\Big|\; \prod [a_{2i}, a_{2i-1}] = \prod \operatorname{Ad}_{c_j} \operatorname{Hol}(\omega_j)\Big\},$$

where $c_1 = 1$, and $\operatorname{Hol}(\omega_j)$ denotes the holonomy of the connection ω about the jth boundary circle over a fixed basepoint. The composition map in our category has the following geometric interpretation. Let Σ be a connected Riemann surface, with non-empty boundary, obtained from a (possibly disconnected) Riemann surface $\hat{\Sigma}$ by gluing along two boundary components $\mathrm{B}_\pm \subseteq \partial\hat{\Sigma}$. Let E denote a bundle over Σ obtained by identifying a bundle $\hat{E}$ on $\hat{\Sigma}$ along $\partial\hat{E}$. Then one may identify the strict moduli space $\mathcal{M}^{\text{strict}}(E)$ with the symplectic reduction of the gauge group $\mathcal{G}(\mathrm{G} \times \mathrm{B})$ acting on $\mathcal{M}^{\text{strict}}(\hat{E})$. Here B is the one-manifold that both components $\mathrm{B}_\pm$ are identified with, and $\mathcal{G}(\mathrm{G} \times \mathrm{B})$ is identified with the gauge group of the trivial bundle over B. The moment map for the $\mathcal{G}(\mathrm{G} \times \mathrm{B})$-action along which the symplectic reduction is carried out is given by $\omega \mapsto \omega_+ - \omega_-$, where $\omega_\pm$ denote the restrictions of the connection $\omega \in \mathcal{M}(\hat{E})$ to the boundary components $\mathrm{B}_\pm$.

3.3. The universal moduli space and the cobordism category

Although the proofs of our two main theorems are largely independent of one another, their content is obviously strongly related. We explain the relation.

To connect our two main theorems, Theorems 1 and 3, consider a closed surface Σ together with a principal G-bundle whose topological type is given by $\gamma \in \pi_1(G)$. Assume that $E \to \Sigma$ admits a flat connection, i.e. that $\gamma \in \pi_1(G)$ is torsion. Then $\mathcal{A}_F(E)$ is the space of flat connections on E, and the definition of $\mathcal{M}(E;\omega)$ in (7) agrees with our previous definition of $\mathcal{M}^G_{g,\gamma}$. The empty 1-manifold $\emptyset$ is an object of $\mathcal{C}^F_G$ and, as in (9) above, we get a map

$$\mathcal{M}^G_{g,\gamma} \to \mathcal{C}^F_G(\emptyset,\emptyset),$$

which identifies $\mathcal{M}^G_{g,\gamma}$ as the (open and closed) subset of $\mathcal{C}^F_G(\emptyset,\emptyset)$ where the topological type of Σ is fixed to be that of a connected genus g surface, and the homotopy class of $c : \Sigma \to BG$

is fixed by the element $\gamma \in \pi_1(G)$. In particular, an element of $\mathcal{M}^G_{g,\gamma}$ defines a loop in the classifying space $B\mathcal{C}^F_G$ that start and end at the vertex $\emptyset \in B\mathcal{C}^F_G$.

Corollary 23. *The induced map to the loop space of the cobordism category,*

$$\iota : \mathcal{M}^G_{g,\gamma} \to \Omega B\mathcal{C}^F_G$$

induces an isomorphism in homology, $H_q(\mathcal{M}^G_{g,\gamma}) \xrightarrow{\cong} H_q(\Omega_\bullet B\mathcal{C}^F_G)$ *for* $g > 2q + 4$.

Proof. Consider the following diagram:

$$\begin{array}{ccc}
\mathcal{M}^G_{g,\gamma} & \xrightarrow{\iota} & \Omega B\mathcal{C}^F_G \\
\Big\downarrow{\scriptstyle \tilde{j}} & & \Big\downarrow{\scriptstyle j} \\
\mathcal{S}_{g,\gamma}(BG) & \xrightarrow{\iota} & \Omega B\mathcal{C}_2(BG)
\end{array}$$

where $\mathcal{C}_2(BG)$ is the cobordism category of oriented surfaces in the background space BG, as described in Section 3.1. By construction, this diagram commutes.

We proved above that the map $j : B\mathcal{C}^F_G \to B\mathcal{C}_2(BG)$ is an equivalence. So the right-hand vertical map in this diagram is an equivalence. By the work of [2] one knows that the bottom horizontal map $\iota : \mathcal{S}_{g,\gamma}(BG) \to \Omega_\bullet B\mathcal{C}_2(BG)$ induces an isomorphism in homology through dimension $\frac{g}{2} - 2$. By Corollary 9, the left-hand vertical map $\tilde{j} : \mathcal{M}^G_{g,\gamma} \to \mathcal{S}_{g,\gamma}(BG)$ is $2(g-1)r$-connected. The result now follows. □

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References

[1] M. Atiyah, R. Bott, The Yang–Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1983) 523–615.

[2] R.L. Cohen, I. Madsen, Surfaces in a background space and the homology of mapping class groups, in: Proc. Summer Institute in Algebraic Geometry, Seattle, WA, 2005, in: Proc. Sympos. Pure Math., American Math. Soc., 2009, in press; preprint math.GT/0601750.

[3] G. Daskalopoulos, The topology of the space of stable bundles on a compact Riemann surface, J. Differential Geom. 36 (3) (1992) 699–746.

[4] S. Galatius, I. Madsen, U. Tillmann, M. Weiss, The homotopy type of the cobordism category, Acta Math., in press, available as arXiv:math/0605249.

[5] G. Harder, M.S. Narasimhan, On the cohomology groups of moduli spaces of vector bundles over curves, Math. Ann. 212 (1975) 215–248.

[6] I. Madsen, U. Tillmann, The stable mapping class group and $Q(\mathbb{CP}^\infty)$, Invent. Math. 145 (3) (2001) 409–544.

[7] I. Madsen, M. Weiss, The stable moduli space of Riemann surfaces: Mumford's conjecture, Ann. of Math. (2) 165 (2007) 843–941.

[8] E. Meinrenken, C. Woodward, Cobordism for Hamiltonian loop group actions and flat connections on the punctured two-sphere, Math. Z. 231 (1999) 133–168, arXiv:dg-ga/9707018.

[9] J. Råde, On the Yang–Mills heat equation in two and three dimensions, J. Reine Angew. Math. 431 (1992) 123–163.