THOM PROSPECTRA FOR LOOPGROUP REPRESENTATIONS

NITU KITCHLOO AND JACK MORAVA

Abstract. This is very much an account of work in progress. We sketch the construction of an Atiyah dual (in the category of $\mathbb{T}$-spaces) for the free loopspace of a manifold; the main technical tool is a kind of Tits building for loop groups, discussed in detail in an appendix. Together with a new localization theorem for $\mathbb{T}$-equivariant K-theory, this yields a construction of the elliptic genus in the string topology framework of Chas-Sullivan, Cohen-Jones, Dwyer, Klein, and others. We also show how the Tits building can be used to construct the dualizing spectrum of the loop group. Using a tentative notion of equivariant K-theory for loop groups, we relate the equivariant K-theory of the dualizing spectrum to recent work of Freed, Hopkins and Teleman.

Introduction

If $P \to M$ is a principal bundle with structure group G then $LP \to LM$ is a principal bundle with structure group

$$LG = \mathrm{Maps}(S^1, G),$$

We will assume in the rest of the paper that the frame bundle of M has been refined to have structure group G, and that the tangent bundle of M is thus defined via some representation V of G. It follows that the tangent bundle of LM is defined by the representation LV of LG. The circle group $\mathbb{T}$ acts on all these spaces by rotation of loops.

This is a report on the beginnings of a theory of differential topology for such objects. Note that if we want the structure group LG to be connected, we need G to be 1-connected; thus $SU(n)$ is preferable to $U(n)$. This helps explain why Calabi-Yau manifolds are so central in string theory, and this note is written assuming this simplifying hypothesis.

Alternately, we could work over the universal cover of LM; then $\pi_2(M)$ would act on everything by decktranslations, and our topological invariants become modules over the Novikov ring $\mathbb{Z}[H_2(M)]$. From the point of view we're developing, these translations may be relevant to modularity, but this issue, like several others, will be backgrounded here.

The circle action on the free loopspace defines a structure much closer to classical differential geometry than one finds on more general (eg) Hilbert manifolds; this action defines something like a Fourier filtration on the tangent space of this infinite-dimensional manifold, which is in some sense locally finite. This leads to a host of new kinds of geometric invariants, such as the Witten genus; but this filtration is

Date: 9 March 2005.
1991 *Mathematics Subject Classification.* 22E6, 55N34, 55P35.
NK is partially supported by NSF grant DMS 0436600, JM by DMS 0406461.

unfamiliar, and has been difficult to work with [17]. The main conceptual result of this note [which was motivated by ideas of Cohen, Godin, and Segal] is the definition of a canonical 'thickening' $L^\dagger M$ of a free loopspace, with the same equivariant homotopy type, but better differential-geometric and analytic properties: the usual tangent bundle to the free loopspace, pulled back over this 'dressed' model, admits a canonical filtration by finite-dimensional equivariant bundles. This thickening involves a contractible LG-space called the affine Tits building $\mathbf{A}(LG)$, which occurs under various guises in nature: it is a homotopy colimit of homogeneous spaces with respect to a finite collection of compact Lie subgroups of LG, and it is also the affine space of principal G-connections on the trivial bundle over S^1; we explore its structure in more detail in the appendix.

The thickening $L^\dagger M$ seems very natural, from a physical point of view: its elements are not the raw geometric loops of pure mathematics, but are rather loops endowed with a choice of connection (depending on the structure group G of interest). This is a conceptual distinction which means little in pure mathematics, but perhaps a lot in physics.

In the first section below, we recall why the Spanier-Whitehead dual of a finite CW-space is a ring-spectrum, and sketch the construction (due to Milnor and Spanier, and Atiyah) of a model for that dual, when the space is a smooth compact manifold. Our goal is to produce an analog of this construction for a free loopspace, which captures as much as possible of its string-topological algebraic structure. In the second section, we introduce the technology used in our construction: pro-spectra associated to filtered infinite-dimensional vector bundles, and the topological Tits building which leads to the construction of such a filtration for the tangent bundle.

In §3 we observe that recent work of Freed, Hopkins, and Teleman on the Verlinde algebra can be reformulated as a conjectural duality between LG-equivariant K-theory of a certain dualizing spectrum for LG constructed from its Tits building, and positive-energy representations of LG. In §4 we use a new strong localization theorem to study the equivariant K-theory of our construction, and we show how this recovers the Witten genus from a string-topological point of view.

We plan to discuss actions of various string-topological operads [15] on our construction in a later paper; that work is in progress.

We would like to thank R. Cohen, V. Godin, S. Stolz, and A. Stacey for many helpful conversations, and we would also like to acknowledge work of G. Segal and S. Mitchell as motivation for many of the ideas in this paper.

1. The Atiyah dual of a manifold

If X is a finite complex, then the function spectrum $F(X, S^0)$ is a **ring-spectrum** (because S^0 is). If X is a manifold M, Spanier-Whitehead duality says that

$$F(M_+, S^0) \sim M^{-TM} \ .$$

If $E \to X$ is a vector bundle over a compact space, we can define its Thom space to be the one-point compactification

$$X^E := E_+ \ .$$

There is always a vector bundle $E_\perp$ over X such that

$$E \oplus E_\perp \cong \mathbf{1}_N$$

is trivial, and following Atiyah, we write

$$X^{-E} := S^{-N} X^{E_\perp}.$$

With this notation, the Thom collapse map for an embedding $M \subset \mathbb{R}^N$ is a map

$$S^N = \mathbb{R}^N_+ \to M^\nu = S^N \ M^{-TM},$$

Moreover, the Thom collapse for the diagonal embedding of M into the zero section of $M_+ \wedge M^\nu$ gives us

$$M_+ \wedge M^\nu \to M^{\nu \oplus TM} = M_+ \wedge \mathbb{R}^N_+ \to \mathbb{R}^N_+ = S^N$$

defining the equivalence with the functional dual. More generally, a smooth map $f : M \to N$ of compact closed orientable manifolds has a Pontrjagin-Thom dual map

$$f_{PT} : N^{-TN} \to M^{-TM}$$

of spectra; in particular, the map $S^0 \to M^{-TM}$ dual to the projection to a point defines a kind of fundamental class, and the dual to the diagonal of M makes M^{-TM} into a ring-spectrum. The ring structure of M^{-TM} has been studied by various authors (see for example [13], [25]).

Prospectus: Chas and Sullivan [11] have constructed a very interesting product on the homology of a free loopspace, suitably desuspended, motivated by string theory. Cohen and Jones [15] saw that this product comes from a ring-spectrum structure on

$$LM^{-TM} := LM^{-e^*TM}$$

where

$$e : LM \to M$$

is the evaluation map at $1 \in S^1$. Unfortunately this evaluation map is not $\mathbb{T}$-equivariant, so the Chas-Sullivan Cohen-Jones spectrum is not in general a $\mathbb{T}$-spectrum. The full Atiyah dual constructed below promises to capture some of this equivariant structure. The Chas-Sullivan Cohen-Johes spectrum and the full Atiyah dual live in rather different worlds: our prospectrum is an equivariant object, whose multiplicative properties are not year clear, while the CSCJ spectrum has good multiplicative properties, but it is not a $\mathbb{T}$-spectrum. In some vague sense our object resembles a kind of center for the Chas-Sullivan-Cohen-Jones spectrum, and we hope that a better understanding of the relation between open and closed strings will make it possible to say something more explict about this.

2. Problems & Solutions

For our constructions, we need two pieces of technology:

Cohen, Jones, and Segal [16](appendix) associate to a filtration

$$\mathbf{E} : \cdots \subset E_i \subset E_{i+1} \subset \dots$$

of an infinite-dimensional vector bundle over X, a pro-object

$$X^{-\mathbf{E}} : \cdots \to X^{-E_{i+1}} \to X^{-E_i} \to \dots$$

in the category of spectra. [A rigid model for such an object can be constructed by taking $\mathbf{E}$ to be a bundle of Hilbert spaces, which are trivializable by Kuiper's theorem. Choose a trivialization $\mathbf{E} \cong H \times X$ and an exhaustive filtration $\{H_k\}$ of H by finite-dimensional vector spaces; then we can define

$$X^{-E_i} = \lim S^{-H_k} X^{H_k \cap E_i^{\perp}} ,$$

with $E_i^{\perp}$ the orthogonal complement of E_i in the trivialized bundle $\mathbf{E}$.] This pro-object will, in general, depend on the choice of filtration. We will be interested in the **direct** systems associated to such a pro-object by a cohomology theory; of course in general the colimit of this system can be very different from the cohomology of the limit of the system of pro-objects.

Example 2.1. If $X = \mathbb{C}P_\infty$, η is the Hopf bundle, and $\mathbf{E}$ is

$$\infty\eta : \cdots \subset (k-1)\eta \subset k\eta \subset (k+1)\eta \subset \dots$$

then the induced maps of cohomology groups are multiplication by the Euler classes of the bundles E_{i+1}/E_i, so

$$H^*(\mathbb{C}P_\infty^{-\infty\eta}, \mathbb{Z}) := \operatorname{colim}\{\mathbb{Z}[t], t - \mathrm{mult}\} = \mathbb{Z}[t, t^{-1}] \ .$$

We would like to apply such a construction to the tangent bundle of a free loopspace. Unfortunately, these tangent bundles do **not**, in general, possess any such nice filtration by finite-dimensional ($\mathbb{T}$-equivariant) subbundles [17]! However, such a splitting **does** exist in a neighborhood of the **constant** loops:

$$M = LM^{\mathbb{T}} \subset LM$$

has normal bundle

$$\nu(M \subset LM) = TM \otimes_{\mathbb{C}} (\mathbb{C}[q, q^{-1}]/\mathbb{C})$$

(at least, up to completions; and assuming things complex for convenience). Here small perturbations of a constant loop are identified with their Fourier expansions

$$\sum_{n \in \mathbb{Z}} a_n q^n,$$

with $q = e^{i\theta}$. The related fact, that TLM is defined by the representation LV of LG looped up from the finite-dimensional representation V of G, will be important below: for LV is **not** a positive-energy representation of LG.

The main step toward our resolution of this problem depends on the following result, proved in §7 below. Such constructions were first studied by Quillen, and were explored further by S. Mitchell [26]. The first author has studied these buildings for a general Kac-Moody group [23]; most of the properties of the affine building used below hold for this larger class.

Theorem 2.2. *There exist a certain finite set of* **compact** *'parabolic' subgroups* H_I *of* LG *(see 7.2), such that the* **topological affine Tits building**

$$\mathbf{A}(LG) := \mathrm{hocolim}_I \; LG/H_I$$

of LG *is* $\mathbb{T}\tilde{\times}LG$-**equivariantly** *contractible. In other words, given any compact subgroup* $K \subset \mathbb{T}\tilde{\times}LG$*, the fixed point space* $\mathbf{A}(LG)^K$ *is contractible.*

Remark 2.3. The group LG admits a universal central extension $\mathbb{L}G$ by a circle group C. The natural action of the rotation group $\mathbb{T}$ on LG lifts to $\mathbb{L}G$, and the $\mathbb{T}$-action preserves the subgroups H_I. Hence $\mathbf{A}(LG)$ admits an action of $\mathbb{T}\tilde{\times}\mathbb{L}G$, with the center acting trivially. We can therefore express $\mathbf{A}(LG)$ as

$$\mathbf{A}(LG) = \mathrm{hocolim}_I \; \mathbb{L}G/\mathbb{H}_I$$

where $\mathbb{H}_I$ is the induced central extension of H_I.

Other descriptions of $\mathbf{A}(LG)$

This Tits building has other descriptions as well. For example:

1. $\mathbf{A}(LG)$ can be seen as the classifying space for proper actions with respect to the class of compact Lie subgroups of $\mathbb{T}\tilde{\times}\mathbb{L}G$. That means that the space $\mathbf{A}(LG)$ is a $\mathbb{T}\tilde{\times}\mathbb{L}G$-CW complex with the isotropy groups of the action being compact Lie groups. Moreover, given any compact Lie group $K \subset \mathbb{T}\tilde{\times}\mathbb{L}G$, the fixed point space $\mathbf{A}(LG)^K$ is contractible. It is not hard to show that these properties uniquely determine $\mathbf{A}(LG)$ upto $\mathbb{T}\tilde{\times}\mathbb{L}G$-homotopy equivalence.

2. It also admits a more differential-geometric description as the smooth infinite dimensional manifold of holonomies on $S^1 \times G$ (see Appendix): Let $\mathcal{S}$ denote the subset of the space of smooth maps from $\mathbb{R}$ to G given by

$$\mathcal{S} = \{g : \mathbb{R} \to G,\; g(0) = 1,\; g(t+1) = g(t) \cdot g(1)\} \,;$$

then $\mathcal{S}$ is homeomorphic to $\mathbf{A}(LG)$. The action of $h(t) \in LG$ on $g(t)$ is given by $hg(t) = h(t) \cdot g(t) \cdot h(0)^{-1}$, where we identify the circle with $\mathbb{R}/\mathbb{Z}$. The action of $x \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ is given by $xg(t) = g(t+x) \cdot g(x)^{-1}$.

3. The description given above shows that $\mathbf{A}(LG)$ is equivalent to the affine space $\mathcal{A}(S^1 \times G)$ of connections on the trivial G-bundle $S^1 \times G$. This identification associates to the function $f(t) \in \mathcal{S}$, the connection $f'(t)f(t)^{-1}$. Conversely, the connection ∇_t on $S^1 \times G$ defines the function $f(t)$ given by transporting the element $(0,1) \in \mathbb{R} \times G$ to the point $(t, f(t)) \in \mathbb{R} \times G$ using the connection ∇_t pulled back to the trivial bundle $\mathbb{R} \times G$.

Remark 2.4. These equivalent descriptions have various useful consequences. For example, the model given by the space $\mathcal{S}$ of holonomies says that given a finite cyclic group $H \subset \mathbb{T}$, the fixed point space $\mathcal{S}^H$ is homeomorphic to $\mathcal{S}$. Moreover, this is a homeomorphism of LG-spaces, where we consider $\mathcal{S}^H$ as an LG-space and identify LG with LG^H in the obvious way. Notice also that $\mathcal{S}^{\mathbb{T}}$ is G-homeomorphic to the model of the adjoint representation of G defined by $\mathrm{Hom}(\mathbb{R}, G)$.

Similarly, the map $\mathcal{S} \to G$ given by evaluation at $t = 1$ is a principal ΩG bundle, and the action of $G = LG/\Omega G$ on the base G is given by conjugation. This allows us to relate our work to that of Freed, Hopkins and Teleman in the following section.

Finally, the description of $\mathbf{A}(LG)$ as the affine space $\mathcal{A}(S^1 \times G)$ implies that the fixed point space $\mathbf{A}(LG)^K$ is contractible for any compact subgroup $K \subseteq \mathbb{T}\tilde{\times}\mathbb{L}G$.

If $E \to B$ is a principal bundle with structure group LG, then (motivated by ideas of [14]) we construct a 'thickening' of B:

Definition 2.5. *The thickening of B associated to the bundle E is the space*

$$B^\dagger(E) = E \times_{LG} \mathbf{A}(LG) = \mathrm{hocolim}_I \ E/H_I \ .$$

We will omit E from the notation, when the defining bundle is clear from context.

Remark 2.6. If $P \to M$ is a principal G bundle, then $LP \to LM$ is a principal LG bundle. In this case, the description above gives $L^\dagger M := LM^\dagger(LP)$ a smooth structure:

$$L^\dagger M = \{(\gamma, \omega) \mid \quad \gamma \in LM, \quad \omega \in \mathcal{A}(\gamma^*(P))\}$$

where $\mathcal{A}(\gamma^*(P))$ is the space of connections on the pullback bundle $\gamma^*(P)$.

Let $\mathbb{T}\tilde{\times}\mathbb{L}G$ be the extension of the central extension of LG by $\mathbb{T}$, acting as rotations, and let U be a unitary representation of this group, of finite type. We propose to construct a Thom $\mathbb{T}\tilde{\times}\mathbb{L}G$-prospectrum $\mathbf{A}(LG)^{-U}$, as a substitute for the nonexistent equivariant prospectrum defined by $-U$, regarded as a vector bundle over a point.

The central extension of LG splits when restricted to the constant loops, so we have a torus $\mathbf{T} := \mathbb{T} \times C \times T \subset \mathbb{T}\tilde{\times}\mathbb{L}G$, where T is a maximal torus of G, and C is the circle of the central extension; $\mathbf{T}$ is in fact a common maximal torus for the family $\mathbb{H}_I$ of parabolics. Let $\check{\mathbf{T}}$ be the character group of this torus, and let $\mathbb{Z}[[\check{\mathbf{T}}]]$ be its completed group ring. On restriction to the subgroup $\mathbb{T}\tilde{\times}\mathbb{H}_I$,

$$U|_{\mathbb{T}\tilde{\times}\mathbb{H}_I} \cong \oplus \, U_I(\alpha)$$

decomposes into a sum of finite dimensional representations; because U is of finite type, the isotypical summands appear only finitely often. Let

$$\mathrm{char}\, U|_{\mathbb{T}\tilde{\times}\mathbb{H}_I} \in \mathbb{Z}[[\check{\mathbf{T}}]]$$

be its character.

For any finite subset $R \subset \check{\mathbf{T}}$ of characters, let

$$U_I(R) = \oplus \, \{U_I(\alpha) \mid \mathrm{char}\, U_I(\alpha) \subset \mathbb{Z}\langle R\rangle\} \subset U_I$$

be the subresentation of U_I with 'support in R', and let S_I^{-U} to be the Thom $\mathbb{T}\tilde{\times}\mathbb{H}_I$-prospectrum associated to the filtered (equivariant) vector bundle $R \mapsto U_I(R)$ over a point. If $I \subset J$ then $\mathbb{H}_I$ maps naturally to $\mathbb{H}_J$, and there is a corresponding morphism

$$\mathbf{U}_J \to \mathbf{U}_I$$

of filtered vector bundles, given by inclusions $U_J(R) \to U_I(R)$. This defines a filtered system of vector bundles over the small category or diagram defined by inclusions of parabolics.

Example 2.7. Consider the standard representation of $\mathbb{T}\tilde{\times}LSU(2)$ on $L\mathbb{C}^2$. Let L_1 and L_2 denote the canonical coordinate summands of $\mathbb{C}^2$, and let q denote the character of the rotation group $\mathbb{T}$. The parabolic subgroups $H_I \subset LSU(2)$ in this case are:

$$H_1 = SU(2) \subset LSU(2) \quad \text{given by constant loops, and}$$

$$H_0 = \{\begin{pmatrix} a & bz^{-1} \\ cz & d \end{pmatrix}\} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)\ ;$$

$T = H_\emptyset = H_0 \cap H_1$ is the maximal torus of $SU(2)$. Notice that the representation $L\mathbb{C}^2$ of the loop group is certainly not of positive energy. However, when restricted to the subgroups $\mathbb{T}\tilde{\times}H_I$, it decomposes as a sum of representations:

$$L\mathbb{C}^2 = \bigoplus_{k\in\mathbb{Z}} (L_1 \oplus L_2)q^k$$

for $\mathbb{T}\tilde{\times}H_1$, $\mathbb{T}\tilde{\times}H_\emptyset$ and as

$$\bigoplus_{k\in\mathbb{Z}} (L_1 \oplus qL_2)q^k$$

for $\mathbb{T}\tilde{\times}H_0$. The resulting filtered vector bundle is in this case just a pushout.

Definition 2.8. *We define* $\mathbf{A}(LG)^{-U}$ *to be the* $\mathbb{T}\tilde{\times}\mathbb{L}G$*-prospectrum*

$$\mathbf{A}(LG)^{-U} = \text{hocolim}_I\ \mathbb{L}G_+ \wedge_{\mathbb{H}_I} S_I^{-U}\ ,$$

where $\mathbb{L}G_+$ *denotes* $\mathbb{L}G$*, with a disjoint basepoint.*

Homotopy colimits in the category of prospectra can be defined in general, using the model category structure of [12].

Remark 2.9. Given any principal LG-bundle $E \to B$, and a representation U of LG, we define the Thom prospectrum of the virtual bundle associated to the representation $-U$ to be

$$B_!^{-U} = E_+ \wedge_{LG} \mathbf{A}(LG)^{-U} = \text{hocolim}_I\ E_+ \wedge_{H_I} S_I^{-U}\ .$$

In particular, if P is the refinement of the frame bundle of M via a representation V of G, then the tangent bundle of LM is defined by the representation LV of LG.

Definition 2.10. *The Atiyah dual* $LM^{-\mathbf{T}LM}$ *of* LM *is the* $\mathbb{T}$*-prospectrum* $L^\dagger M^{-LV}$.

We will explore this object further in §6. Note that its underlying non-equivariant object maps naturally to (a thickening of) the Chas-Sullivan spectrum.

3. The dualizing spectrum of $\mathbb{L}G$

The dualizing spectrum of a topological group K is defined [24] as the K-homotopy fixed point spectrum:

$$D_K = K_+^{hK} = F(EK_+, K_+)^K$$

where K_+ is the suspension spectrum of the space K_+, endowed with a right K-action. The dualizing spectrum D_K admits a K-action given by the residual left K-action on K_+. If K is a compact Lie group, then it is known [24] that D_K is the one point compactification of the adjoint representation $Ad(K)_+$. It is also known that there is a $K \times K$-equivariant homotopy equivalence

$$K_+ \cong F(K_+, D_K)\ .$$

It follows from the compactness of K_+ that for any free K_+-spectrum E, we have the K-equivariant homotopy equivalence

$$E \cong F(K_+, E \wedge_{K_+} D_K) \ .$$

It is our plan to understand the dualizing spectrum for the (central extension of the) loop group.

Theorem 3.1. *There is an equivalence*

$$D_{LG} \cong \mathrm{holim}_I \ LG_+ \wedge_{H_I} Ad(H_I)_+$$

of left LG-spectra.

Proof. We have the sequence of equivalences:

$$D_{LG} = F(ELG_+, LG_+)^{LG} \cong F(ELG_+ \wedge \mathbf{A}(LG)_+, LG_+)^{LG} \ .$$

The final space may be written as

$$\mathrm{holim}_I \ F(ELG_+ \wedge_{H_I} LG_+, LG_+)^{LG} = \mathrm{holim}_I \ F(ELG_+, LG_+)^{H_I} \ .$$

Now recall the equivalence of $H_I \times H_I$-spectra:

$$LG_+ \cong F(H_{I+}, LG_+ \wedge_{H_I} D_{H_I}) \ . \qquad (1)$$

Taking H_I-homotopy fixed points implies a left H_I-equivalence

$$F(ELG_+, LG_+)^{H_I} = (LG_+)^{hH_I} \cong LG_+ \wedge_{H_I} Ad(H_I)_+ \ ;$$

where we have used equation (1) at the end. Replacing this term into the homotopy limit completes the proof. □

Similarly, we have:

Theorem 3.2. *There is an equivalence*

$$D_{\mathbb{L}G} \cong \mathrm{holim}_I \ \mathbb{L}G_+ \wedge_{\mathbb{H}_I} Ad(\mathbb{H}_I)_+$$

of left $\mathbb{L}G$-spectra.

Remark 3.3. The diagram underlying D_{LG} or $D_{\mathbb{L}G}$ can be constructed in the category of spaces. Given an inclusion $I \subseteq J$, the orbit of a suitable element in $Ad(\mathbb{H}_J)$ gives an embedding $\mathbb{H}_J/\mathbb{H}_I \subset Ad(\mathbb{H}_J)$, and the Pontrjagin-Thom construction for this embedding defines an $\mathbb{H}_J$-equivariant map

$$Ad(\mathbb{H}_J)_+ \longrightarrow \mathbb{H}_{J+} \wedge_{\mathbb{H}_I} Ad(\mathbb{H}_I)_+$$

which extends to the map

$$\mathbb{L}G_+ \wedge_{\mathbb{H}_J} Ad(\mathbb{H}_J)_+ \longrightarrow \mathbb{L}G_+ \wedge_{\mathbb{H}_I} Ad(\mathbb{H}_I)_+$$

required for the diagram. Moreover, composites of these maps can be made compatible up to homotopy.

A conjectural relationship with the work of Freed, Hopkins and Teleman

The discussion below assumes the existence of $\mathbb{L}G$-equivariant K-theory, as defined in [19] (see Appendix). Given a space X with a proper $\mathbb{L}G$-action, Freed-Hopkins-Teleman define an $\mathbb{L}G$-equivariant spectrum over X. The equivariant K-theory groups are defined as the homotopy groups of the space of sections of this spectrum. In this section, we will assume that these K-theory groups can be defined for *spectra* X with proper $\mathbb{L}G$-action. We will primarily be interested with $\mathbb{L}G$-CW spectra with proper isotropy.

This hypothesis provides us with a convenient language. We expect to return to the underlying technical issues in a later paper.

The center of $\mathbb{L}G$ acts trivially on $D_{\mathbb{L}G}$, defining a second grading on $K^*_{\mathbb{L}G}(D_{\mathbb{L}G})$; we will use a formal variable z to keep track of the grading, so

$$K^*_{\mathbb{L}G}(D_{\mathbb{L}G}) = \bigoplus_n K^{*,n}_{\mathbb{L}G}(D_{\mathbb{L}G}) z^n.$$

The spectral sequence for the cohomology of a cosimplicial spectrum, in the case of $K^*_{\mathbb{L}G}(D_{\mathbb{L}G})$, has

$$E_2^{i,j} = \operatorname{colim}^i_I \; K^j_{\mathbb{H}_I}(Ad(\mathbb{H}_I)_+) \ .$$

This spectral sequence respects the second grading given by powers of z. As before, we may decompose $K^*_{\mathbb{H}_I}(Ad(\mathbb{H}_I)_+)$ under the action of the center of $\mathbb{L}G$ (contained in $\mathbb{H}_I$)

$$K^*_{\mathbb{H}_I}(Ad(\mathbb{H}_I)_+) = \bigoplus_n K^{*,n}_{\mathbb{H}_I}(Ad(\mathbb{H}_I)_+) z^n$$

We therefore have a spectral sequence converging to $K^{*,n}_{\mathbb{L}G}(D_{\mathbb{L}G})$, with

$$E_2^{i,j,n} = \operatorname{colim}^i_I \; K^{j,n}_{\mathbb{H}_I}(Ad(\mathbb{H}_I)_+) \ .$$

For $n > 0$, we will show presently that this spectral sequence collapses to give

$$K^{*,n}_{\mathbb{L}G}(D_{\mathbb{L}G}) = \operatorname{colim}_I \; K^{*,n}_{\mathbb{H}_I}(Ad(\mathbb{H}_I)_+)$$

Therefore, this group admits a natural Thom class given by the system $\{Ad(\mathbb{H}_I)_+\}$, of spinor bundles for the adjoint representations of the parabolics $\mathbb{H}_I$. We therefore have a (global) Thom isomorphism:

$$K^{*+r+1,n}_{\mathbb{L}G}(D_{\mathbb{L}G}) \cong \operatorname{colim}_I \; Rep^{*,n}(\mathbb{H}_I) \ .$$

where $Rep^{*,n}(\mathbb{H}_I)$ is the subgroup of $Rep^*(\mathbb{H}_I)$ corresponding to $K^{*,n}_{\mathbb{H}_I}(Ad(\mathbb{H}_I)_+)$ under the (local) Thom isomorphism. We will also show that the above colimit may naturally be identified with the free abelian group generated by the regular, dominant characters of level n. This free abelian group can further be identified with the Grothendieck group of level $n-h$ positive energy representations of $\mathbb{L}G$, where h denotes the dual Coxeter number. This Grothendieck group is known as the Verlinde algebra (of level $n-h$). Therefore, we get an (abstract) isomorphism between $K^{*+r+1,n}_{\mathbb{L}G}(D_{\mathbb{L}G})$ and the Verlinde algebra of level $n-h$. In [19], Freed-Hopkins-Teleman give a geometric meaning to the Thom isomorphism above, which explains the shift in level by showing that the Thom class has internal level h.

Under the assumptions on $K_{\mathbb{L}G}$ given above, we get:

Theorem 3.4. *For $n > 0$, the groups $K^{*,n}_{\mathbb{L}G}(D_{\mathbb{L}G})$ are two-periodic. We have a (Thom) isomorphism of groups:*

$$V_{n-h} \cong K^{r+1,n}_{\mathbb{L}G}(D_{\mathbb{L}G})$$

where V_k is the Verlinde algebra of level k, h is the dual Coxeter number of G, and r is its rank. Moreover, $K^{r,n}_{\mathbb{L}G}(D_{\mathbb{L}G}) = 0$.

Before we prove the collapse of the spectral sequence, let us consider an example:

Example 3.5. To illustrate this in an example, recall the case of $G = SU(2)$. In this case $r = 1$, $h(G) = 2$. Here the groups $\mathbb{H}_I$ are given by

$$\mathbb{H}_0 = SU(2) \times S^1, \quad \mathbb{H}_1 = S^1 \times SU(2), \quad \mathbb{H}_0 \cap \mathbb{H}_1 = T = S^1 \times S^1 \ .$$

The respective representation rings may be identified by restriction with subalgebras of $K_T(pt) = \mathbb{Z}[u^{\pm 1}, z^{\pm 1}]$:

$$K_{\mathbb{H}_0}(pt) = \mathbb{Z}[u + u^{-1}, (z/u)^{\pm 1}], \quad K_{\mathbb{H}_1}(pt) = \mathbb{Z}[z^{\pm 1}, u + u^{-1}] \ .$$

Now consider the two pushforward maps involved in the colimit:

$$\varphi_0 : K_T(pt) \to K_{\mathbb{H}_0}(pt), \quad \varphi_1 : K_T(pt) \to K_{\mathbb{H}_1}(pt)$$

A quick calculation shows that for $k > 0$, we have

$$\varphi_j(z^k) = \begin{cases} (z/u)^k Sym^k(u + u^{-1}), & j = 0 \\ z^k, & j = 1 \end{cases}$$

$$\varphi_j(z^k u^{-1}) = \begin{cases} (z/u)^k Sym^{k-1}(u + u^{-1}), & j = 0 \\ 0, & j = 1 \ , \end{cases}$$

where $Sym^k(V)$ denotes the k-th symmetric power of the representation V, e.g. $Sym^k(u + u^{-1}) = u^k + \cdots + u^{-k}$.

The colimit is the cokernel of

$$\varphi_1 \oplus \varphi_0 : K_T(pt) \longrightarrow K_{\mathbb{H}_1}(pt) \oplus K_{\mathbb{H}_0}(pt)$$

restricted to positive powers of z. Now consider the decomposition

$$\mathbb{Z}[u^{\pm 1}, z] = \mathbb{Z}[u + u^{-1}, z] \oplus u^{-1}\mathbb{Z}[u + u^{-1}, z] \ .$$

It is easy to check from this that the cokernel for nontrivial powers of z is isomorphic to the cokernel of φ_0 restricted to $u^{-1}\mathbb{Z}[u + u^{-1}, z]$ and hence is

$$\bigoplus_{k \geq 0} \frac{\mathbb{Z}[u + u^{-1}]}{\langle Sym^{k+1}(u + u^{-1}) \rangle} (z/u)^{k+2}$$

which agrees with the classical result [18].

We now get to the collapse of the spectral sequence. It is sufficient to establish:

Proposition 3.6. *Assume $n > 0$, then $\operatorname{colim}^i_I K^{r+1,n}_{\mathbb{H}_I}(Ad(\mathbb{H}_I)_+)$ is trivial if $i > 0$. For $i = 0$, this group is isomorphic to the free abelian group generated by regular, dominant, level n characters of $\mathbb{L}G$.*

Proof. To simplify the notation, we will abbreviate $K^{r+1,n}_{\mathbb{H}_I}(Ad(\mathbb{H}_I)_+)$ by $K(I)$, hence the letter K denotes the functor $I \mapsto K(I)$. The strategy in proving the above proposition is to show that the functor K decomposes into a sum of functors $K = \bigoplus K_J$, where all the functors K_J have trivial higher derived colimits. We then show that $\mathrm{colim} K_J$ is also trivial for all but one of the functors, and we identify its colimit as the free abelian group generated by the dominant regular characters of level n.

Let $R(T)(n)$ denote the free abelian group generated by characters of $\mathbb{L}G$ of level n. Recall that the fundamental domain for the action of the affine Weyl group $\tilde{W}$ on the set of characters of level n, is the set of characters in the affine alcove Δ (at height n). The affine alcove is an affine r-simplex in the (dual) Lie algebra of the maximal torus of $\mathbb{L}G$. Hence we may index the walls of Δ by the category $\mathcal{C}$ of proper subsets of the $r+1$-element set $\{0,\ldots,r\}$, where $\Delta_I \subseteq \Delta_J$ if $J \subset I$. We get a corresponding decomposition:

$$R(T)(n) = \bigoplus_{J \in \mathcal{C}} R_J(T)(n)$$

where $R_J(T)(n)$ is the free abelian group generated by the characters that are $\tilde{W}$-translates of characters in the interior of the face Δ_I. Let $R^J(T)(n)$ denote the free abelian group generated by the characters in the interior of the face Δ_J, so we have an isomorphism:

$$R^J(T)(n) \otimes \mathbb{Z}[\tilde{W}/W_J] \cong R_J(T)(n), \quad e^\lambda \otimes w \mapsto e^{w^{-1}\lambda}$$

where W_J is the isotropy of the wall Δ_J (and is also the Weyl group of $\mathbb{H}_J$). Consider the push forward map $\pi_I : R(T)(n) \to K(I)$. By pushforward, we mean with respect to the $\hat{A}$-orientation. Therefore, the character of $\pi_I(e^\lambda)$ is given by:

$$\frac{\sum_{w \in W_I} (-1)^{sgn(w)} e^{w(\lambda)}}{A(I)}, \quad A(I) = \prod (e^{\alpha/2} - e^{-\alpha/2}),$$

the product in the Weyl denominator $A(I)$ being taken over the positive roots of H_I. The map π_I is surjective, and one may define K_J to be the functor $I \mapsto \pi_I(R_J(T)(n))$. It is not hard to see that direct sum decomposition of $R(T)(n)$ remains direct when we push forward to $K(I)$, hence we get a direct sum decomposition of functors:

$$K = \bigoplus_{J \in \mathcal{C}} K_J$$

We now proceed to calculate $\mathrm{colim}^i K_J$ by splitting it out of another functor L_J. We define the functor $I \mapsto L_J(I)$ by

$$L_J(I) = R^J(T)(n) \otimes \mathbb{Z}(sgn) \otimes_{\mathbb{Z}[W_J]} \mathbb{Z}[\tilde{W}/W_I]$$

where $\mathbb{Z}(sgn)$ denotes the sign representation of $\mathbb{Z}[W_J]$. One may check that the following map is a retraction of functors:

$$L_J(I) \to K_J(I), \quad e^\lambda \otimes 1 \otimes w \mapsto (-1)^w \pi_I(e^{w^{-1}\lambda})$$

where $(-1)^w$ denotes the sign of the element w. Hence, the groups $\mathrm{colim}^i K_J$ are retracts of the homology of the complex $R^J(T)(n) \otimes \mathbb{Z}(sgn) \otimes_{\mathbb{Z}[W_J]} C_*$, where C_* is the simplicial chain complex of the affine hyperplane given by the $\tilde{W}$ orbit of Δ. Since W_J is a finite subgroup, this simplicial complex is W_J-equivariantly contractible. Thus C_* is W_J-equivariantly equivalent to the constant complex $\mathbb{Z}$ in dimension zero. It follows that $\mathrm{colim}^i K_J = 0$ if $i > 0$, and colim K_J is a retract of $R^J(T)(n) \otimes \mathbb{Z}(sgn) \otimes_{\mathbb{Z}[W_J]} \mathbb{Z}$. Furthermore, if J is nonempty, then the groups $R^J(T)(n) \otimes \mathbb{Z}(sgn) \otimes_{\mathbb{Z}[W_J]} \mathbb{Z}$ are two torsion, hence the map to K_J is trivial. This shows that colim $K_J = 0$ if J is not the empty set. Finally, to complete the proof, one simply observes that for the empty set, the above retraction is an isomorphism, and hence colim $K_\phi = R^\phi(T)(n)$, which is the free abelian group generated by the level n dominant regular weights of $\mathbb{L}G$. □

Remark 3.7. We can calculate the equivariant K-homology $K_{\mathbb{L}G*}(\mathbf{A}(LG))$ using the same spectral sequence. This establishes an isomorphism between $K^*_{\mathbb{L}G}(D_{\mathbb{L}G})$ and $K_{\mathbb{L}G*}(\mathbf{A}(LG))$. Results above imply that the latter group calculates the Verlinde algebra. Infact, the results of [19] may be stated in terms of K-homology, and our spectral sequence argument may be seen as an alternate proof of their results. Recall also that $\mathbf{A}(LG)$ is the classifying space for **proper** actions (i.e. with compact isotropy) so these results also bear an interesting relationship to the Baum-Connes conjecture [7]

Question. For the manifold LM, with frame bundle LP, we can construct a spectrum

$$D_{LM} := \mathrm{holim}_I \; LP_+ \wedge_{H_I} Ad(H_I)_+$$

It would be very interesting to understand something about $K_{\mathbb{T}}(D_{LM})$.

4. Localization Theorems

If E is a $\mathbb{T}$-equivariant complex-oriented multiplicative cohomology theory, and X is a $\mathbb{T}$-space, we have contravariant (j^*) and covariant ($j^!$) homomorphisms associated to the fixedpoint inclusion

$$j : X^{\mathbb{T}} \subset X \;,$$

satisfying

$$j^* j^!(x) = x \cdot e_{\mathbb{T}}(\nu) \;;$$

if the Euler class of the normal bundle ν is invertible, this leads to a close relation between the cohomology of X and $X^{\mathbb{T}}$.

More generally, if $f : M \to N$ is an equivariant map, then its Pontrjagin-Thom transfer is related to the analogous transfer defined by its restriction

$$f^{\mathbb{T}} : M^{\mathbb{T}} \to N^{\mathbb{T}}$$

to the fixedpoint spaces, by a 'clean intersection' formula:

$$j^*_N \circ f^!(-) = f^{\mathbb{T}!}(j^*_M(-) \cdot e_{\mathbb{T}}(\nu(f)|_{M^{\mathbb{T}}})) \;.$$

Definition 4.1. *The fixed-point orientation defined by the Thom class*

$$\mathrm{Th}^\dagger(\nu(f^{\mathbb{T}})) = \mathrm{Th}(\nu(f^{\mathbb{T}})) \cdot e_{\mathbb{T}}(\nu(f)|_{M^{\mathbb{T}}}$$

for the normal bundle of the inclusion of fixed-point spaces is the product of the usual Thom class with the equivariant Euler class of the full normal bundle restricted to the fixed-point space.

Recall that we have the formula $f^{\mathbb{T}!}(-) = f^{\mathbb{T}*}_{PT}(- \cdot \mathrm{Th}(\nu(f^{\mathbb{T}})))$, where $f^{\mathbb{T}}_{PT}$ is the Thom collapse map:

$$f^{\mathbb{T}}_{PT} : \mathbb{R}^k_+ \wedge N_+ \to (M^{\mathbb{T}})^{\nu(f^{\mathbb{T}})}$$

corresponding to $f^{\mathbb{T}}$ (having first replaced $f^{\mathbb{T}}$ by an embedding of $M^{\mathbb{T}}$ into $N^{\mathbb{T}} \times \mathbb{R}^k$ for large k). In our new notation the clean intersection formula becomes

$$j^*_N \circ f^! = f^{\mathbb{T}\dagger} \circ j^*_M$$

with a new Pontrjagin-Thom transfer

$$f^{\mathbb{T}\dagger}(-) = f^{\mathbb{T}*}_{PT}(- \cdot \mathrm{Th}^{\dagger}(\nu(f^{\mathbb{T}}))) \; .$$

In the case of most interest to us (free loopspaces), we identified the normal bundle above, in §2; using that description, we have

$$e_{\mathbb{T}}(\nu(M \subset LM)) = \prod_{0 \neq k \in \mathbb{Z}; i} (e(L_i) +_E [k](q)),$$

where the L_i are the line bundles in a formal decomposition of TM, q is the Euler class of the standard one-dimensional complex representation of $\mathbb{T}$, and $+_E$ is the sum with respect to the formal group law defined by the orientation of E. It may not be immediately obvious, but it turns out that such a formula implies that the fixed-point orientation defined above will have good multiplicative properties.

Such Weierstrass products sometimes behave better when ‘renormalized’, by dividing by their values on constant bundles [2]. If E is $K_{\mathbb{T}}$ with the usual complex (Todd) orientation, we have

$$e(L) +_K [k](q) = 1 - q^k L \; ;$$

but for our purposes things turn out better with the Atiyah-Bott-Shapiro **spin** orientation; in that case the corresponding Euler class is

$$(q^k L)^{1/2} - (q^k L)^{-1/2} \; .$$

The square roots make sense under the simple-connectivity assumptions on G mentioned in the introduction:

To be precise, let V be a representation of LG with an intertwining action of $\mathbb{T}$. We restrict ourselves to finite type representations V which (for lack of a better name) we call **symmetric**, i.e. such that V is equivalent to the representation of LG obtained by composing V with the involution of LG which reverses the orientation of the loops. The restriction of the representation V to the constant loops $\mathbb{T} \times G \subset \mathbb{T} \tilde{\times} LG$ has a decomposition

$$V = V^{\mathbb{T}} \oplus \sum_{k \neq 0} V_k \; q^k$$

where V_k are representations of G, and q denotes the fundamental representation of $\mathbb{T}$. Let $V(m)$ be the finite dimensional subrepresentation

$$V(m) = V^{\mathbb{T}} \oplus \sum_{0 < |k| \leq m} V_k \; q^k \; ;$$

the symmetry assumption implies that $V_k = V_{-k}$ as representations of G, so this can be rewritten

$$V(m) = V^{\mathbb{T}} \oplus \sum_{0 < k \leq m} V_k \; (q^k \oplus q^{-k}) \; .$$

At this point we need the following

Proposition 4.2. *If G is a compact Lie group, and W is an m-dimensional complex spin representation of G, then the representation $\tilde{W} = W \otimes (q^k \oplus q^{-k})$ of $\mathbb{T} \times G$ admits a canonical spin structure.*

Proof. The representation $q^k \oplus q^{-k}$ of $\mathbb{T}$ admits a unique spin structure. Since W is also endowed with a spin structure, the representation $W \otimes (q^k \oplus q^{-k})$ admits a canonical spin structure defined by their tensor product. □

Remark 4.3. *If G is simply connected, then any representation W of G admits a unique spin structure.*

This justifies the square roots of the formal line bundles appearing in the restriction of TLM to M. The resulting renormalized Euler class

$$\prod_{k \neq 0} \frac{(q^k L)^{1/2} - (q^k L)^{-1/2}}{q^{k/2} - q^{-k/2}}$$

is a product of terms of the form

$$(1 - q^k L)(-q^k L)^{-1/2}(q^{-k}L)^{1/2}(1 - q^k L^{-1})$$

divided by terms of the form

$$(1 - q^k)(-q^k)^{-1/2}(q^{-k})^{1/2}(1 - q^k)\ ,$$

(where now all k's are positive) yielding a unit

$$\epsilon_{\mathbb{T}}(L) = \prod_{k \geq 1} \frac{(1 - q^k L)(1 - q^k L^{-1})}{(1 - q^k)^2}$$

in the ring $\mathbb{Z}[L^{\pm}][[q]]$.

Following 4.1, we can reformulate the localization theorem in terms of a new orientation, obtained by multiplying the ABS Thom class by this unit, to get **precisely** the Mazur-Tate normalization

$$\sigma(L, q) = (L^{1/2} - L^{-1/2}) \prod_{k \geq 1} \frac{(1 - q^k L)(1 - q^k L^{-1})}{(1 - q^k)^2}$$

for the Weierstrass sigma-function as Thom class for a line bundle L. This extends by the splitting principle to define the orientation giving the Witten genus [32].

5. One moral of the story

Since the early 80's physicists have been trying to interpret

$$M \mapsto K_{\mathbb{T}}(LM)$$

as a kind of elliptic cohomology theory; but of course we know better, because we know that mapping-space constructions (such as free loop spaces) don't preserve cofibrations.

Now it is an easy exercise in commutative algebra to prove that

$$\mathbb{Z}((q)) := \mathbb{Z}[[q]][q^{-1}]$$

is flat over

$$K_{\mathbb{T}} = \mathbb{Z}[q^{\pm}] \ ,$$

for the completion of a Noetherian ring, eg $\mathbb{Z}[q]$, at an ideal, eg (q), defines a flat [6](§10.14) $\mathbb{Z}[q]$-algebra $\mathbb{Z}[[q]]$. Flat modules pull back to flat modules [8] (Ch I §2.7), so it follows that $\mathbb{Z}((q))$ is flat over the localization

$$\mathbb{Z}[q^{\pm}] := \mathbb{Z}[q][q^{-1}] = \mathbb{Z}[q, q^{-1}] \ .$$

The Weierstrass product above is a genuine formal power series in q, so for questions involving the Witten genus it is formally easier to work with the functor defined on finite $\mathbb{T}$-CW spaces by

$$X \mapsto K^*_{\mathbb{T}}(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) := K^*_{\hat{\mathbb{T}}}(X) \ .$$

This takes cofibrations to long exact sequence of $\mathbb{Z}((q))$-modules. Its real virtue, however, is that it satisfies a strong localization theorem:

Theorem 5.1. *If X is a finite $\mathbb{T}$-CW space, then restriction to the fixedpoints defines an* **isomorphism**

$$j^* : K_{\hat{\mathbb{T}}}(X) \cong K_{\hat{\mathbb{T}}}(X^{\mathbb{T}}) \ .$$

Proof, by skeletal induction; based on the

Lemma 5.2. *If $C \subset \mathbb{T}$ is a proper closed subgroup, then*

$$K^*_{\hat{\mathbb{T}}}(\mathbb{T}/C) = 0 \ .$$

Proof. If C is cyclic of order $n \neq 1$, then

$$K_{\hat{\mathbb{T}}}(\mathbb{T}/C) = K_{\mathbb{T}}(\mathbb{T}/C) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = \mathbb{Z}[q]/(q^n - 1) \otimes_{\mathbb{Z}[q]} \mathbb{Z}((q))$$

is zero, since

$$-1 = (q^n - 1) \cdot \sum_{k \geq 0} q^{nk} = 0 \ .$$

On the other hand, if $C = \{0\}$, then

$$K_{\hat{\mathbb{T}}}(T) = \mathbb{Z} \otimes_{\mathbb{Z}[q]} \mathbb{Z}((q)) \ ,$$

with $\mathbb{Z}$ a $\mathbb{Z}[q]$-module via the specialization $q \to 1$; but by a similar argument, the resulting tensor product again vanishes. □

The functor $K^*_{\hat{\mathbb{T}}}$ extends to an equivariant cohomology theory on the category of $\mathbb{T}$-CW spaces, which sends a general (large) object X to the pro-$\mathbb{Z}((q))$-module

$$\{K^*_{\hat{\mathbb{T}}}(X_i) \mid X_i \in \text{finite } \mathbb{T} - \text{CW} \subset X\} \ ,$$

[5] (appendix). We can thus extend the claim above:

Corollary 5.3. *For a general $\mathbb{T}$-CW-space X, the restriction-to-fixedpoints map*

$$K^*_{\hat{\mathbb{T}}}(X) \to K^*_{\hat{\mathbb{T}}}(X^{\mathbb{T}})$$

is an isomorphism of pro-objects. Moreover, if the fixedpoint space $X^{\mathbb{T}}$ is a finite CW-space, then the pro-object on the left is isomorphic to the **constant** *pro-object on the right.*

The free loopspace LX of a CW-space X is weakly $\mathbb{T}$-homotopy equivalent to a $\mathbb{T}$-CW-space, by a map which preserves the fixed-point structure [25] (§1.1).

Theorem 5.4.

$$M \mapsto K_{\hat{\mathbb{T}}}(LM) := K_{\mathrm{Tate}}(M)$$

is *a cohomology theory, after all!*

Remark 5.5. This seems to be what the physicists have been trying to tell us all along: they probably thought (as the senior author did [27]) that the formal completion was a minor technical matter, not worth making any particular fuss about. Of course our construction is a completion of a much smaller (elliptic) cohomology theory, whose coefficients are modular forms, with the completion map corresponding to the q-expansion. The geometry underlying modularity is still [10] quite mysterious.

Remark 5.6. A cohomology theory defined on finite spectra extends to a cohomology theory on all spectra [1]; moreover, any two extensions are equivalent, and the equivalence is unique up to phantom maps. For the case at hand, it is clear that this cohomology theory is equivalent to the formal extension $K((q))$, where K is complex K-theory, with q a parameter in degree zero. Hovey and Strickland [20] have shown that an evenly graded spectrum does not support phantom maps, so our cohomology theory is uniquely equivalent, as a homotopy functor, to $K((q))$.

However, there is more to our construction than a simple homotopy functor: it comes with a natural (fixed-point) orientation, which defines a systematic theory of Thom isomorphisms. In the terminology of [4], it is represented by an **elliptic** spectrum, associated to the Tate curve over $\mathbb{Z}((q))$; its natural orientation is defined by the σ-function of §4.

Remark 5.7. Let $H_{\mathbb{T}}$ denote $\mathbb{T}$-equivariant singular Borel cohomology with rational coefficients; then $H^*_{\mathbb{T}}(pt) = \mathbb{Q}[t]$, where t has degree two. We have a localization theorem

$$H^*_{\mathbb{T}}(X)[t^{-1}] = H^*_{\mathbb{T}}(X^{\mathbb{T}})[t^{-1}] = H^*(X^{\mathbb{T}}) \otimes \mathbb{Q}[t^{\pm}]$$

for finite complexes, and can therefore play the same game as before, and observe that there is a unique extension $\hat{H}_{\mathbb{T}}(-)$ of $H_{\mathbb{T}}(-)[t^{-1}]$ to all $\mathbb{T}$-spectra, with a localization theorem valid for an arbitrary $\mathbb{T}$-spectrum X:

$$\hat{H}^*_{\mathbb{T}}(X) = \hat{H}^*_{\mathbb{T}}(X^{\mathbb{T}}) \ .$$

Jones and Petrack [21] have constructed such a theory over the real numbers, together with the analogous fixedpoint orientation - which in their case is (a rational version of) the $\hat{A}$-genus.

Results of this sort (which relate oriented equivariant cohomology theories on free loopspaces to cohomology theories on the fixedpoints, with related (but distinct) formal groups, is part of an emerging understanding of what homotopy theorists call ‘chromatic redshift’ phenomena, cf. [2, 3, 31].

Remark 5.8. Our completion of equivariant K-theory is the natural repository for characters of **positive-energy** representations of loop groups; it is **not** preserved by the orientation-reversing involution $\lambda \mapsto \lambda^{-1}$ of $\mathbb{T}$. It is in some sense a **chiral** completion.

Remark 5.9. The completion theorem above is a specialization of Segal's original localization theorem [29], which says that $K_{\mathbb{T}}(X)$, considered as a sheaf over the multiplicative groupscheme Spec $K_{\mathbb{T}} = \mathbb{G}_m$ (cf. [28]), has for its stalks over generic (ie nontorsion) points, the K-theory of the fixed point space. The Tate point

$$\text{Spec } \mathbb{Z}((q)) \to \text{Spec } \mathbb{Z}[q^{\pm}]$$

is an example of such a generic point, perhaps too close to zero (or infinity) to have received the attention it seems to deserve.

In fact, we can play a similar game for any oriented theory $E_{\mathbb{T}}$. Let $\mathcal{E}$ denote the union of the one-point compactifications of all representations of $\mathbb{T}$ which do not contain the trivial representation (cf. [25]). Then the theory $E_{\mathbb{T}} \wedge \mathcal{E}$ satisfies a strong localization theorem.

6. Toward Pontrjagin-Thom duality

One might hope for a construction which associates to a map $f : M \to N$ of manifolds (with suitable properties), a morphism

$$Lf^{PT} : LN^{-\mathbf{T}LN} \to LM^{-\mathbf{T}LM}$$

of prospectra. This seems out of reach at the moment, but some of the constructions sketched above can be interpreted as partial results in this direction.

In particular, there is at least a **cohomological** candidate for a PT dual

$$j^{PT} : LM^{-\mathbf{T}LM} \to M^{-TM}$$

to the fixedpoint inclusion $j : M \subset LM$. To describe it, we should first observe that there is a morphism

$$M^{-TLM} \to L^{\dagger}M^{-TLM}$$

of prospectra, constructed by pulling back the tangent bundle of LM along the fixedpoint inclusion. To be more precise we need to note that the $\mathbb{T}$-fixedpoints of the thickened loopspace is a bundle $L^{\dagger}M^{\mathbb{T}} \to M$ with contractible fiber; the choice of a section defines a composition

$$\tilde{j} : M^{-TLM} \sim ((L^{\dagger}M)^{\mathbb{T}})^{-TLM} \to L^{\dagger}M^{-TLM} \, ;$$

of course $M^{-TLM} = (M^{-TM})^{-\nu}$, where $\nu = T_M \otimes (\mathbb{C}[q^{\pm}]/\mathbb{C})$ is the normal bundle described in §2.

Now according to the localization theorem above, the map induced on $K_{\hat{\mathbb{T}}}$ by $\tilde{j}$ is an isomorphism, so it makes sense to define

$$j^! := (\tilde{j}^*)^{-1} \circ \phi_{\nu}^{-1} : K_{\hat{\mathbb{T}}}(M^{-TM}) \to K_{\hat{\mathbb{T}}}(M^{-TM-\nu}) \to K_{\hat{\mathbb{T}}}(LM^{-\mathbf{T}LM}) \, ,$$

where ϕ_{ν} is the Thom pro-isomorphism associated to the filtered vector bundle ν.

A good general theory of PT duals would provide us with a commutative diagram

$$\begin{array}{ccc} LN^{-\mathbf{T}LN} & \xrightarrow{Lf^{PT}} & LM^{-\mathbf{T}LM} \\ \downarrow j_N^{PT} & & \downarrow j_M^{PT} \\ N^{-TN} & \xrightarrow{f^{PT}} & M^{-TM} \end{array} ,$$

so it follows from the constructions above that

$$Lf^! := j_N^! \circ f^! \circ (j_M^!)^{-1}$$

defines a formally consistent theory of PT duals for $K_{\hat{\mathbb{T}}}$.

For example, (the evaluation at 1 of) the composition

$$K_{\mathrm{Tate}}(M) = K_{\hat{\mathbb{T}}}(LM) \cong K_{\hat{\mathbb{T}}}(LM^{-\mathbf{T}LM}) \to K_{\hat{\mathbb{T}}}(M^{-TM}) \to K_{\mathbb{T}}(S^0) \cong K_{\mathrm{Tate}}(pt)$$

is (the q-expansion of) the Witten genus; more generally, our *ad hoc* construction $Lf^!$ for $K_{\hat{\mathbb{T}}}$ agrees with the covariant construction $f^\dagger$ defined for the underlying manifold by the fixedpoint (or σ) orientation.

The **un**completed K-theory $K_{\mathbb{T}}(LM^{-\mathbf{T}LM})$ is also accessible, through the spectral sequence of a colimit, but our understanding of it is at an early stage. It is of course not a cohomological functor of M, but it does not seem unreasonable to hope that some of its aspects may be within reach through similar PT-like constructions.

These fragmentary constructions suffice to show that $K_{\hat{\mathbb{T}}}(LM^{-\mathbf{T}LM})$ has enough of a Frobenius (or ambialgebra) structure to define a two-dimensional topological field theory, which assigns to a closed surface of genus g, the class

$$\pi^\dagger \; \epsilon_{\mathbb{T}}(TM)^g \in \mathbb{Z}((q)) \; ,$$

with $\epsilon_{\mathbb{T}}$ the characteristic class defined in §4, and $\pi^\dagger$ the pushforward of M to a point defined by the fixed-point (σ) orientation. When $g = 0$ this is the Witten genus of M, and when $g = 1$ it is the Euler characteristic.

Finally: our construction is, from its beginnings onward, formulated in terms of **closed** strings. Stolz and Teichner [30] have produced a deeper approach to a theory of elliptic objects, which promises to incorporate interesting aspects of **open** strings as well. However, their theory is in some ways quite complicated; and our hope is that their global theory, combined with the quite striking computational simplicity of the very local theory sketched here, will lead to something **really** interesting.

7. Appendix: The Tits Building of a loop group

Let G be a simply-connected compact Lie group of rank n. Let LG denote the loop group of G. For the sake of convenience, we will work with a smaller (but equivalent) model for LG, which we now describe:

Let $G_{\mathbb{C}}$ be the complexification of the group G. Since $G_{\mathbb{C}}$ has the structure of a complex affine variety, we may define the group $L_{alg}G_{\mathbb{C}}$ to be the group of polynomial maps from $\mathbb{C}^*$ to $G_{\mathbb{C}}$. Let $L_{alg}G$ be the subgroup of $L_{alg}G_{\mathbb{C}}$ consisting of maps taking the unit circle into $G \subset G_{\mathbb{C}}$. The inclusion $L_{alg}G \subset LG$ is a homotopy equivalence in the category of $\mathbb{T}$-spaces. In this section, we start our constructions with $L_{alg}G$. We then use these to draw conclusions about the (smooth) Tits building $\mathbf{A}(LG)$. We begin with some preliminaries about compact Lie groups.

Fix a maximal torus T of G, and consider the adjoint action of $Lie(T)$ on the complexified Lie algebra $\mathcal{G}_{\mathbb{C}}$ of the group G. Let us briefly recall the structure of this representation [9]. The nonzero (infinitesimal) characters of the adjoint representation restricted to $Lie(T)$ are known as roots, which we denote by the letter R. We may pick (and we do) a linearly independent set α_i, $1 \leq i \leq n$ of roots (called simple roots) such that $R = R_+ \coprod -R_+$, where any element of R_+ is a positive integral linear combination of the simple roots. There exists a unique root α_0 (called the highest root) with the property that $\alpha_0 - \alpha$ is a nonnegative linear combination of the simple roots for any root α. Let $\mathcal{G}_\alpha \subset \mathcal{G}_{\mathbb{C}}$ denote the eigenspace of a root α. It is known that $\mathcal{G}_\alpha$ is one dimensional, and that the Lie algebra generated by $\mathcal{G}_\alpha$ and $\mathcal{G}_{-\alpha}$ is canonically isomorphic to $sl_2(\mathbb{C})$. On exponentiating this subalgebra, we get a homomorphism of complex Lie groups $\varphi_\alpha : SL_2(\mathbb{C}) \to G_{\mathbb{C}}$. Since G was simply connected, it can be shown that this map is injective and restricts to an injection $\varphi_\alpha : SU(2) \to G$. Let φ_i, $0 \leq i \leq n$ denote the maps so constructed for the roots α_i, $0 \leq i \leq n$.

We use these maps to define compact subgroups G_i of $L_{alg}G$ as follows:

$$G_i = \{z \mapsto \varphi_i \begin{pmatrix} a & b \\ c & d \end{pmatrix}\} \qquad \text{if} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2), \quad i > 0$$

$$G_0 = \{z \mapsto \varphi_0 \begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix}\} \quad \text{if} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2),$$

Remark 7.1. Note that each G_i is a compact subgroup of $L_{alg}G$ isomorphic to $SU(2)$. Moreover, G_i belongs to the subgroup G of constant loops if $i \geq 1$. The circle group $\mathbb{T}$ preserves each G_i, acting trivially on G_i for $i \geq 1$, and nontrivially on G_0.

Definition 7.2. *For any proper subset $I \subset \{0, 1, \ldots, n\}$, define the parabolic subgroup $H_I \subset L_{alg}G$ to be the group generated by T and the groups G_i, $i \in I$. For the empty set, we define H_I to be T. It follows from 7.1 that each H_I is preserved under the action of $\mathbb{T}$.*

Remark 7.3. The groups H_I are compact Lie [26]. Moreover, H_I is isomorphic to its image in G, under the map $e : L_{alg}G \to G$ that evaluates a loop at $1 \in S^1$. Notice that for $I = \{1, \ldots, n\}$, $H_I = G$. Notice also that $\mathbb{T}$ acts nontrivially on H_I if and only if $0 \in I$.

We are now ready to define the Tits building $\mathbf{A}(L_{alg}G)$.

Definition 7.4. *Let $\mathbf{A}(L_{alg}G)$ be the homotopy colimit:*

$$\mathbf{A}(L_{alg}G) = \operatorname{hocolim}_{I \in \mathcal{C}} L_{alg}G/H_I$$

where $\mathcal{C}$ denotes the poset category of proper subsets of $\{0, 1, \ldots, n\}$.

We now come to the main theorem:

Theorem 7.5. *The space $\mathbf{A}(L_{alg}G)$ is $\mathbb{T}\tilde{\times} L_{alg}G$-equivariantly contractible. In other words, given a compact subgroup $K \subset \mathbb{T}\tilde{\times} L_{alg}G$, then the fixed point space $\mathbf{A}(L_{alg}G)^K$ is contractible.*

Proof. A proof of the contractibility of $\mathbf{A}(L_{alg}G)$ was given in [26]. We use some of the ideas from that paper, but our proof is different in flavour.

Mitchell expresses the space $\mathbf{A}(L_{alg}G)$ as the following:

$$\mathbf{A}(L_{alg}G) = (L_{alg}G/T \times \Delta)/\sim$$

where Δ is the n-simplex, and $(aT, x) \sim (bT, y)$ if and only if $x = y \in \mathring{\Delta}_I$ and $aH_I = bH_I$. Here we have indexed the walls of Δ by the category $\mathcal{C}$, and denoted the interior of Δ_I by $\mathring{\Delta}_I$.

Let $\mathbb{R} \oplus L_{alg}\mathcal{G}$ be the Lie algebra of the extended loop group $\mathbb{T}\tilde{\times}L_{alg}G$. Consider the affine subspace

$$\mathbf{A} = 1 \oplus L_{alg}\mathcal{G} \subset \mathbb{R} \oplus L_{alg}\mathcal{G}\ .$$

The adjoint action of $L_{alg}G$ on $\mathbf{A}$ is given by

$$Ad_{f(z)}(1, \lambda(z)) = (1, Ad_{f(z)}(\lambda(z)) + zf'(z)f(z)^{-1})$$

This action extends to an affine action of $\mathbb{T}\tilde{\times}L_{alg}G$. The identification of $\mathbf{A}$ with $\mathbf{A}(L_{alg}G)$ is given as follows. First consider the affine alcove:

$$\{(1, h) \in 1 \oplus Lie(T) \,|\, \alpha_i(h) \geq 0,\ i > 0,\ \alpha_0(h) \leq 1\}\ .$$

We may identify this space with Δ using the roots α_i. So, for example, the codimension 1 face Δ_i for $1 \leq i \leq n$ is identified with the subset of the alcove $\{(1, h) \,|\, \alpha_i(h) = 0\}$, for $i \neq 0$, and Δ_0 is identified with the subspace $\{(1, h) \,|\, \alpha_0(h) = 1\}$. General facts about Loop groups [22, 26] show that the surjective map

$$L_{alg}G \times \Delta \longrightarrow \mathbf{A}, \quad (f(z), y) \mapsto Ad_{f(z)}(y)$$

has isotropy H_I on the subspace Δ_I. Hence it factors through a $\mathbb{T}\tilde{\times}L_{alg}G$-equivariant homeomorphism between $\mathbf{A}(L_{alg}G)$ and the affine space $\mathbf{A}$. Notice that any compact subgroup $K \subset \mathbb{T}\tilde{\times}L_{alg}G$ admits a fixed point on $\mathbf{A}(L_{alg}G)$. Hence, the space $\mathbf{A}(L_{alg}G)^K$ is also affine. This completes the proof. □

Remark 7.6. *The affine space $\mathbf{A}$ above should be thought of as the space of (algebraic) connections on the trivial G-bundle on S^1. The action of the group $L_{alg}G$ then corresponds to the action of the (algebraic) gauge group. This analogy can be taken one step further to define the space of (algebraic) holonomies:*

$$\mathcal{S}_{alg} = \{g : \mathbb{R} \to G \,|\, g(t) = f(e^{2\pi it}) \cdot \exp(tX);\ f(z) \in \Omega_{alg}G, X \in Lie(G)\}$$

topologized as a quotient of $\Omega_{alg}G \times Lie(G)$, with the action of $\mathbb{T}\tilde{\times}L_{alg}G$ given by left multiplication (see discussion before 2.4).

In fact, in [26] *Mitchell shows that $\mathbf{A}(L_{alg}G)$ is $L_{alg}G$ equivariantly homeomorphic to the space $\mathcal{S}_{alg}$ (attributing the result to Quillen).*

We now define the smooth Tits building

Definition 7.7. *Let $\mathbf{A}(LG)$ be the homotopy colimit:*

$$\mathbf{A}(LG) = \text{hocolim}_{I \in \mathcal{C}}\ LG/H_I = LG \times_{L_{alg}G} \mathbf{A}(L_{alg}G)\ .$$

Theorem 7.8. *The smooth Tits building $\mathbf{A}(LG)$ is $\mathbb{T}\tilde{\times}LG$-equivariantly contractible.*

Proof. By the above remark, the space $\mathbf{A}(LG)$ is LG-equivariant homeomorphic to the space $LG \times_{L_{alg}G} \mathcal{S}_{alg}$. This is easily seen to be LG-equivariantly homeomorphic to the space of smooth maps:

$$\mathcal{S} = \{g : \mathbb{R} \to G \,|\, g(t) = f(e^{2\pi i t}) \cdot \exp(tX);\ f(z) \in \Omega G, X \in Lie(G)\}$$

Equivalently, we may describe $\mathcal{S}$ as

$$\mathcal{S} = \{g : \mathbb{R} \to G,\ g(0) = 1,\ g(t+1) = g(t) \cdot g(1)\}\ ;$$

from which it follows (see 2.4) that $\mathcal{S}$ is LG-equivariantly homeomorphic to the space of connections on the trivial G-bundle on S^1, which we denote by $\mathcal{A}(S^1 \times G)$. This sequence of LG-equivariant homeomorphisms from $\mathbf{A}(LG)$ to $\mathcal{A}(S^1 \times G)$ is compatible with the action of $\mathbb{T}$. The proof is now complete, since the space of connections $\mathcal{A}(S^1 \times G)$ is clearly $\mathbb{T} \tilde{\times} LG$-equivariantly contractible. □

References

1. F. Adams, A variant of E.H. Brown's representability theorem, Topology, Vol. 10 (1971) 185-198.
2. M. Ando, J. Morava, A Renormalized Riemann-Roch formula and the Thom isomorphism for the free loopspace, in the Milgram Festschrift, Contemp. Math. 279 (2001)
3. ——, ———, H. Sadofsky, Completions of $\mathbb{Z}/(p)$-Tate cohomology of periodic spectra, Geometry & Topology 2 (1998) 145 - 174
4. —–, M. Hopkins, N. Strickland, Elliptic spectra, the Witten genus and the theorem of the cube, Inv. Math. 146 (2001) 595 - 687
5. M. Artin, B. Mazur **Etale homotopy**, Springer LNM 100 (1969)
6. M. Atiyah, I. MacDonald, **Commutative Algebra**
7. P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras, p. 240 - 291 in C^*-**algebras**: 1943-1993, Contemporary Math 167, AMS (1994)
8. N. Bourbaki, **Algebre Commutatif**
9. T. Brocker, T. tom Dieck, **Representations of Compact Lie groups**, Springer GTM 98.
10. JL Brylinski, Representations of loop groups, Dirac operators on loop space, and modular forms, Topology 29 (1990) 461 - 480
11. M. Chas, D. Sullivan, String topology, available at `math.AT/9911159`
12. J.D. Christensen, D.C. Isaksen, Duality and prospectra (in progress)
13. R. Cohen, Multiplicative properties of Atiyah duality, available at `math.AT/0403486`
14. ———, V. Godin, A polarized view of string topology, available at `math.AT/0303003`
15. ———, J.D.S Jones, A homotopy-theoretic realization of string topology, available at `math.GT/0107187`
16. ———, ———, G. Segal, Floer's infinite-dimensional Morse theory and homotopy theory, in **The Floer memorial volume**, Progr. Math 133, Birkhäuser (1995)
17. ———, A. Stacey, Fourier decomposition of loop bundles, available at `math.AT/0210351`
18. D. Freed, The Verlinde Algebra is Twisted Equivariant K-Theory, available at `math.RT/0101038`
19. D. Freed, M. Hopkins, C. Teleman, Twisted K-theory and loop group representations I, available at `math.AT/0312155`
20. M. Hovey, N.P. Strickland, Morava K-theories and localization, Mem. Amer. Math. Soc. 139 (1999), No.666.
21. J.D.S Jones, S.B. Petrack, The fixed point theorem in equivariant cohomology, Transactions of the Amer. Math. Soc., Vol. 322, No. 1 (1990) 35-49.
22. V.G. Kac, **Infinite dimensional Lie algebras**, Cambridge University Press, 1985
23. N. Kitchloo, **Topology of Kac-Moody groups**, Thesis, M.I.T., 1998.
24. J. Klein, The dualizing spectrum of a topological group, Math. Ann 319 (2001) 421 - 456
25. G. Lewis, J.P. May, M. Steinberger, **Equivariant stable homotopy theory**, Springer LNM 1213 (1986)
26. S.A. Mitchell, Quillen's theorem on buildings and the loops on a symmetric space, Enseign. Math. 34 (1988) 123-166

27. J. Morava, Forms of K-theory, Math Zeits. 201 (1989)
28. I. Rosu, Equivariant K-theory and equivariant cohomology, Math. Zeits. 243 (2003) 423-448.
29. G. Segal, Equivariant K-theory, Inst. Hautes Études Sci. Publ. Math. No. 34 (1968) 129-151.
30. S. Stolz, P. Teichner, What is an elliptic object? available at `math.ucsd.edu`
31. T. Torii, On degeneration of one-dimensional formal group laws and stable homotopy theory, AJM 125 (2003) 1037-1077
32. D. Zagier, Note on the Landweber-Stong elliptic genus, in **Elliptic curves and modular forms in algebraic topology**, ed. P. Landweber, Springer 1326 (1988)

Department of Mathematics, UCSD, LaJolla, CA 92093.
Department of Mathematics, Johns Hopkins University, Baltimore, Maryland 21218.
E-mail address: `nitu@math.ucsd.edu, jack@math.jhu.edu`