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COHOMOLOGY SPLITTINGS OF STIEFEL MANIFOLDS

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Abstract

The complex Stiefel manifolds admit a stable decomposition as Thom spaces of certain bundles over Grassmannians. The purpose of the paper is to identify the splitting in any complex oriented cohomology theory.

Introduction

The space of k-frames in $\mathbb{C}^n$, $V_{n,k}$, admits an obvious free action of the unitary group U_k. Consequently one has a principal U_k-bundle with the total space being $V_{n,k}$ and the base space being the Grassmann manifold of k-planes in $\mathbb{C}^n$, $G_{n,k}$. Given a representation ρ of U_k, one constructs an associated vector bundle over $G_{n,k}$ via the Borel construction on the above principal bundle. Let $G^{\rho}_{n,k}$ denote the Thom space of this bundle. Let U_n^+ denote the unitary group with a disjoint base-point. Miller [**4**] proves that there is a stable homotopy equivalence

$$U_n^+ \cong \bigvee_{k=0}^{n} G_{n,k}^{\mathrm{ad}_k}$$

where ad_k denotes the adjoint action of U_k on its Lie algebra. An observation of naturality allows him to conclude that

$$U^+ \cong \bigvee_{k=0}^{\infty} BU_k^{\mathrm{ad}_k}$$

where U is the infinite unitary group. This splitting can be seen as a special case of a more general result on the splitting of Stiefel manifolds. Let $V_{n+t,n}$ be the Stiefel manifold of n-frames in $\mathbb{C}^{n+t}$. Miller [**4**] has shown that there is a stable homotopy equivalence

$$V_{n+t,n}^+ \cong \bigvee_{k=0}^{n} G_{n,k}^{\mathrm{ad}_k \oplus \mathrm{tcan}_k}$$

where ad_k denotes the adjoint action of U_k on its Lie algebra and tcan_k denotes t copies of the canonical representation can_k of U_k on $\mathbb{C}^k$. Once we identify $V_{n+t,n}$ with the coset space U_{n+t}/U_t, naturality gives us

$$U/U_t^+ \cong \bigvee_{k=0}^{\infty} BU_k^{\mathrm{ad}_k \oplus \mathrm{tcan}_k}.$$

Notice that the case $t = 0$ is the splitting of the unitary group described earlier. Another version of these splittings is described in [**1**]. This paper is self-contained.

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Section 1 recalls the splitting as constructed in [**1**]. The rest of the paper is devoted to identifying the above splittings in any complex-oriented cohomology theory. Our method uses, in an essential way, the description of the multiplication map on U^+ in terms of the splitting of the unitary group.

In Section 3 we describe the stable nature of the multiplication map on the unitary group. Notice that the diagonal inclusion map $U_i \times U_j \hookrightarrow U_{i+j}$ is covered by the obvious map $\mathrm{ad}_i \oplus \mathrm{ad}_j \hookrightarrow \mathrm{ad}_{i+j}$. Hence we get a map

$$BU_i^{\mathrm{ad}_i} \wedge BU_j^{\mathrm{ad}_j} \xrightarrow{\mu} BU_{i+j}^{\mathrm{ad}_{i+j}}.$$

Let us denote the splitting maps by $\sigma_k : BU_k^{\mathrm{ad}_k} \longrightarrow U^+$. Let E be any complex-oriented cohomology theory and

$$M : U^+ \wedge U^+ \longrightarrow U^+$$

be the multiplication map on the unitary group. We have the following theorem.

THEOREM A. *The diagram in Figure* 1 *commutes in* $\tilde{E}^*$.

$$\begin{array}{ccc} BU_j^{\mathrm{ad}_i} \wedge BU_i^{\mathrm{ad}_j} & \xrightarrow{\mu} & BU_{i+j}^{\mathrm{ad}_{i+j}} \\ \big\downarrow{\scriptstyle \sigma_i \wedge \sigma_j} & & \big\downarrow{\scriptstyle \sigma_{i+j}} \\ U^+ \wedge U^+ & \xrightarrow{M} & U^+ \end{array}$$

FIGURE 1.

In Section 4 we recall the Hopf-algebra structure on $\tilde{E}^*(U^+)$. The following two theorems are standard. We have chosen to re-prove them for the sake of completeness. The Hopf-algebra $\tilde{E}^*(U^+)$ is an inverse limit of Hopf-algebras $\tilde{E}^*(U_n^+)$. The structure of $\tilde{E}^*(U_n^+)$ is given by the following.

THEOREM B. *$\tilde{E}^*(U_n^+)$ has the structure of an exterior algebra as a graded Hopf-algebra over E^*. More precisely, we have $\tilde{E}^*(U_n^+) = \Lambda(a_0^{(n)}, a_1^{(n)}, \ldots, a_{n-1}^{(n)})$ where $a_i^{(n)}$ is a primitive element of homogeneous degree $2i+1$. Under the inclusion $U_{n-1} \hookrightarrow U_n$ given by*

$$v \longmapsto \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$$

in the standard basis, the element $a_i^{(n)}$ maps to $a_i^{(n-1)}$ for $i < n-1$ and 0 for $i = n-1$. Moreover, under the splitting map

$$\sigma_1 : \Sigma(\mathbb{C}P^{n-1^+}) = G_{n,1}^{\mathrm{ad}_1} \longrightarrow U_n^+$$

the element $a_i^{(n)}$ maps to the suspension of the element x^i, where x is the orientation in $E^2(\mathbb{C}P^{n-1})$.

There is the analogous result for Stiefel manifolds: the algebra $\tilde{E}^*(U/U_t^+)$ is an inverse limit of algebras $\tilde{E}^*(V_{n+t,n}^+)$. The structure of $\tilde{E}^*(V_{n+t,n}^+)$ is given by the following theorem.

THEOREM C. *$\tilde{E}^*(V^+_{n+t,n})$ can be identified with the subalgebra of $\tilde{E}^*(U^+_{n+t})$ given by $\tilde{E}^*(V^+_{n+t,n}) = \Lambda(a^{(n)}_t, a^{(n)}_{t+1}, \ldots, a^{(n)}_{n-1+t})$. Under the map $V_{n-1+t,n-1} \hookrightarrow V_{n+t,n}$ induced by $U_{n-1+t} \hookrightarrow U_{n+t}$, the element $a^{(n)}_i$ maps to $a^{(n-1)}_i$ for $t \leqslant i < n-1+t$ and 0 for $i = n-1+t$.*

By this theorem we can view $\tilde{E}^*(U/U^+_t)$ as a suitably completed exterior algebra on generators a_i of homogeneous degree $2i+1$ for $i \geqslant t$. It is also shown in Section 4 that $\tilde{E}^*(BU_k^{\mathrm{ad}_k \oplus t\mathrm{can}_k})$ is a free module of rank one over $E^*(BU_k)$ generated by a specific Thom class u_k. This Thom class is defined by

$$u_k = \sigma^*_k(a_{k-1+t} \wedge a_{k-2+t} \wedge \ldots \wedge a_t).$$

where $\sigma_k : BU_k^{\mathrm{ad}_k \oplus t\mathrm{can}_k} \longrightarrow U/U^+_t$ is the splitting map. Recall that $E^*(BU_k)$ is a power-series ring in the Conner–Floyd Chern classes c_i of the universal k-plane bundle (associated to the standard representation of U_k on $\mathbb{C}^k$).

Let λ be a sequence of integers $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k \geqslant 0$. The *diagram* of λ is defined as the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \lambda_i$. In particular, if $\lambda_1 = 0$, we have the 'empty' diagram. The *conjugate* of λ is the sequence $\lambda'_1 \geqslant \lambda'_2 \geqslant \ldots \geqslant \lambda'_{\lambda_1} \geqslant 0$ whose diagram is the transpose of the diagram of λ, or equivalently $\lambda'_i = \mathrm{Card}\{j : \lambda_j \geqslant i\}$. We have the following theorem.

THEOREM D. *σ^*_k factors as the projection onto the component of weight k in the completed exterior algebra, $\tilde{E}^*(U/U^+_t)$, followed by an isomorphism with $\tilde{E}^*(BU_k^{\mathrm{ad}_k \oplus t\mathrm{can}_k})$. In terms of the generators specified above*

$$\sigma^*_k(a_{n_1} \wedge a_{n_2} \wedge \ldots \wedge a_{n_k}) = u_k c_k^{-t} \det(c_{m_i - i + j})_{1 \leqslant i,j \leqslant n_1+1-k}, \quad n_1 > n_2 > \ldots > n_k \geqslant t$$

where the sequence of integers $(m_1, m_2, \ldots, m_{n_1+1-k})$ is conjugate to the sequence $(n_1+1-k, n_2+2-k, \ldots, n_k)$. If $n_1+1-k = 0$, then we define $\det(c_{m_i-i+j}) = 1$. It is to be understood that $c_{m_i-i+j} = 0$ whenever $m_i - i + j < 0$.

For example consider the case when $t = 1$, $k = 3$ and $n_1 = 5, n_2 = 3, n_3 = 2$. In this case $(n_1+1-k, n_2+2-k, \ldots, n_k) = (3,2,2)$ and the sequence conjugate to it is $(m_1, m_2, \ldots, m_{n_1+1-k}) = (3,3,1)$, so by Theorem D

$$\sigma^*_3(a_5 \wedge a_3 \wedge a_2) = u_3 c_3^{-1} \det \begin{pmatrix} c_3 & c_4 & c_5 \\ c_2 & c_3 & c_4 \\ c_{-1} & c_0 & c_1 \end{pmatrix} = u_3 c_3^{-1} \begin{pmatrix} c_3 & 0 & 0 \\ c_2 & c_3 & 0 \\ 0 & 1 & c_1 \end{pmatrix} = u_3 c_3 c_1.$$

It is interesting to observe that for Theorem D to make sense, c^t_k must always divide $\det(c_{m_i-i+j})$.

The organization of this paper is as follows. In the first few sections we will identify the splitting for the unitary group in singular cohomology. Towards that end, Section 2 will contain the proof of the fact that the map σ^*_k factors through the component of weight k in the exterior algebra $\tilde{H}^*(U^+)$. This is done by analyzing the eigenspace decomposition in cohomology of the power map on the unitary group. In Section 3 we interpret the multiplication on the unitary group in terms of the geometric splitting. Theorem A is proved in this section. Section 4 contains the proofs of the structure theorems on the E-cohomology of U^+ and U/U^+_t that were stated as Theorems B and C. We also establish the structure of the E-cohomology of $BU_k^{\mathrm{ad}_k \oplus t\mathrm{can}_k}$ stated earlier. In Section 5 we begin by establishing Theorem D for the special case $t = 0$ using an induction argument that rests on results of Sections 2, 3 and 4. We then invoke a commutative diagram that relates the splitting of U^+ with the splitting of U/U_t^+ to obtain Theorem D in full generality. Section 1 contains a description of the geometric splitting of the unitary group as described in [**1**]. The splitting for the Stiefel manifolds can be found in [**1**] and is omitted from Section 1. The construction in Section 1 is referred to throughout the paper and readers are encouraged to acquaint themselves with it. In this paper it will be convenient to work with reduced cohomology and hence the reader should read the word 'E-cohomology' as reduced E-cohomology. We shall, however, make a point to denote reduced cohomology by $\tilde{E}^*$ whenever we refer to it symbolically. For reduced singular cohomology (with integer coefficients) we shall suppress the letter H and call it simply 'cohomology'.

1. *The splitting*

Let us identify U_n as the space of all hermitian inner-product preserving $\mathbb{C}$-linear endomorphisms of $\mathbb{C}^n$. Define a filtration of U_n by closed sets

$$F_k\, U_n = \{\psi : \mathrm{Rank}(\psi - \mathrm{Id}) \leqslant k\}.$$

Define

$$\Gamma_{n,k} = \{(\psi, V) : \psi|_{V^\perp} = \mathrm{Id}|_{V^\perp}\} \subseteq U_n \times G_{n,k}. \tag{1.1}$$

This is a submanifold, and the obvious smooth map $\pi_1 : \Gamma_{n,k} \longrightarrow U_n$ has image equal to $F_k\, U_n$. If $\psi \in F_k\, U_n - F_{k-1}\, U_n$ then it has a unique pre-image in $\Gamma_{n,k}$. Hence this manifold can be seen as a desingularization of $F_k\, U_n$. The projection $\pi_2 : \Gamma_{n,k} \longrightarrow G_{n,k}$ is clearly a fiberbundle. We also have a section

$$\iota : G_{n,k} \longrightarrow \Gamma_{n,k}$$

sending V to (ψ, V), where

$$\psi|_V = -\mathrm{Id}|_V, \quad \psi|_{V^\perp} = \mathrm{Id}|_{V^\perp}. \tag{1.2}$$

An equivalent construction of $\Gamma_{n,k}$ is as follows. Let U_k act on itself by conjugation

$$\mu.\psi = \mu\psi\mu^{-1}. \tag{1.3}$$

Write U_k^{c} for this U_k-space.

LEMMA 1.1. *$\Gamma_{n,k}$ is diffeomorphic over $G_{n,k}$ to $V_{n,k} \times_{U_k} U_k^{\mathrm{c}}$ where $V_{n,k}$ is the Stiefel manifold of k-frames in $\mathbb{C}^n$ seen as a principal U_k bundle over $G_{n,k}$.*

Proof. Map $U_n \times U_k \longrightarrow \Gamma_{n,k}$ by

$$(\alpha, \psi) \longmapsto \left(\alpha \begin{pmatrix} \psi & 0 \\ 0 & 1 \end{pmatrix} \alpha^{-1}, \alpha V_0\right)$$

where $V_0 \subseteq \mathbb{C}^n$ is the subspace spanned by the first k standard basis vectors. This passes to a diffeomorphism

$$U_n/(1 \times U_{n-k}) \times_{U_k} U_k^{\mathrm{c}} \cong \Gamma_{n,k}. \qquad \square$$

Notice that the action of U_k preserves the filtration on U_k^{c}, so we obtain a filtration on $V_{n,k} \times_{U_k} U_k^{\mathrm{c}} \cong \Gamma_{n,k}$. The projection $\pi_1 : \Gamma_{n,k} \longrightarrow U_n$ is filtration preserving inducing a relative diffeomorphism

$$V_{n,k} \times_{U_k} (U_k^{\mathrm{c}}, F_{k-1}\, U_k^{\mathrm{c}}) \cong (F_k\, U_n, F_{k-1}\, U_n). \tag{1.4}$$

Identifying ad_k with the space of all skew-hermitian $k \times k$ matrices, we have the following lemma.

LEMMA 1.2. *The Cayley transform gives a U_k-equivariant diffeomorphism*

$$\mathrm{ad}_k \xrightarrow{\psi} U_k^c - F_{k-1} U_k^c, \quad \psi(x) = (x/2-1)(x/2+1)^{-1}.$$

Proof. One can easily check that the map $y \longmapsto 2(1+y)(1-y)^{-1}$ is the inverse. □

From (1.4) and Lemma 1.2, we get a homeomorphism

$$\tau_k : F_k U_n / F_{k-1} U_n \cong G_{n,k}^{\mathrm{ad}_k}. \tag{1.5}$$

Composing the section ι with π_1 gives us an embedding of $G_{n,k}$ into the submanifold $F_k U_n - F_{k-1} U_n$. By Lemma 1.2, this submanifold is a tubular neighborhood of $G_{n,k}$ diffeomorphic to $E(\mathrm{ad}_k)$ and τ_k is the corresponding homeomorphism on the one-point compactification of these spaces. On composing with the projection and adding a disjoint basepoint, we get the Pontrjagin–Thom collapse map

$$h_k : F_k U_n^+ \longrightarrow G_{n,k}^{\mathrm{ad}_k}.$$

We are now ready to construct the splitting maps. Let ζ_k be the space of $k \times k$ hermitian matrices. We have the well-known polarization identity (refer to the exercises in [**6**, p. 136] for a proof).

LEMMA 1.3. *The smooth map*

$$U_k \times \zeta_k \xrightarrow{\cong} \mathrm{Gl}(k, \mathbb{C}) \subset \mathrm{End}(\mathbb{C}^k), \quad (\psi, z) \longmapsto \psi \exp(-z)$$

is a diffeomorphism onto the set of invertible linear maps.

Once we notice that $\mathrm{End}(\mathbb{C}^k) \cong \mathrm{ad}_k \oplus \zeta_k$, we have the corresponding collapse map

$$\mathrm{ad}_k^+ \wedge \zeta_k^+ \xrightarrow{\sigma} U_k^+ \wedge \zeta_k^+ \tag{1.6}$$

where '$+$' denotes the one-point compactification. The above map is compatible with the adjoint action of U_k on ad_k and the conjugation action on ζ_k and on U_k (we suppress the notation 'c' for the latter). Let $u : U_k^+ \longrightarrow \mathrm{ad}_k^+$ be the collapsing map onto $U_k / F_{k-1} U_k$ identified with ad_k^+ via Lemma 1.2. Then the map $(u \wedge 1) \circ \sigma : \mathrm{ad}_k^+ \wedge \zeta_k^+ \longrightarrow \mathrm{ad}_k^+ \wedge \zeta_k^+$ is the collapse map corresponding to the equivariant smooth embedding

$$\mathrm{ad}_k \oplus \zeta_k \longrightarrow \mathrm{ad}_k \oplus \zeta_k, \quad (x, z) \longmapsto (x/2-1)(x/2+1)^{-1} \exp(-z)$$

whose derivative at $(0,0)$ is the identity. It is easy to show that the map $(u \wedge 1) \circ \sigma$ is equivariantly homotopic to the identity. Thus the map (1.6) serves as the splitting of the top cell. The rest of the splitting is a 'fiberwise' version of (1.6). Let $V_{n,k}$ be seen as the principal U_k bundle of k-frames in $\mathbb{C}^n$ over the Grassmannian, $G_{n,k}$. We have a fiberwise map between the associated bundles

$$V_{n,k} \times_{U_k} (\mathrm{ad}_k^+ \wedge \zeta_k^+) \longrightarrow V_{n,k} \times_{U_k} (U_k^+ \wedge \zeta_k^+).$$

Notice that the associated bundle $V_{n,k} \times_{U_k} \zeta_k$ can be seen as all pairs $\{(V, \phi) : V \in G_{n,k}, \phi|_{V\perp} = 0, \phi|_V \text{ is hermitian}\}$. Similarly, the space $G_{n,k} \times \zeta_n$ can be seen as all pairs

$$\begin{array}{ccc} \vee G_{n,k}^{\mathrm{ad}_k} & \longrightarrow & \vee G_{m,k}^{\mathrm{ad}_k} \\ \big\downarrow & & \big\downarrow \\ U_n^+ & \xrightarrow{\;f\;} & U_m^+ \end{array}$$

FIGURE 2.

$$\begin{array}{ccc} \vee BU_k^{\mathrm{ad}_k} & \longrightarrow & \vee BU_k^{\mathrm{ad}_k \oplus \mathrm{tcan}_k} \\ \big\downarrow & & \big\downarrow \\ U^+ & \longrightarrow & U/U_t^+ \end{array}$$

FIGURE 3.

$\{(V,\phi): V \in G_{n,k}, \phi \in \mathrm{End}(\mathbb{C}^n)$ is hermitian$\}$. Clearly, $V_{n,k} \times_{U_k} \zeta_k^+$ can be embedded into the trivial bundle $G_{n,k} \times \zeta_n^+$ and the map extended by smashing with the identity on the complement. After collapsing the section at infinity to a point and identifying $V_{n,k} \times_{U_k} U_k$ with $\Gamma_{n,k}$, we have an unstable representative for our splitting map:

$$\sigma_k : \Sigma^{n^2} G_{n,k}^{\mathrm{ad}_k} \longrightarrow \Sigma^{n^2} \Gamma_{n,k}^+ \xrightarrow{\pi_1} \Sigma^{n^2} U_n^+. \tag{1.7}$$

This map clearly factors through $F_k U_n^+$ and the composition with $h_k : F_k U_n^+ \longrightarrow G_{n,k}^{\mathrm{ad}_k}$ is easily seen to have a fiberwise homotopy to the identity.

We have therefore constructed a splitting which has an unstable representative

$$\vee \sigma_k : \Sigma^{n^2} \bigvee_{k=0}^{n} G_{n,k}^{\mathrm{ad}_k} \longrightarrow \Sigma^{n^2} U_n^+. \tag{1.8}$$

REMARK 1.4. If $n \leqslant m$ and $f : U_n \longrightarrow U_m$ is the map given by

$$\psi \longmapsto \begin{pmatrix} \psi & 0 \\ 0 & 1 \end{pmatrix}$$

in the standard basis, then notice that this construction is natural enough so that the diagram in Figure 2 commutes, where the top horizontal arrow is the obvious inclusion.

Remark 1.4 on naturality gives us splittings of the infinite unitary group

$$\bigvee_{k \geqslant 0} BU_k^{\mathrm{ad}_k} \longrightarrow U^+. \tag{1.9}$$

REMARK 1.5. Consider the map

$$\kappa : \Sigma(\mathbb{C}P^{n-1^+}) \longrightarrow U_n^+$$

defined as follows. Let (s,l) be any element in $\Sigma(\mathbb{C}P^{n-1^+})$, where s is the suspension coordinate seen as an element in U_1 and l is a line in $\mathbb{C}^n$. Define $\kappa(s,l)$ to be the unitary transformation that is given by multiplication with s on the one-dimensional vector subspace given by l and identity on its complement. It is not hard to show that κ is homotopic to the map σ_1. Thus the map σ_1 can be defined unstably.

REMARK 1.6. Miller [4] and M. Crabb [1] have also constructed stable splittings of Stiefel manifolds. Let us denote the Stiefel manifold of n-frames in $\mathbb{C}^{n+t}$ by $V_{n+t,n}$. Let can_k denote the canonical representation of U_k on $\mathbb{C}^k$, then there is a stable equivalence

$$V^+_{n+t,n} \cong \bigvee_{k=0}^{n} G^{\mathrm{ad}_k \oplus t\mathrm{can}_k}_{n,k}$$

and using naturality

$$U/U^+_t \cong \bigvee_{k=0}^{\infty} BU^{\mathrm{ad}_k \oplus t\mathrm{can}_k}_k.$$

The stably homotopy commutative diagram in Figure 3 follows from the naturality of the constructions given in [1] where the top horizontal arrow is the obvious inclusion and the bottom one is the standard projection.

2. *The self map*

We begin this section with a technical lemma (suggested to us by the referee) which will be useful at various points in the paper.

LEMMA 2.1. *Let $f: X \longrightarrow Y$ be a map of torsion-free connective spectra of finite type such that $H_*(f, \mathbb{Q}) = 0$. Then for any complex-oriented ring spectrum E, we have $1 \wedge f: E \wedge X \longrightarrow E \wedge Y$ is null. In particular, $E_* f = 0$ and $E^* f = 0$.*

Proof. Since X and Y are torsion-free spectra, the Atiyah–Hirzebruch spectral sequence computing $MU_* X$ and $MU_* Y$ collapses (since it injects into its rationalization). This shows that $MU_* X$ and $MU_* Y$ are free modules over MU_*. From the natural isomorphism $\mathbb{Q} \otimes MU_* X = MU_* \otimes H_*(X, \mathbb{Q})$, we see that $f_* : \mathbb{Q} \otimes MU_* X \longrightarrow \mathbb{Q} \otimes MU_* Y$ is trivial. However $MU_* Y$ was torsion free, and hence $f_* : MU_* X \longrightarrow MU_* Y$ is also trivial.

A standard argument shows that the functor $Y \longmapsto \mathrm{Hom}_{MU_*}(MU_* X, MU_* Y)$ agrees with the homology theory $Y \longmapsto [X, MU \wedge Y]$. From this it follows trivially that $1 \wedge f: MU \wedge X \longrightarrow MU \wedge Y$ is null. Consequently, for any MU-module spectra E we also have $1 \wedge f: E \wedge X \longrightarrow E \wedge Y$ is null. In particular, $E_* f = 0$.

Working in the naive category $\mathscr{C}$ of E-module spectra, notice that we have the isomorphism $E^* X = \mathscr{C}(E \wedge X, E)^*$. This shows that $E^* f = 0$, proving the lemma. □

REMARK 2.2. Notice that given a diagram of torsion-free spectra of finite type that commutes in homology, Lemma 2.1 says that it commutes in E-cohomology and E-homology as well, for any complex-oriented ring spectrum E.

Assume for the purposes of this section that $\tilde{H}^*(U^+_n)$ is an exterior algebra as a Hopf-algebra. This fact is standard and we shall not prove it.

LEMMA 2.3. *In cohomology, $\sigma_k : G^{\mathrm{ad}_k}_{n,k} \longrightarrow U^+_n$ factors as the projection onto the component of weight k in the exterior algebra $\tilde{H}^*(U^+_n)$ followed by an isomorphism with $\tilde{H}^*(G^{\mathrm{ad}_k}_{n,k})$.*

The vector space ad_k of all skew-hermitian matrices with the adjoint action of U_k admits the following invariant norm. If $s \in \mathrm{ad}_k$ has eigenvalues s_j, we define

$$\|s\| = \max_j |s_j|. \tag{2.1}$$

Let $D(k)$ and $S(k)$ be the unit disk and unit sphere in ad_k. Consider the relative homeomorphism given by the exponential map

$$\mathrm{Exp}: (D(k), S(k)) \longrightarrow (U_k, F_k\, U_k) \tag{2.2}$$

where $\mathrm{Exp}(s) = -e^{\pi s}$. Let m be any integer different from ± 1. We have the m-power function, $P_m: U_k \longrightarrow U_k$, that raises every matrix to its mth power. Clearly, P_m commutes with the conjugation action of U_k on itself and is filtration preserving. Under the relative homeomorphism Exp, P_m corresponds to a pointed map

$$p_m: D(k)/S(k) \longrightarrow D(k)/S(k) \tag{2.3}$$

that commutes with the adjoint action of U_k on $D(k)/S(k)$ and has degree m^k. We can therefore extend p_m to a self map of $G_{n,k}^{\mathrm{ad}_k}$ which we call ϕ_m. Similarly P_m passes to a fiberwise self map, ρ_m, of $\Gamma_{n,k}$.

Notice that ϕ_m has the effect of multiplication by m^k in cohomology. One should also note that P_m has the effect of multiplying the primitives in cohomology by m and hence multiplies all elements in the homogeneous component of weight k in the cohomology of U_n by m^k. This tells us that to prove Lemma 2.3, it is sufficient to show that the diagram in Figure 4 commutes in cohomology.

$$\begin{array}{ccc} G_{n,k}^{\mathrm{ad}_k} & \xrightarrow{\ \phi_m\ } & G_{n,k}^{\mathrm{ad}_k} \\ \downarrow{\scriptstyle \sigma_k} & & \downarrow{\scriptstyle \sigma_k} \\ U_n^+ & \xrightarrow{\ P_m\ } & U_n^+ \end{array}$$

FIGURE 4.

Recall from (1.7) that σ_k factors through the map $\pi_1: \Gamma_{n,k}^+ \longrightarrow U_n^+$ which commutes with the power maps. Hence the proof of the lemma reduces to showing the following.

PROPOSITION 2.4. *The diagram in Figure 5 commutes in cohomology.*

$$\begin{array}{ccc} G_{n,k}^{\mathrm{ad}_k} & \xrightarrow{\ \phi_m\ } & G_{n,k}^{\mathrm{ad}_k} \\ \downarrow & & \downarrow \\ \Gamma_{n,k}^+ & \xrightarrow{\ \rho_m\ } & \Gamma_{n,k}^+ \end{array}$$

FIGURE 5.

Proof. Notice that the result is clear for the case $k = n$ for dimensional reasons. The proof for the general case can be packaged in terms of spectral sequences.

Consider the relative Serre spectral sequence with E_2 term given by $H^*(G_{n,k}; \tilde{H}^*(\mathrm{ad}_k^+))$ converging to the cohomology of $G_{n,k}^{\mathrm{ad}_k}$ and the spectral sequence with E_2 term given by $H^*(G_{n,k}; \tilde{H}^*(U_k^+))$ converging to the cohomology of $\Gamma_{n,k}^+$. We can set up the diagram in Figure 5 in terms of these spectral sequences. Since the map between $G_{n,k}^{\mathrm{ad}_k}$ and $\Gamma_{n,k}$ is fiberwise, it induces a map of spectral sequences. The map on the E_2 term

$$E_2^{p,q} = H^p(G_{n,k}; \tilde{H}^q(U_k^+)) \longrightarrow E_2^{p,q} = H^p(G_{n,k}; \tilde{H}^q(\mathrm{ad}_k^+))$$

is given by the coefficient homomorphism

$$\tilde{H}^q(U_k^+) \longrightarrow \tilde{H}^q(\mathrm{ad}_k^+)$$

corresponding to the fiberwise splitting of the top cell. The observation at the beginning of the proof tells us that the diagram of E_2 terms (Figure 6) commutes.

$$\begin{array}{ccc} H^p(G_{n,k}; \tilde{H}^q(\mathrm{ad}_k^+)) & \xleftarrow{\phi_m^*} & H^p(G_{n,k}; \tilde{H}^q(\mathrm{ad}_k^+)) \\ \uparrow & & \uparrow \\ H^p(G_{n,k}; \tilde{H}^q(U_k^+)) & \xleftarrow{\rho_m^*} & H^p(G_{n,k}; \tilde{H}^q(U_k^+)) \end{array}$$

FIGURE 6.

Consequently, the similar diagram of E_∞ terms also commutes. In other words, the diagram in Figure 5 commutes on the level of associated quotients corresponding to the E_∞ term. To show that it commutes honestly we need to make two observations, namely, $E_\infty^{p,q}$ for the cohomology of $G_{n,k}^{\mathrm{ad}_k}$ is concentrated in the row $q = k^2$ and the $E_\infty^{p,q}$ terms for the cohomology of $\Gamma_{n,k}^+$ are zero for $q > k^2$. In particular, this says that all the elements in $\tilde{H}^*(G_{n,k}^{\mathrm{ad}_k})$ are detected on the first non-trivial associated quotient and the map

$$\tilde{H}^*(\Gamma_{n,k}^+) \longrightarrow \tilde{H}^*(G_{n,k}^{\mathrm{ad}_k})$$

factors through the corresponding associated quotient of $\tilde{H}(\Gamma_{n,k}^+)$. These observations allow the question of the diagram commuting honestly in cohomology to be verified on the level of associated quotients. □

REMARK 2.5. Since all maps in Figure 5 were defined fiberwise (after a suitable suspension), if one could show that the collapse map of (1.6) commutes with the power maps up to an equivariant stable homotopy, it would imply that the diagram commutes stably. Unfortunately, this seems far from obvious. We therefore content ourselves with showing that Figure 5 commutes only in cohomology (and consequently by Remark 2.2, in any complex-oriented cohomology theory).

3. *The multiplication*

Notice that the diagonal inclusion map $U_i \times U_j \hookrightarrow U_{i+j}$ is covered by the obvious map $\mathrm{ad}_i \oplus \mathrm{ad}_j \hookrightarrow \mathrm{ad}_{i+j}$. Hence we get a map

$$BU_i^{\mathrm{ad}_i} \wedge BU_j^{\mathrm{ad}_j} \xrightarrow{\mu} BU_{i+j}^{\mathrm{ad}_{i+j}}. \tag{3.1}$$

In this section, we propose to show that the multiplication, M, on U^+ is compatible with the above map in the following sense.

$$\begin{array}{ccc} BU_i^{\mathrm{ad}_i} \wedge BU_j^{\mathrm{ad}_j} & \xrightarrow{\mu} & BU_{i+j}^{\mathrm{ad}_{i+j}} \\ \downarrow \sigma_i \wedge \sigma_j & & \uparrow r_{i+j} \\ U^+ \wedge U^+ & \xrightarrow{M} & U^+ \end{array}$$

FIGURE 7.

$$\begin{array}{ccc} BU_i^{\mathrm{ad}_i} \wedge BU_j^{\mathrm{ad}_j} & \xrightarrow{\mu} & BU_{i+j}^{\mathrm{ad}_{i+j}} \\ \downarrow \tilde{\sigma}_i \wedge \tilde{\sigma}_j & & \uparrow \tau_{i+j} \\ \tilde{F}_i U \wedge \tilde{F}_j U & \xrightarrow{M} & \tilde{F}_{i+j} U \end{array}$$

FIGURE 8.

LEMMA 3.1. *The diagram in Figure* 7 *commutes up to stable homotopy, where the map* r_{i+j} *is the retraction to the splitting map* σ_{i+j}.

Before we proceed, let us observe that the multiplication is a filtration-preserving map. In particular, the multiplication on the unitary group induces a map

$$M: \tilde{F}_i U \wedge \tilde{F}_j U \longrightarrow \tilde{F}_{i+j} U$$

where we abbreviate $F_k U / F_{k-1} U$ as $\tilde{F}_k U$. Let us also abbreviate as $\tilde{\sigma}_k$ the map

$$BU_k^{\mathrm{ad}_k} \xrightarrow{\sigma_k} F_k U^+ \longrightarrow \tilde{F}_k U.$$

Notice that by construction $\tilde{\sigma}_k$ is nothing other than τ_k^{-1} (cf. (1.5)). Let us now begin with a key proposition.

PROPOSITION 3.2. *The diagram in Figure* 8 *commutes up to homotopy, where* τ_k *is the homeomorphism of* (1.5).

Proof. Let us pick our model for BU_k to be the infinite Grassmannian of k-planes in $\mathbb{C}^\infty$. Consider the subspace $BU_{i,j} = \{(V, W) \in BU_i \times BU_j : V \perp W\}$. Let $\mathrm{ad}_{i,j}$ denote the pullback of the external sum of the adjoint bundles $E(\mathrm{ad}_i) \times E(\mathrm{ad}_j)$ under the inclusion $BU_{i,j} \hookrightarrow BU_i \times BU_j$. By construction the map $BU_{i,j} \longrightarrow BU_{i+j}$ given by $(V, W) \longmapsto V \oplus W$ is covered by a bundle map, $\lambda : \mathrm{ad}_{i,j} \longrightarrow E(\mathrm{ad}_{i+j})$. The reader can easily check that the diagram in Figure 9 commutes on the nose, where σ is the map $(\tilde{\sigma}_i \wedge \tilde{\sigma}_j) \circ \iota$ and ι is the inclusion $BU_{i,j}^{\mathrm{ad}_{i,j}} \hookrightarrow BU_i^{\mathrm{ad}_i} \wedge BU_j^{\mathrm{ad}_j}$. Now since $BU_{i,j}$ is a

$$\begin{array}{ccc} BU_{i,j}^{\mathrm{ad}_{i,j}} & \xrightarrow{\lambda} & BU_{i+j}^{\mathrm{ad}_{i+j}} \\ \downarrow \sigma & & \uparrow \tau_{i+j} \\ \tilde{F}_i U \wedge \tilde{F}_j U & \xrightarrow{M} & \tilde{F}_{i+j} U \end{array}$$

FIGURE 9.

model for the classifying space of $U_i \times U_j$, there is a deformation retract from $BU_i^{\mathrm{ad}_i} \wedge BU_j^{\mathrm{ad}_j}$ to $BU_{i,j}^{\mathrm{ad}_{i,j}}$. Figure 9 along with this observation provide us with a proof of the proposition. □

Proof of Lemma 3.1. The proof is essentially an exercise in diagram chasing. Consider the diagram in Figure 10.

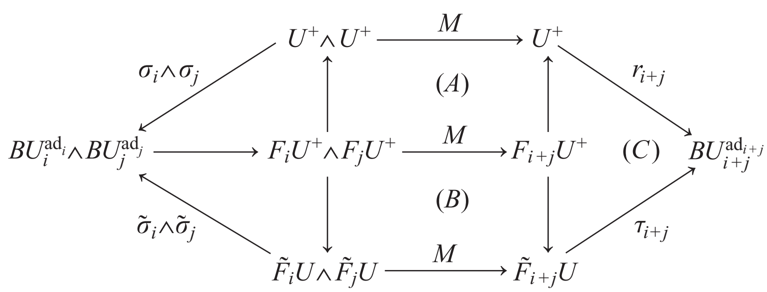

FIGURE 10.

The left side of the diagram commutes by definition. Squares (A) and (B) commute by virtue of the fact that the multiplication on U^+ is filtration preserving. Finally, square (C) commutes because

$$\tau_k \circ \tilde{\sigma}_k : BU_k^{\mathrm{ad}_k} \longrightarrow \tilde{F}_k U \longrightarrow BU_k^{\mathrm{ad}_k}$$

is homotopic to the identity map. One can express the outer arrows at the bottom of the diagram in terms of Proposition 3.2. Comparing this with the outer arrows at the top of the diagram gives a proof of Lemma 3.1. □

REMARK 3.3. What Lemma 3.1 really identifies is the 'top' component of the multiplication in terms of the splitting. Lemma 2.3 tells us that this is the only component that contributes to the multiplication in cohomology: the composite map

$$BU_i^{\mathrm{ad}_i} \wedge BU_j^{\mathrm{ad}_j} \xrightarrow{\sigma_i \wedge \sigma_j} U^+ \wedge U^+ \xrightarrow{M} U^+ \xrightarrow{r_k} BU_k^{\mathrm{ad}_k}$$

is zero in cohomology unless $k = i+j$. Using Remark 2.2, we can now deduce the following.

THEOREM 3.4. *If E is any complex-oriented cohomology theory, then the diagram in Figure* 11 *commutes in E-cohomology.*

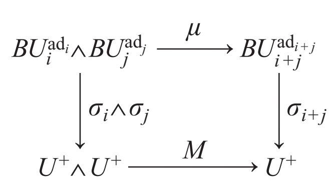

FIGURE 11.

4. *Some structure theorems*

Let E be a complex-oriented cohomology theory. We begin this section by analyzing the E-cohomology of U^+. Let us start with the following theorem.

THEOREM 4.1. *As a Hopf-algebra* $\tilde{E}^*(U_n^+) = \Lambda(a_0^{(n)}, a_1^{(n)}, \ldots, a_{n-1}^{(n)})$ *where* $a_i^{(n)}$ *are primitive elements of homogeneous degree* $2i+1$. *Under the inclusion* $U_{n-1} \hookrightarrow U_n$ *given by*

$$v \longmapsto \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$$

in the standard basis, the element $a_i^{(n)}$ *maps to* $a_i^{(n-1)}$ *for* $i < n-1$ *and* 0 *for* $i = n-1$. *Moreover, under the splitting map* σ_1, *the element* $a_i^{(n)}$ *maps to the suspension of the element* x^i, *where* x *is the orientation in* $E^2(\mathbb{C}P^{n-1})$.

Proof. This theorem is well known for cohomology (see for example [**5**]). Consider the cohomology suspension

$$\sigma : \tilde{E}^*(BU_n) \longrightarrow \tilde{E}^{*-1}(U_n) \tag{4.1}$$

induced by the map $\Sigma U_n \longrightarrow BU_n$ adjoint to the equivalence $U_n \longrightarrow \Omega BU_n$. It is a standard argument to show that the image of σ consists of primitive elements.

Let P be the subgroup of $E^*(U_n)$ spanned by the elements $\sigma(c_k)$, $1 \leqslant k \leqslant n$, where c_k are the Conner–Floyd Chern classes in $\tilde{E}^*(BU_n)$. It is clear that these classes are compatible under natural transformations of complex-oriented cohomology theories. As easy argument using the naturality of the Atiyah–Hirzebruch spectral sequence shows that $MU^*(U_n)$ is an exterior algebra on the classes $\sigma(c_k)$. Since $E^*(U_n) = E^* \otimes MU^*(U_n)$, we deduce that $E^*(U_n)$ is also an exterior algebra on the classes $\sigma(c_k)$. Now, using Remark 1.5 and the Atiyah–Hirzebruch spectral sequence it follows that the composite map

$$f : P \hookrightarrow \tilde{E}^*(U_n^+) \xrightarrow{\sigma_1^*} \tilde{E}^*(\Sigma(\mathbb{C}P^{n-1^+}))$$

is an isomorphism. Thus we may define the element $a_i^{(n)}$ as $f^{-1}(\Sigma x^i)$. The rest of the theorem is a restatement of the fact that the splitting of U_n^+ is natural in n (cf. Remark 1.4). □

It is easy to see that $\tilde{E}^*(U^+) = \lim_{\leftarrow n} \tilde{E}^*(U_n^+)$. Thus the graded Hopf-algebra $\tilde{E}^*(U^+)$ can be viewed as an exterior algebra (suitably completed) on generators a_i of degree $2i+1$ for $i \geqslant 0$.

Now consider the Stiefel manifold $V_{n+t,n}$. If we identify $V_{n+t,n}$ with the space U_{n+t}/U_t, then it is an easy verification using the Atiyah–Hirzebruch spectral sequence that the projection map $U_{n+t} \longrightarrow V_{n+t,n}$ is monic in E-cohomology and that the following theorem holds.

THEOREM 4.2. $\tilde{E}^*(V_{n+t,n}^+)$ *can be identified with the subalgebra of* $\tilde{E}^*(U_{n+t}^+)$ *given by* $\tilde{E}^*(V_{n+t,n}^+) = \Lambda(a_t^{(n)}, a_{t+1}^{(n)}, \ldots, a_{n-1+t}^{(n)})$. *Moreover, under the inclusion* $V_{n-1+t,n-1} \hookrightarrow V_{n+t,n}$ *induced by* $U_{n-1+t} \hookrightarrow U_{n+t}$, *the element* $a_i^{(n)}$ *maps to* $a_i^{(n-1)}$ *for* $t \leqslant i < n-1+t$ *and* 0 *for* $i = n-1+t$.

From Theorem 4.2 we can identify the E-cohomology of U/U_t^+ with the subalgebra of $\tilde{E}^*(U^+)$ generated by a_i for $i \geqslant t$. Now let us turn to the E-cohomology of $BU_k^{\mathrm{ad}_k \oplus \mathrm{tcan}_k}$. We prove the following theorem.

THEOREM 4.3. *The E-cohomology of $BU_k^{\mathrm{ad}_k \oplus \mathrm{tcan}_k}$ is a free module of rank one over $E^*(BU_k)$ generated by a Thom class u_k. The Thom class can be given as*

$$u_k = \sigma_k^*(a_{k-1+t} \wedge a_{k-2+t} \wedge \ldots \wedge a_t)$$

where $\sigma_k : BU_k^{\mathrm{ad}_k \oplus \mathrm{tcan}_k} \longrightarrow U/U_t^+$ is the splitting map.

Proof. To show that u_k serves as a Thom class, one wants to show that $i^*(u_k) \in \tilde{E}^{k^2+2tk}(S^{\mathrm{ad}_k \oplus \mathrm{tcan}_k})$ is a generator where $i : S^{\mathrm{ad}_k \oplus \mathrm{tcan}_k} \hookrightarrow BU_k^{\mathrm{ad}_k \oplus \mathrm{tcan}_k}$ is the inclusion of the (compactified) fiber. Notice first that if E is singular cohomology then this theorem is trivial since using Remark 1.6 we already know that σ_k^* is an isomorphism from the homogeneous component of degree k in the exterior algebra $\tilde{H}^*(U/U_t^+)$ to $\tilde{H}^*(BU_k^{\mathrm{ad}_k \oplus \mathrm{tcan}_k})$ and every generator in $\tilde{H}^{k^2+2tk}(BU_k^{\mathrm{ad}_k \oplus \mathrm{tcan}_k})$ serves as a Thom class. Now consider the Atiyah–Hirzebruch spectral sequence for $E^*(G_{m,k}^{\mathrm{ad}_k \oplus \mathrm{tcan}_k})$. Invoking the result for singular cohomology we know that the class in the E_2 term for $E^*(G_{m,k}^{\mathrm{ad}_k \oplus \mathrm{tcan}_k})$ representing u_k restricts to a generator in the E_2 term for $E^*(S^{\mathrm{ad}_k \oplus \mathrm{tcan}_k})$. Since the Atiyah–Hirzebruch spectral sequences collapse at E_2, it follows that $i^*(u_k)$ restricts to a generator in $\tilde{E}^{k^2+2tk}(S^{\mathrm{ad}_k \oplus \mathrm{tcan}_k})$. □

NOTE 4.4. Notice that Theorem 4.3 says that the vector bundle $\mathrm{ad}_k \oplus \mathrm{tcan}_k$ is oriented in complex-cobordism. It is interesting to observe that $\mathrm{ad}_k \oplus \mathrm{tcan}_k$ need not possess any complex structure. It need not even be an even-dimensional bundle.

5. *The cohomology splitting*

In this section we begin by proving the formula stated as Theorem D in the introduction for the special case $t = 0$. We then invoke the statement of Remark 1.6 to prove it for the general case.

Let T_k be the standard maximal torus of U_k given by all diagonal matrices. Let e_i, $i \leqslant k$ denote the standard one-dimensional complex representation of this torus given by its ith diagonal entry. It is clear that the restriction of the adjoint representation, ad_k, to the maximal torus decomposes as

$$\mathrm{ad}_k|_{T_k} = k \bigoplus (e_i \otimes e_j^*), \; i < j \leqslant k.$$

The trivial k-dimensional part represents the Lie algebra of the maximal torus itself. Let x_i denote the Euler class of the bundle over BT_k associated to the representation e_i. Let us denote the representation $\bigoplus (e_i \otimes e_j^*)$ by ρ_k. Consider the composite map

$$\delta_k : \Sigma^k BT_k^+ \xrightarrow{\Sigma^k s} \Sigma^k BT_k^{\rho_k} \cong BT_k^{k \oplus \rho_k} \longrightarrow BU_k^{\mathrm{ad}_k} \tag{5.1}$$

where s is the inclusion of $BT_k^+ \hookrightarrow BT_k^{\rho_k}$ as the zero section. This map is injective in E-cohomology. Theorem 3.4 can now be rewritten as the following.

LEMMA 5.1. *The diagram in Figure 12 commutes in E-cohomology where the top horizontal homeomorphism is induced by the obvious diagonal map $T_i \times T_j \xrightarrow{\cong} T_{i+j}$ of the standard maximal tori.* □

$$\begin{array}{ccc} \Sigma^i BT_i^+ \wedge \Sigma^j BT_j^+ & \xrightarrow{\cong} & \Sigma^{i+j} BT_{i+j}^+ \\ \Big\downarrow (\sigma_i \circ \delta_i) \wedge (\sigma_j \circ \delta_j) & & \Big\downarrow \sigma_{i+j} \circ \delta_{i+j} \\ U^+ \wedge U^+ & \xrightarrow{M} & U^+ \end{array}$$

FIGURE 12.

By Theorem 4.1 the map $\sigma_1 \circ \delta_1$ has the effect of mapping the primitives a_i in $\tilde{E}^*(U^+)$ to the elements Σx^i of $\tilde{E}^*(\Sigma(\mathbb{C}P^{\infty+}))$. Since $\tilde{E}^*(U^+)$ is an exterior algebra on these generators, we can now verify the following lemma.

LEMMA 5.2. *If $n_1 > n_2 > \ldots > n_k \geqslant 0$, then*

$$(\sigma_k \circ \delta_k)^* (a_{n_1} \wedge a_{n_2} \wedge \ldots \wedge a_{n_k}) = \Sigma^k \det \begin{pmatrix} x_1^{n_1} & x_1^{n_2} & \cdot & \cdot & x_1^{n_k} \\ x_2^{n_1} & x_2^{n_2} & \cdot & \cdot & x_2^{n_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_k^{n_1} & x_k^{n_2} & \cdot & \cdot & x_k^{n_k} \end{pmatrix}.$$

Proof. By Theorem 4.1, the case $k = 1$ is clear. We proceed by induction. Assuming the result for $k-1$, let us look at Figure 12 when $i = 1$ and $j = k-1$. Consider the element $a_{n_1} \wedge a_{n_2} \wedge \ldots \wedge a_{n_k} \in \tilde{E}^*(U^+)$. Since the a_i are primitives, the image of this element under the left composite arrow is

$$\sum_{i=1}^{k} (-1)^{i-1} (\sigma_1 \circ \delta_1)^* a_{n_i} \otimes (\sigma_{k-1} \circ \delta_{k-1})^* (a_{n_1} \wedge a_{n_2} \wedge \ldots \wedge \hat{a}_{n_i} \wedge \ldots \wedge a_{n_k}).$$

This element of $\tilde{E}^*(\Sigma(\mathbb{C}P^{\infty+})) \otimes \tilde{E}^*(\Sigma^{k-1}(BT_{k-1}^+))$ is by assumption

$$\sum_{i=1}^{k} (-1)^{i-1} \Sigma x^{n_i} \otimes \Sigma^{k-1} \det(x_r^{p_s^i})$$

where $(p_1^i, p_2^i, \ldots, p_{k-1}^i)$ is the sequence $(n_1, n_2, \ldots, \hat{n}_i, \ldots, n_k)$. However under the identification $\Sigma^i BT_i^+ \wedge \Sigma^j BT_j^+ \cong \Sigma^{i+j} BT_{i+j}^+$, this expression is nothing other than $\Sigma^k \det(x_i^{n_j})$ expanded along the top row! □

The above determinant is an alternating polynomial of k variables and hence it can be written as some symmetric polynomial $s_{n_1, n_2, \ldots, n_k}$ times the element $\Sigma^k \tilde{u}_k$ where $\tilde{u}_k$ is defined as

$$\tilde{u}_k = \prod_{1 \leqslant i < j < \leqslant k} (x_i - x_j).$$

Notice that the class $\Sigma^k \tilde{u}_k = \delta_k^*(u_k)$ where u_k was the Thom class was defined in Theorem 4.3 for the case $t = 0$. To recover σ_k, we must first express $s_{n_1, n_2, \ldots, n_k}$ in terms of the elementary symmetric functions e_i where $1 \leqslant i \leqslant k$ and then replace e_i by the Conner–Floyd Chern classes c_i. The symmetric functions $s_{n_1, n_2, \ldots, n_k}$ are called Schur functions in the literature. They form an additive basis of the symmetric polynomials if we allow the indexing set, $(n_1, n_2, \ldots, n_k)$, to range over all strictly decreasing k-tuples of non-negative integers. A comprehensive treatment on Schur functions can be found in I. G. Macdonald's classic, 'Symmetric functions and Hall polynomials' [**3**].

Let $(m_1, m_2, \ldots, m_{n_1+1-k})$ be the sequence conjugate to $(n_1+1-k, n_2+2-k, \ldots, n_k)$. The Jacobi–Trudi identity states that

$$s_{n_1, n_2, \ldots, n_k} = \det(e_{m_i-i+j})_{1 \leqslant i, j \leqslant n_1+1-k}.$$

A proof of this identity can be found in [**3**, Section I.3]. This fact along with Lemma 5.2 implies that

$$\sigma_k^*(a_{n_1} \wedge a_{n_2} \wedge \ldots \wedge a_{n_k}) = u_k \det(c_{m_i-i+j})_{1 \leqslant i, j \leqslant n_1+1-k}, \quad n_1 > n_2 > \ldots > n_k \geqslant 0$$

where u_k is the Thom class for the bundle ad_k which restricts to the element $\Sigma^k \tilde{u}_k$ under the map $\sigma_k \circ \delta_k$. The proof of the special case is complete. The proof for the general case is easily derived from the statement of Remark 1.6 and the injectivity of the projection map $U^+ \longrightarrow U/U_t^+$ in E-cohomology.

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