COHOMOLOGY OPERATIONS AND THE NIL-HECKE ALGEBRA

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CONTENTS

1. INTRODUCTION

The Nil-Hecke Algebra can be seen as an algebra of natural operations on the cohomology rings of flag varieties of a compact Lie group (or a Kac-Moody group [2]). This algebra has recently appeared in the context of categorification of quantum groups and link homology ([3, 5, 8]). It is therefore a natural question to ask if the action of the Nil-Hecke algebra extends to the action of yet larger algebras. A natural choice of extension is given by cohomology operations. Of particular interest to categorification is the Steenrod algebra and the Landweber-Novikov algebra being the cohomology operations in mod-p cohomology and cobordism respectively.

This note originated in an attempt to describe the structure of Nil-Hecke modules that extend to admit actions of the Steenrod algebra. We have recently become aware of a similar attempt at understanding this action in [1]. In work under progress we will describe the action of the Landweber-Novikov algebra on link homology.

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2. THE NIL-HECKE RING NH AND THE STEENROD ALGEBRA

The Nil-Hecke algebra NH is an algebra of operations on the cohomology of flag varieties of Kac-Moody groups [2]. In this section we work with the primary field $\mathbb{F} = \mathbb{F}_p$ for some prime p. The Nil-Hecke ring NH is defined as a sub-ring of operators acting on the polynomial algebra: $\mathbb{F}[x_1, \ldots, x_n]$, with the variables x_i given a grading of $|x_i| = 2$ if p is odd, and $|x_i| = 1$ if $p = 2$. The Nil-Hecke ring is generated by polynomials in

$\mathbb{F}[x_1, \ldots, x_n]$ considered as left multiplication operators, and the divided difference operators A_i for $1 \leq i < n$. The action of NH on $\mathbb{F}[x_1, \ldots, x_n]$ is linear over the symmetric algebra $\mathbb{F}[\sigma_1, \ldots, \sigma_n]$, where σ_i is our notation for the i-th symmetric function.

Let r_i for $1 \leq i < n$ be the standard reflections generating the n-th symmetric group Σ_n seen as the Weyl group of $\mathrm{U(n)}$. Then identifying $\mathbb{F}[x_1, \ldots, x_n]$ with the cohomology of the classifying space of the maximal torus of $\mathrm{U(n)}$ (after re-grading the variables if $p = 2$), we get an action of Σ_n on $\mathbb{F}[x_1, \ldots, x_n]$. Let $\alpha_i = x_i - x_{i+1}$ be the element we call the "i-th simple root".

It is well known that the action of A_i on an element $\beta \in \mathbb{F}[x_1, \ldots, x_n]$ is defined uniquely by the the formula:

$$\alpha_i \cup A_i(\beta) = \beta - r_i(\beta).$$

The operators A_i also satisfy a twisted derivation property with respect to the product on $\mathbb{F}[x_1, \ldots, x_n]$:

$$A_i(\beta \cup \gamma) = A_i(\beta) \cup \gamma + r_i(\beta) \cup A_i(\gamma).$$

From these formulas, we easily derive the formula:

Claim 2.1. *One has the following relations among operators on* $\mathbb{F}[x_1, \ldots, x_n]$*:*

$$A_i \mathcal{P} = (1 + {\alpha_i}^{p-1}) \cup \mathcal{P} A_i,$$

where $\mathcal{P} = \sum_i \mathcal{P}^i$ *is the total reduced* p*-th power operation (using the convention* $\mathcal{P}^i = \mathrm{Sq}^i$ *if* $p = 2$*), and* $(1 + {\alpha_i}^{p-1}) \cup$ *denotes the operator given by left multiplication by* $(1 + {\alpha_i}^{p-1})$*.*

One may twist the Steenrod algebra on the polynomial algebra by algebraically modeling the cohomology of Thom spaces over the classifying space of the maximal torus. Borrowing notation from the section 4, let t $\mathrm{T(V)}$ denote a rank one free module over $\mathbb{F}[x_1, \ldots, x_n]$, generated by a (Thom) class μ. Let us assume that the Steenrod Algebra acts on $\mathrm{T(V)}$. This action is determined by an invertible power series V that satisfies:

$$\mathcal{P}\mu = V \cup \mu = (\sum_k v_k) \cup \mu, \quad \mathcal{P}(x\,\mu) = V \cup \mathcal{P}(x)\,\mu.$$

Notice that the classes v_k play the role of mod-p Stiefel-Whitney classes (see section 4). By our convention, the class v_k has degree $|v_k| = 2k(p-1)$ if p is odd, and $|v_k| = k$ if $p = 2$. We may also extend the NH action on $\mathrm{T(V)}$ by canonically identifying $\mathrm{T(V)}$ with $\mathbb{F}[x_1, \ldots, x_n]$.

The actions of the Steenrod algebra and NH do not commute. We have:

Claim 2.2. *One has the following relations among operators on* $\mathrm{T(V)}$*:*

$$A_i \mathcal{P} = \frac{A_i(V)}{V} \cup \mathcal{P} + \frac{r_i(V)}{V}(1 + {\alpha_i}^{p-1}) \cup \mathcal{P} A_i.$$

Rearranging the terms, we get a formula for $\mathcal{P}\, A_i$*:*

$$\mathcal{P} A_i = \frac{V}{r_i(V)(1 + {\alpha_i}^{p-1})} \cup A_i \mathcal{P} - \frac{A_i(V)}{r_i(V)(1 + {\alpha_i}^{p-1})} \cup \mathcal{P}.$$

Proof. Recall the twisted derivation property of the operators A_i given by:

$$A_i(\beta \cup \gamma) = A_i(\beta) \cup \gamma + r_i(\beta) \cup A_i(\gamma),$$

where r_i is the action of the reflection $r_i \in \Sigma_n$. Now on $\mathrm{T(V)}$ we have using the twisted derivation property:

$$A_i\mathcal{P}(x\,\mu) = A_i\,(V \cup \mathcal{P}(x))\,\mu = A_i(V) \cup \mathcal{P}(x)\,\mu + r_i(V)\,A_i\mathcal{P}(x)\,\mu.$$

Now using the previous claim, we may rewrite the right hand side as:

$$A_i(V) \cup \mathcal{P}(x)\,\mu + r_i(V)(1+{\alpha_i}^{p-1})\mathcal{P}A_i(x)\,\mu = \frac{A_i(V)}{V} \cup \mathcal{P}(x\,\mu) + \frac{r_i(V)}{V}(1+{\alpha_i}^{p-1}) \cup \mathcal{P}A_i(x\,\mu).$$

□

The above claim allows us to:

Definition 2.3. *Define an action of $\mathcal{A}$ on NH as follows: Given $D \in \mathrm{NH}$, define the action of the total p-th power operation $\mathcal{P}$ on D as $\mathcal{P} * D \in \mathrm{NH}$ with:*

$$\mathcal{P} * D := \mathcal{P}\,D\,\chi(\mathcal{P}),$$

where χ is the antipode map on the Steenrod algebra. Note that $\mathcal{P} * D$ is an endomorphism of $\mathrm{T(V)}$ as a module over the symmetric algebra.

Remark 2.4. *The antipode map χ has the property that $\mathcal{P}\chi(\mathcal{P}) = \chi(\mathcal{P})\mathcal{P} = \mathrm{Id}$. It follows that the action of $\mathcal{A}$ on NH satisfies the Cartan product identity with respect to the action of NH on $\mathrm{T(V)}$:*

$$(\mathcal{P} * D)(\mathcal{P}(x)) = \mathcal{P}\,D(x), \qquad x \in \mathrm{T(V)}\,.$$

From this it follows that we also have the Cartan product identity with respect to composition:

$$\mathcal{P} * (D_1 D_2) = (\mathcal{P} * D_1)(\mathcal{P} * D_2).$$

Lemma 2.5. *Given the operator $D = A_i$, we have the following identities of operators in NH:*

$$\mathcal{P} * A_i = \frac{V}{r_i(V)(1+{\alpha_i}^{p-1})} \cup A_i - \frac{A_i(V)}{r_i(V)(1+{\alpha_i}^{p-1})} \cup \mathrm{Id}\,.$$

In particular, using the Cartan product identity, one obtains a formula for $\mathcal{P} * (A_{i_1} \cdots A_{i_k})$. The above formula also shows that $\mathcal{P}^k * A_i$ depends only on the first k Steifel-Whitney classes: $v_1, \ldots, v_k$.

Proof. Using the previous claim in the second equality below, we have:

$$\mathcal{P} * A_i = \mathcal{P}A_i\,\chi(\mathcal{P}) = \frac{V}{r_i(V)(1+{\alpha_i}^{p-1})} \cup A_i\mathcal{P}\chi(\mathcal{P}) - \frac{A_i(V)}{r_i(V)(1+{\alpha_i}^{p-1})} \cup \mathcal{P}\chi(\mathcal{P}).$$

Now using the fact that $\mathcal{P}\chi(\mathcal{P}) = \mathrm{Id}$, the right hand side becomes:

$$\frac{V}{r_i(V)(1+{\alpha_i}^{p-1})} \cup A_i - \frac{A_i(V)}{r_i(V)(1+{\alpha_i}^{p-1})} \cup \mathrm{Id}\,.$$

□

Example 2.6. Let us compute the value of $\mathcal{P}^1 * A_i$ using the above formula. For this computation, we simply need to extract the degree zero part of the above expression. This can easily be seen to give:

$$\mathcal{P}^1 * A_i = (v_1 - r_i(v_1) - {\alpha_i}^{p-1}) \cup A_i - A_i(v_1) \cup \mathrm{Id}\,.$$

The above expression can be further simplified to give:

$$\mathcal{P}^1 * A_i = {\alpha_i}^{p-1} \cup (A_i(v_1) - 1) \cup A_i - A_i(v_1) \cup \mathrm{Id}\,.$$

Remark 2.7. *Everything described above generalizes to arbitrary lie groups and large primes* [2].

3. The example of a stable (local) family:

In seeking a family of modules $\mathrm{T(V)}$ that are compatible with respect to restriction in n, consider a two-parameter family of line bundles over $(\mathbb{C}P^\infty)^{\times n}$ ($(\mathbb{R}P^\infty)^{\times n}$, when $p=2$), generated by the tensor powers of the bundles determined by characters: ρ and Δ. Here ρ denotes the "Coxeter character'" given by $\rho = \sum h_i^*$. In other words, ρ has the (locality) property $\rho(h_i) = 1$ for all i. Such a ρ is well-defined up to the central (diagonal) character Δ which has the property $\Delta(h_i) = 0$ for all i. Let us fix a choice for ρ.

From general facts about Lie groups, we observe for any i, that:

$$r_i(\rho) = \rho - \alpha_i, \quad r_i(\Delta) = \Delta \quad \Rightarrow \quad A_i(\rho) = 1, \quad A_i(\Delta) = 0.$$

Let $\tau(c,d) = \rho^{\otimes c} \otimes \Delta^{\otimes d}$ be the line bundle given by tensoring c-copies of the Coxeter character with d-copies of the Diagonal character. Notice that the total Stiefel-Whitney class of $\tau(c,d)$ is given by:

$$V(c,d) = 1 + c\rho^{p-1} + d\Delta^{p-1}.$$

For this family of bundles, we obtain:

Lemma 3.1. *For the local family $\tau(c,d)$, the twisted Steenrod action on* NH *is given by:*

$$\mathcal{P} * A_i = \frac{1 + c\rho^{p-1} + d\Delta^{p-1}}{(1 + c(\rho-\alpha_i)^{p-1} + d\Delta^{p-1})(1 + {\alpha_i}^{p-1})} \cup A_i + \frac{c(\rho^{p-2} + \rho^{p-3}\alpha_i + \rho^{p-4}\alpha_i^2 + \cdots + \alpha_i^{p-2})}{(1 + c(\rho-\alpha_i)^{p-1} + d\Delta^{p-1})(1 + {\alpha_i}^{p-1})}.$$

4. Realizing Algebraic Stiefel-Whitney Classes for $p = 2$

Claim 4.1. *Let cohomology be taken by $\mathbb{F}_2$-coefficients. Given a space X, let $\mathrm{T}(X)$ denote a rank one free module over $H^*(X)$ with a generator μ, and endowed with the left action of the Steenrod algebra $\mathcal{A}$ induced by the $\mathcal{A}$ action on $H^*(X)$:*

$$\mathrm{Sq}^n(x\,\mu) = \sum_{i+j=n} \mathrm{Sq}^i(x)\ \mathrm{Sq}^j(\mu).$$

Define algebraic Stiefel-Whitney classes $v_j \in H^j(X)$ by the relation (see remark 4.2 below):

$$v_j\,\mu = \mathrm{Sq}^j(\mu).$$

Then the classes v_i satisfy the Wu relations in $H^*(X)$:

$$\mathrm{Sq}^j(v_i) = \sum_{t=0}^{j} \binom{i+t-j-1}{t} v_{j-t}\, v_{i+t}.$$

Proof. We work by double induction on i and j. Notice that $v_0 = 1$ since $\mathrm{Sq}^0 = Id$, and hence $\mathrm{Sq}^j(v_0) = 0$ for all $j > 0$. Similarly, $\mathrm{Sq}^0(v_n) = v_n$ for all n. We assume by induction that the Wu relations hold for all $i < n$ and arbitrary j, and for $i = n$ and $j < m$. Now consider the Adem relation for $m < 2n$:

$$\mathrm{Sq}^m\, \mathrm{Sq}^n = \sum_{k=0}^{[m/2]} \binom{n-k-1}{m-2k} \mathrm{Sq}^{m+n-k}\, \mathrm{Sq}^k .$$

Applying this relation to the generator μ, we get the equality:

$$\mathrm{Sq}^m(v_n\, \mu) = \sum_{k=0}^{[m/2]} \binom{n-k-1}{m-2k} \mathrm{Sq}^{m+n-k}(v_k\, \mu).$$

We may use the definiton of the $\mathcal{A}$-action on $T(X)$ to rewrite the above equality as:

$$\sum_{i+j=m} \mathrm{Sq}^i(v_n)\, v_j = \sum_{k=0}^{[m/2]} \sum_{r+s=m+n-k} \binom{n-k-1}{m-2k} \mathrm{Sq}^r(v_k)\, v_s.$$

Notice that the left hand side has the term $\mathrm{Sq}^m(v_n)$. All other terms in the above equality are expressible as polynomials in v_i using the induction hypothesis. Hence we obtain a homogeneous polynomial $f(v_i)$ that satisfies the following relation for $m < 2n$:

$$\mathrm{Sq}^m(v_n) = f(v_i).$$

In the universal case of $X = \mathrm{BO}$, the classes v_i are the universal Stiefel-Whitney classes which are algebraically independent. Hence the above relation must be the Wu relation. This establishes the induction step for $i = n$ and arbitrary m. Now for $i = n + 1$ we may start the induction at $j = 0$ and proceed as before. $\square$

Remark 4.2. *Let $V = 1 + v_1 + \cdots$ denote the generating function of the algebraic Stiefel-Whitney classes. Then we may write the action of the total Steenrod square $\mathrm{Sq} = 1 + \mathrm{Sq}^1 \cdots$ on $x\,\mu$ as:*

$$\mathrm{Sq}(x\,\mu) = V \cup \mathrm{Sq}(x)\,\mu.$$

Corollary 4.3. *Assume that $\mathrm{T}(X)$ is as above. Then one has a canonical map φ^* of unstable algebras over the Steenrod algebra:*

$$\varphi^* : H^*(\mathrm{BO}) = \mathbb{F}_2[w_1, w_2, \ldots] \longrightarrow H^*(X), \quad \varphi^*(w_i) = v_i.$$

Notice that φ^* may not be induced by a map of spaces in general.

Theorem 4.4. *Assume X is a product of real projective spaces $X = (\mathbb{R}P^\infty)^{\times k}$, and $\mathrm{T}(X)$ is as above. Then there exists a virtual vector bundle $\varphi : X \longrightarrow \mathrm{BO}$, so that $\mathrm{T}(X) = H^*(\mathrm{T}(\varphi))$, where $\mathrm{T}(\varphi)$ denotes the Thom spectrum of of the bundle φ.*

Proof. By the Sullivan conjecture [4] (Thm 0.4), any map of unstable algebras over the Steenrod algebras $\varphi : H^*(Y) \longrightarrow H^*(\mathrm{BA})$ can be uniquely (up to homotopy) realized by a map of spaces $\varphi : \mathrm{BA} \longrightarrow Y$, where A is an elementary abelian 2-group, and Y is a simply connected space of finite type. In our application, we would like to take $Y = \mathrm{BO}$. Since BO is not simply connected, we decompose it as $\mathrm{BO} = \mathrm{B}\mathbb{Z}/2 \times \mathrm{BSO}$. Now any map $H^*(\mathrm{B}\mathbb{Z}/2) \longrightarrow H^*(\mathrm{BA})$ is clearly realized by a homotopy unique map of spaces. Hence we may reduce to the simply connected case of $Y = \mathrm{BSO}$. Applying this to the previous corollary, we obtain the map $\varphi : X \longrightarrow \mathrm{BO}$. By construction of the classes v_i, it follows that $H^*(\mathrm{T}(\varphi))$ is canonically isomorphic to $\mathrm{T}(X)$. $\square$

Corollary 4.5. *Let X be the space $(\mathbb{R}P^\infty)^{\times k}$. Then the set of $\mathcal{A}$ module structures on $\mathrm{T}(X)$ is in bijection with the group $[X, \mathrm{BO}]$. It is well know that there is a ring isomorphism:*

$$[X, \mathrm{BO}] = \tilde{\mathrm{K}}^0(\mathbb{R}P^\infty) = x\,\mathbb{Z}[[x]]/\langle x^2 - 2x\rangle,$$

where x denotes the restriction of the universal bundle over $\mathbb{C}P^\infty$. It follows that $[\mathbb{R}P^\infty, \mathrm{BO}]$ is isomorphic to the 2-adic integers $\hat{\mathbb{Z}}_2$. This result generalizes to $X = (\mathbb{R}P^\infty)^{\times k}$ to show that $[X, \mathrm{BO}]$ is a free module over $\hat{\mathbb{Z}}_2$ on a basis indexed by non-trivial square free monomials in k-variables. In particular, it has a basis of cardinality $2^k - 1$.

Remark 4.6. *An elementary argument using the Wu relations shows that the classes v_{2^n} generate all the classes v_k over the Steenrod algebra. There are however, relations between the classes v_{2^n} over the Steenrod algebra* [7].

5. Appendix: The Steenrod Algebra of reduced p-th powers and the Landweber-Novikov algebra:

Let $\mathbb{F} = \mathbb{F}_p$ be the primary field of characteristic p. Let $r\mathcal{A}$ denote the sub algebra of the mod-p Steenrod algebra generated by the reduced p-th powers. The dual Hopf-algebra can be expressed as: $r\mathcal{A}^* = \mathcal{O}\,\mathrm{Aut}(\mathbb{G}_a)$, where $\mathcal{O}\,\mathrm{Aut}(\mathbb{G}_a)$ denotes is defined as the commutative Hopf algebra of functions on the group of automorphisms of the additive formal group $\mathbb{G}_a$. More precisely, $\mathcal{O}\,\mathrm{Aut}(\mathbb{G}_a) = \mathbb{F}[\xi_0^{\pm 1}, \xi_1, \xi_2, \cdots]$ as a ring, with ξ_i being the coordinate function given by the coefficient of X^{p^i} on the affine group scheme over $\mathbb{F}$, with R-points given by the group formal power series (under composition):

$$\mathrm{Aut}(\mathbb{G}_a)(R) = \{f(X) = \sum_{i\geq 0} a_i X^{p^i}, a_i \in R, a_0 \in R^\times\}.$$

Note that the variable ξ_0 simply provides a grading for any comodule over $\mathcal{A}^*$. Consider the composition map:

$$\mu : \mathrm{Aut}(\mathbb{G}_a) \times \mathbb{G}_a \longrightarrow \mathbb{G}_a\,.$$

Taking functions, we get:

$$\mu^* : \mathcal{O}(\mathbb{G}_a) = \mathbb{F}_2[X] \longrightarrow \mathbb{F}_2[X, \xi_0^{\pm 1}, \xi_2, \cdots] = \mathcal{O}(\mathrm{Aut}(\mathbb{G}_a) \times \mathbb{G}_a), \qquad \mu^*(X) = \sum_{i\geq 0} \xi_i\, X^{p^i}.$$

Consider a multi-index $I = (I_1, I_2, \ldots, I_n)$ with $I_k \geq 0$. It follows from this that the dual monomials $(\xi^I)^* := (\xi_1^{I_1}\xi_2^{I_2}\cdots\xi_n^{I_n})^*$ act trivially on X unless I has the property $\sum I_k = 1$. The corresponding primitive classes ξ_n^* form a Lie sub-algebra of $\mathcal{A}$. These classes are known as the Milnor primities, $\mathrm{P}_n = \xi_n^*$. Their action on $\mathcal{O}(\mathbb{G}_a)$ is described as:

$$\mathrm{P}_n(X) = \langle \mu^*(X), \xi_n^* \rangle = X^{p^n}.$$

The Steenrod reduced powers are defined by $\mathcal{P}^i = (\xi_1^i)^* = (\xi^{I_i})^*$, where $I_i = (i, 0, 0, \ldots)$. We have:

$$\mathcal{P}(X) = \sum_{i\geq 0} \mathcal{P}^i(X) = X + X^p.$$

The Landweber-Novikov Algebra:

The cohomology operations on complex cobordism are closely related to the algebra of reduced-p powers as described above [6, 9].

We define the algebra $\mathcal{O}(\mathbb{S}) = \mathbb{Z}[s_0^{\pm 1}, s_1, \ldots]$ as the commutative Hopf algebra of functions on the group $\mathbb{S}$ of invertible formal power series under composition (i.e. formal diffeomorphisms of the affine line). Therefore, the R-valued points of $\mathbb{S}$ are given by:

$$\mathbb{S}(R) = \{f(X) = \sum_{i\geq 0} b_i X^{i+1}, b_i \in R, b_0 \in R^{\times}\},$$

where, as before, s_i is the co-ordinate function given by the coefficient of X^{i+1}. Now let $A = MU_*$ denote the cobordism ring. By results of Quillen, the ungraded ring underlying A is isomorphic to the Lazard ring that represents the one dimensional formal group laws. Define $\Gamma := A \otimes \mathcal{O}(\mathbb{S})$. Then the pair (A, Γ) is the Hopf algebroid that represents the groupoid of one dimensional formal group laws. This can be interpreted as saying that the pair $(Spec(A), Spec(\Gamma))$ is a groupoid. This is none other than the groupoid of formal group laws, and as such it is isomorphic to the co-operations in cobordism MU_*MU.

For our purposes, the Landweber-Novikov algebra $\mathcal{L} = \mathrm{Hom}(\mathcal{O}(\mathbb{S}), \mathbb{Z})$ will be the subalgebra of cohomology operations on complex cobordism MU given by the continuous dual of $\mathcal{O}(\mathbb{S})$. The reader should be warned that our notation is slightly different that that in the literature. By the Landweber-Novikov algebra, one typically means the completion $A \otimes \hat{\mathcal{L}}$ given by all cohomology operations in cobordism.

The groupoid $(Spec(A), Spec(\Gamma))$ acts on the universal formal group $\mathbb{G}$, as described by the diagram:

$$\begin{array}{ccc} Spec(\Gamma) \times_{Spec(A)} \mathbb{G} & \xrightarrow{\mu} & \mathbb{G} \\ \downarrow & & \downarrow \\ Spec(\Gamma) & \xrightarrow{\eta_R} & Spec(A). \end{array}$$

where η_R represents the "target" map on the morphisms $Spec(\Gamma)$. Taking functions:

$$\mu^* : \mathcal{O}(\mathbb{G}) = A[[X]] \longrightarrow \Gamma[[X]] = A[s_0^{\pm 1}, s_1, \ldots][[X]] = \mathcal{O}(Spec(\Gamma) \times_{Spec(A)} \mathbb{G}).$$

It is easy to see that:

$$\mu^*(X) = \sum_{i\geq 0} s_i \, X^{i+1}.$$

As before, it follows from this that the dual monomials $(s^I)^* := (s_1^{I_1} s_2^{I_2} \cdots s_n^{I_n})^*$ act trivially on X unless the multi-index I has the property $\sum I_k = 1$. The corresponding primitive classes $s_n^* := \mathrm{L}_n$ form a Lie sub-algebra of $\mathcal{L}$. Their action on $\mathcal{O}(\mathbb{G})$ is described as:

$$\mathrm{L}_n(X) = \langle \mu^*(X), s_n^* \rangle = X^{n+1} \quad \Rightarrow \quad \mathrm{L}_n = X^{n+1} \frac{\partial}{\partial X}.$$

This implies that one has a map from the universal enveloping algebra of the positive Virasoro $\mathbb{W} := \mathcal{U}(\mathrm{L}_n, n \geq 0)$:

$$\zeta : \mathbb{W} \longrightarrow \mathcal{L}.$$

An easy dimensional count shows that ζ is a rational isomorphism. It follows that $\mathcal{L}$ is an integral form of the algebra of differential operators on the formal line that can be explicitly described [9].

Remark 5.1. *The map ζ extends to a (rationally dense) map:*

$$\zeta : A \otimes \mathbb{W} \longrightarrow MU^*(MU).$$

However, the sub-algebra $A \subset MU^*(MU)$, given by scalar multiplication operations, is not central. The action of $\mathbb{W}$ on A can be described as an integral form of the Bosonic Fock space representation of $\mathbb{W}$ given by the polynomial algebra on the vector space: $\mathbb{Q}[z^{\pm}]/\,\mathbb{Q}[z]$, with L_n acting as the operator $z^{n+1}\frac{d}{dz}$ (see [6]).

Since automorphisms of the additive formal group law are, in particular, automorphisms of the formal affine line, we have an obvious inclusion:

Claim 5.2. *There is a map of group schemes over $\mathbb{F}$:*

$$\mathbb{G}(\iota) : \mathrm{Aut}(\mathbb{G}_a) \longrightarrow \mathbb{S}, \qquad \mathcal{O}\mathbb{G}(\iota) : \mathcal{O}(\mathbb{S}) \longrightarrow \mathcal{O}\,\mathrm{Aut}(\mathbb{G}_a).$$

In particular, there is an injective map of Hopf-algebras from the Hopf-algebra $r\mathcal{A}$ to the reduction of the Landwever-Novikov algebra. This map is given by the linear dual of $\mathcal{O}\mathbb{G}(\iota)$:

$$r\mathcal{A} \longrightarrow \mathcal{O}(\mathbb{S})^* \otimes \mathbb{F} = \mathcal{L} \otimes \mathbb{F}.$$

Remark 5.3. *In the examples of the Steenrod algebra and the Landweber-Novikov algebra discussed above, the functions on the formal group can be interpreted as elements in the (respective) reduced cohomology of $\mathbb{C}P^\infty$. In the case of the Steenrod algebra for $p = 2$, one has $\mathcal{O}(\mathbb{G}_a) = \tilde{H}^*(\mathbb{R}P^\infty, \mathbb{F}_2)$.*

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