ERRATUM: DOMINANT K-THEORY AND INTEGRABLE HIGHEST WEIGHT REPRESENTATIONS OF KAC-MOODY GROUPS

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CONTENTS

EXPLAINATION:

The proof of Lemma 4.3 involved an (incorrect) argument to detect characters in irreducible representations after multiplication with the Weyl denominator. However, due to possible cancellation of terms on multiplication with the Weyl denominator, one may fail to detect some characters. This makes the statement of Lemma 4.3 incorrect. Consequently, the statement of Claim 4.5 may also be incorrect. All other statements in the article are unaffected by this error. Claim 4.6 and Theorem 4.7 (which depend on Lemma 4.3 at present) require an alternate proof. These proofs are provided in this erratum.

CORRECTED PROOFS

Given a compact Lie subgroup $G \subseteq K(A)$, the restriction of $\mathcal{H}$ may not contain all irreducible representations of G. Let us fix a G-stable Hilbert space $\mathcal{L}(G)$ [S]. By definition, $\mathcal{L}(G)$ contains every G-representation infinitely often. So we may fix an equivariant isometry $\mathcal{H} \subseteq \mathcal{L}(G)$ (notice that any two isometries are equivariantly isotopic). We may stabilize fredholm operators on $\mathcal{H}$ by the identity operator on the complement to obtain:

$$\mathrm{St} : \mathcal{F}(\mathcal{H}) \longrightarrow \mathcal{F}(\mathcal{L}(G)).$$

Let $\mathrm{K}_G^*(X)$ denote standard equivariant K-theory represented by the space $\mathcal{F}(\mathcal{L}(G))$ [S].

Claim 4.6. *Let $G \subseteq K(A)$ be a compact subgroup. Then the stabilization map is an injection:*

$$St : {}^{A}\mathbb{K}_G^*(S^0) \longrightarrow K_G^*(S^0).$$

Furthermore, given a proper orbit $X = K(A)_+ \wedge_G S^0$ for some compact Lie subgroup $G \subseteq K(A)$, we have a canonical isomorphism:

$$\mathbb{K}_{K(A)}^*(X) = DR_G[\beta^{\pm 1}],$$

where β is the Bott class in degree 2, and DR_G lies in degree 0. In particular, the odd Dominant K-cohomology of X is trivial.

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Proof. Let $\mathcal{M}$ be a minimal G-invariant complement of $\mathcal{H}$ inside $\mathcal{L}(G)$, so that $\mathcal{H} \oplus \mathcal{M}$ is an G-stable Hilbert space inside $\mathcal{L}(G)$. In particular, $\mathcal{M}$ and $\mathcal{H}$ share no nonzero G-representations. On the level of G-fixed points on the space of Fredholm operators, we have an inclusion:

$$\mathcal{F}^G(\mathcal{H} \oplus \mathcal{M}) = \mathcal{F}^G(\mathcal{H}) \times \mathcal{F}^G(\mathcal{M}) \longrightarrow \mathcal{F}^G(\mathcal{L}(G)).$$

Moreover, it is easy to see that this map is a homotopy equivalence. Hence, we have an injection:

$$\mathrm{St} : {}^A\mathbb{K}^0_G(S^m) = \pi_m \mathcal{F}^G(\mathcal{H}) \longrightarrow \pi_m \mathcal{F}^G(\mathcal{L}(G)) = \mathrm{K}^0_G(S^m).$$

Furthermore, decomposing $\mathcal{H}$ into its irreducible isotypical summands, shows that the group $\pi_0(\mathcal{F}^G(\mathcal{H}))$ is free on the irreducible G-summands in $\mathcal{H}$. In other words, it is isomorphic to DR_G. This gives the identification of $\mathbb{K}^0_{K(A)}(X)$ we claimed. □

The following theorem may be seen as a Thom isomorphism theorem for Dominant K-theory. It would be interesting to know the most general conditions on an equivariant vector bundle that ensure the existence of a Thom class.

Theorem 4.7. *Let $G \subseteq K(A)$ be a subgroup of the form $K_J(A)$ for some $J \in \mathcal{S}(A)$, and let r be the rank of $K(A)$. Let $\mathfrak{g}$ denote the Adjoint representation of $K_J(A)$. Then there exists a fundamental irreducible representation λ of the Clifford algebra Cliff $(\mathfrak{g} \otimes \mathbb{C})$ that serves as a Thom class in ${}^A\mathbb{K}^r_G(S^{\mathfrak{g}})$ (a generator for ${}^A\mathbb{K}^*_G(S^{\mathfrak{g}})$ as a free module of rank one over DR_G), where $S^{\mathfrak{g}}$ denotes the one point compactification of $\mathfrak{g}$.*

Proof. We begin by giving an explicit description of λ. Fix an invariant inner product B on $\mathfrak{g}$, and let Cliff($\mathfrak{g} \otimes \mathbb{C}$) denote the corresponding complex Clifford algebra. One has the triangular decomposition: $\mathfrak{g} \otimes \mathbb{C} = \eta_+ \oplus \eta_- \oplus \mathfrak{h}$, where $\eta_\pm$ denote the nilpotent subalgebras. The inner product extends to a Hermitian inner product on $\mathfrak{g} \otimes \mathbb{C}$, which we also denote B, and for which the triangular decomposition is orthogonal .

Recall that $\mathfrak{h}$ contains the lattice $\mathfrak{h}_\mathbb{Z}$ containing the coroots h_i for $i \in I$. Fix a dual set $h^*_i \in \mathfrak{h}^*_\mathbb{Z}$. We may decompose $\eta_\pm$ further into root spaces indexed on the roots generated by the simple roots in the set J:

$$\eta_\pm = \sum_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha,$$

where $\Delta_\pm$ denotes the positive (resp. negative) roots for $K_J(A)$. Now fix a Weyl element ρ_J, defined by:

$$\rho_J = \sum_{j \in J} h^*_j.$$

It is easy to see using character theory that the irreducible G-module L_{ρ_J}, with highest weight ρ_J belongs to $\mathcal{H}$, and has character:

$$\mathrm{ch}\, \mathrm{L}_{\rho_J} = e^{\rho_J} \prod_{\alpha \in \Delta_+} (1 + e^{-\alpha}) = e^{\rho_J} \mathrm{ch}\, \Lambda^*(\eta_-),$$

where $\Lambda^*(\eta_-)$ denotes the exterior algebra on η_-. In particular, the vector space L_{ρ_J} is naturally $\mathbb{Z}/2$-graded and belongs to $\mathcal{H}$. The exterior algebra $\Lambda^*(\eta_-)$ can naturally be identified with the fundamental Clifford module for the Clifford algebra Cliff($\eta_+ \oplus \eta_-$). Let $\mathbb{S}(\mathfrak{h})$ denote an irreducible Clifford module for Cliff($\mathfrak{h}$). It is easy to see that the action of Cliff($\mathfrak{g} \otimes \mathbb{C}$) on $\mathbb{S}(\mathfrak{h}) \otimes \Lambda^*(\eta_-)$ extends uniquely to an action of $G \ltimes$ Cliff($\mathfrak{g} \otimes \mathbb{C}$) on λ, with highest weight ρ_J, where:

$$\lambda = \mathbb{C}_{\rho_J} \otimes \mathbb{S}(\mathfrak{h}) \otimes \Lambda^*(\eta_-) = \mathbb{S}(\mathfrak{h}) \otimes \mathrm{L}_{\rho_J}.$$

The Clifford multiplication parametrized by the base space $\mathfrak{g}$, naturally describes λ as an element in ${}^A\mathbb{K}^r_G(S^{\mathfrak{g}})$.

We now show that λ is a free generator of rank one for ${}^A\mathbb{K}^r_G(S^{\mathfrak{g}})$, as a DR_G-module. Let L_μ be an irreducible generator of DR_G. Consider the element $\lambda \otimes \mathrm{L}_\mu \in {}^A\mathbb{K}^r_G(S^{\mathfrak{g}})$, and restrict it to ${}^A\mathbb{K}^r_T(S^{\mathfrak{h}_\mathbb{R}})$, along the action map:

$$\varphi : G_+ \wedge_T S^{\mathfrak{h}_\mathbb{R}} \longrightarrow S^{\mathfrak{g}},$$

where $T \subseteq G$ is the maximal torus. Identifying ${}^A\mathbb{K}^r_T(S^{\mathfrak{h}_\mathbb{R}})$, with DR_T, and using the character formula, we see that $\lambda \otimes \mathrm{L}_\mu$ has virtual character given by:

$$\sum_{w \in W_J(A)} (-1)^w e^{w(\rho_J + \mu)}.$$

This correspondence shows that ${}^A\mathbb{K}^r_G(S^{\mathfrak{g}})$ contains a subgroup DR^+_G generated by positive dominant characters τ, with the property: $\{\tau \in \mathrm{D} \,|\, \tau(h_j) > 0, \; j \in J\}$.

It remains to show that all elements in ${}^A\mathbb{K}^r_G(S^{\mathfrak{g}})$ are in this subgroup. For this we will make an explicit computation of ${}^A\mathbb{K}^r_G(S^{\mathfrak{g}})$ using a homotopy decomposition of $S^{\mathfrak{g}}$ as a G-space as constructed in [CK]. This method of computation uses the Bousfield-Kan spectral sequence for the cohomology of a homotopy colimit. This spectral sequence will be used extensively throughout this paper.

It will be more convenient to study the unit sphere $S(\mathfrak{g})$ in the representation $\mathfrak{g}$. Notice that one has an equivariant cofiber sequence:

$$S(\mathfrak{g})_+ \longrightarrow S^0 \longrightarrow S^{\mathfrak{g}}.$$

Therefore, the calculation of ${}^A\mathbb{K}_G(S^{\mathfrak{g}})$ will follow from a similar calculation for $S(\mathfrak{g})$.

It is shown in [CK] that $S(\mathfrak{g})$ is a suspension of the following space (the suspension coordinates correspond to the rank of the center of G):

$$X(G) := \mathrm{hocolim}_{S \subsetneq J}\, G/K_I(A).$$

Filtering $X(G)$ by the equivariant skeleta, we get a convergent spectral sequence (E_n, d_n), $|d_n| = (n, 1-n)$, and with E_2 term given by:

$$E_2^{p,*} = \varprojlim{}^{p\,A}\mathbb{K}^*_G(G/K_\bullet(A)) = \varprojlim{}^p\mathrm{DR}_\bullet[\beta^{\pm 1}] \Rightarrow {}^A\mathbb{K}^p_G(X(G)))[\beta^{\pm 1}],$$

where β denotes the invertible Bott class, and we have simplified the notation $\mathrm{DR}_\bullet$ to denote the functor $S \mapsto \mathrm{DR}_{K_S(A)}$ for $S \subsetneq J$.

Now using character formula, we see that DR_S is isomorphic to the $W_S(A)$-invariant characters: $\mathrm{DR}_T^{W_S(A)}$. In other words, we have:

$$\mathrm{DR}_S = \mathrm{DR}_T^{W_S(A)} = \mathrm{Hom}_{W_J(A)}(W_J(A)/W_S(A), \mathrm{DR}_T).$$

Consider the $W_J(A)$-equivariant spherical Davis complex: Σ_J defined as:

$$\Sigma_J = \mathrm{hocolim}_{S \subsetneq J}\, W_J(A)/W_S(A).$$

It follows that $\varprojlim{}^p \mathrm{DR}_\bullet$ is canonically isomorphic to the equivariant cohomology (as defined in [D2]) of $\Sigma_J \subset \mathfrak{h}$, with values in the ring DR_T:

$$\varprojlim{}^p \mathrm{DR}_\bullet = \mathrm{H}^p_{W_J(A)}(\Sigma_J, \mathrm{DR}_T).$$

Now recall that the set of dominant weights D has a decomposition indexed by subsets $K \subseteq I$:

$$\mathrm{D} = \coprod \mathrm{D}_K, \quad \text{where} \quad \mathrm{D}_K = \{\lambda \in \mathrm{D} \,|\, \lambda(h_k) = 0, \iff k \in K\}.$$

Let R^K_T denote the ideal in DR_T generated by the weights belonging to the subset D_K. We get a corresponding decomposition of DR_T as a $W(A)$-module:

$$\mathrm{DR}_T = \bigoplus \mathrm{DR}^K_T, \quad \text{where} \quad \mathrm{DR}^K_T \cong \mathbb{Z}[W(A)/W_K(A)] \otimes \mathrm{R}^K_T.$$

We therefore have an induced decomposition of the functor $\mathrm{DR}_\bullet = \bigoplus \mathrm{DR}^K_\bullet$ indexed by $K \subseteq I$. On taking derived functors we have:

$$\varprojlim{}^p \mathrm{DR}_\bullet = \bigoplus \varprojlim{}^p \mathrm{DR}^K_\bullet = \bigoplus \mathrm{H}^p_{W_J(A)}(\Sigma_J, \mathbb{Z}[W(A)/W_K(A)]) \otimes \mathrm{R}^K_T.$$

Now the left $W_J(A)$-space $W(A)/W_K(A)$ is a disjoint union over double cosets:

$$W(A)/W_K(A) = \coprod_{w \in {}^J W^K} W_J(A)/W_{K_w}(A), \quad \text{where} \quad W_{K_w}(A) = W_J(A) \cap w\, W_K(A)\, w^{-1}.$$

Since Σ_J is a compact simplicial complex, it is easy to see directly or using [D1, D2] that:

$$\mathrm{H}^p_{W_J(A)}(\Sigma_J, \mathbb{Z}[W_J(A)]) = \mathrm{H}^p(\Sigma_J, \mathbb{Z}) = 0, \text{ if } p \neq \{0, |J|-1\}, \text{ and } = \mathbb{Z} \text{ if } p = \{0, |J|-1\}.$$

In addition, given a non-empty subset $K_w \subseteq J$,

$$\mathrm{H}^p_{W_J(A)}(\Sigma_J, \mathbb{Z}[W_J(A)/W_{K_w}(A)]) = \mathrm{H}^p(\Sigma_J/W_{K_w}(A), \mathbb{Z}) = 0 \text{ if } p \neq 0, \; = \mathbb{Z} \text{ if } p = 0.$$

The last equality above follows form the fact that $\Sigma_J/W_{K_w(A)}$ is the fundamental domain of the $W_{K_w(A)}$-action on Σ_J. Hence it is the intersection of Σ_J with a cone that lies in a half-quadrant. In particular, it is a retract of the cone and therefore contractible.

It follows that the spectral sequence has only two columns $p = \{0, |J|-1\}$, with:

$$E_2^{0,*} = \bigoplus_{\{K, {}^J W^K\}} \mathrm{R}^K_T[\beta^{\pm 1}] = \mathrm{DR}_G[\beta^{\pm 1}], \qquad E_2^{|J|-1,*} = \bigoplus_{\{K, {}^J \overline{W}^K\}} \mathrm{R}^K_T[\beta^{\pm 1}] = \mathrm{DR}^+_G[\beta^{\pm 1}],$$

where ${}^J\overline{W}^K$ is the subset of elements $w \in {}^J W^K$ for which $K_w = \emptyset$. This spectral sequence must collapse because DR_G is detected by the pinch map $S(\mathfrak{g})_+ \longrightarrow S^0$. It follows that ${}^A\mathbb{K}^*_G(S^{\mathfrak{g}})$ is isomorphic to $\mathrm{DR}^+_G[\beta^{\pm 1}]$, which is what we wanted to show.

□

REFERENCES

[A1] M. Atiyah, *Bott periodicity and the index of elliptic operators*, Quart. J. Math. Oxford (2), 19 (1968), 113–140.

[A2] M. Atiyah, *K-theory*, Benjamin, 1967.

[AS] M. Atiyah, G. Segal, *Twisted K-theory*, available at `math.KT/0407054`.

[BGH] J. Brown, O. Ganor, C. Helfgott, *M-theory and E_{10}: Billiards, Branes and Imaginary Roots*, available at `hep-th/0401053`

[BK] A.K. Bousfield, D.M. Kan, *Homotopy limits, completions, and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer, 1972.

[CK] N. Castellana, N. Kitchloo, *A homotopy construction of the adjoint representation for Lie groups*, Math. Proc. Camb. Phil. Soc. 133 (2002), 399–409.
[DN] T. Damour, H. Nicolai, *Eleven dimensional supergravity and the $\mathbb{E}_{10}/K(\mathbb{E}_{10})$ σ-model at low A_9 levels.*, available at `hep-th/0410245`.
[DHN] T. Damour, M. Henneaux, H. Nicolai, *Cosmological Billiards*, available at `hep-th/0212256`
[D1] M. W. Davis, *The cohomology of a Coxeter group with group ring coefficients*, Duke Math. J. 91 (1998), 297–314.
[D2] M. W. Davis, J. Dymara, T. Januszkiewicz, B. Okun, *Cohomology of Coxeter groups with group ring coefficients: II*, Algebraic and Geometric Topology, 6 (2006), 1001–1030.
[D3] M. W. Davis, *The Geometry and Topology of Coxeter Groups*, available at `http://www.math.ohio-state.edu/~mdavis/`
[D4] M. W. Davis, *Buildings are CAT(0)*, Geometry and Cohomology in Group Theory, London Math. Soc. Lecture Notes 252, Cambridge Univ. Press (1998), 108–123.
[FHT] D. Freed, M. Hopkins, C. Teleman, *Loop groups and twisted K-theory III*, available at `math.AT/0312155`.
[H] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.
[J] B. Julia, *On Infinite Dimensional Symmetry Groups in Physics*, in Recent developments in quantum field theory, ed. J. Ambjorn et al., North-Holland, 1985.
[K1] V. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 1990.
[K2] V. Kac, D. Peterson, *Defining relations of certain infinite-dimensional groups*, Asterisque Numero Hors Srie (1985), 165–208.
[K3] V. Kac, D. Peterson, *On geometric invariant theory for infinite-dimensional groups*, Algebraic Groups, Utrecht 1986, Springer LNM 1271, 1986.
[K4] V. Kac, D. Peterson, *Regular functions on certain infinite-dimensional groups*, Arithmetic and Geometry, II, Birkhauser, Boston Progress in Mathematics, Vol. 36, (1983), 141–166.
[Kl] J. Klein, *The dualizing spectrum of a topological group*, Math. Ann. 319, (2001), 421–456.
[KM] N. Kitchloo, J. Morava, *Thom prospectra and Loop group representations*, Elliptic Cohomology, London Math. Soc. Lecture Note Series (342), Cambridge Univ. Press (2007).
[Ku] S. Kumar, *Kac-Moody groups, their Flag Varieties and Representation Theory*, Birkhauser, Boston Progress in Mathematics, Series, 204, 2002.
[KW] V. Kac, S. Wang, *On automorphisms of Kac-Moody algebras and groups*, Adv. Math. 92, (1992), No.2, 129–195.
[LMS] G. Lewis, J.P. May, M. Steinberger, *Equivariant stable homotopy theory*, Springer LNM 1213, 1986.
[PS] A. Presley, G. Segal, *Loop Groups*, Oxford University Press, 1986.
[S] G. Segal, *Equivariant K-theory*, Publ. Math. I.H.E.S., Paris, 34, (1968).
[T] J. Tits, *Uniqueness and presentation of Kac-Moody groups over fields*, Journal of Algebra, 105, (1987), 542–573.

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