Finding the Dimension and Basis of the Image and Kernel of a Linear Transformation

Sinan Ozdemir

1 Introduction

Recall that the *basis* of a Vector Space is the smallest set of vectors such that they span the entire Vector Space.

ex. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^3$ because you can create any vector in $\mathbb{R}^3$ by a linear combination of those three vectors ie. $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ can be written as the linear combination $a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. But the standard basis is obvious. Are there other vectors that form a basis of $\mathbb{R}^3$? The answer is yes, but we will get there later.

Also recall that the *Dimension* of a Vector Space is the number of elements in the basis of the Vector Space. For example, the dimension of $\mathbb{R}^3$ is 3.

2 The Good Stuff

Keeping these definitions in mind, let's turn our attention to finding the basis and dimension of images and kernels of linear transformation. Let's begin by first finding the image and kernel of a linear transformation.

To find the image of a transformation, we need only to find the linearly independent column vectors of the matrix of the transformation. Recall that if a set of vectors $v_1, v_2, ..., v_n$ is linearly independent, that means that the linear combination

$$c_1v_1 + c_2v_2 + ... + c_nv_n = 0$$

where c_i are all scalars, has only one solution, and that is that all c_i's are 0.

So, to find out which columns of a matrix are independent and which ones are redundant, we will set up the equation $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$, where v_i is the i^{th} column of the matrix and see if we can make any relations.

ex. Consider the matrix $\begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{pmatrix}$ which defines a linear transformation from $\mathbb{R}^4 -> \mathbb{R}^3$. We can set up the equation $c_1\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 7 \\ 5 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 3 \\ 3 \\ 0 \end{pmatrix} + c_4\begin{pmatrix} 4 \\ 9 \\ 1 \\ 8 \end{pmatrix} = 0$ to find out which vectors are linearly dependent. We now need to find out what the scalars of this system are. We expand the system to obtain the equations

$$\begin{gathered} c_1 + 3c_2 + c_3 + 4c_4 = 0 \\ 2c_1 + 7c_2 + 3c_3 + 9c_4 = 0 \\ c_1 + 5c_2 + 3c_3 + c_4 = 0 \\ c_1 + c_2 + 0c_3 + 8c_4 = 0 \end{gathered}$$

We solve this system by setting up the coefficient matrix, and augmenting it with the solution vector, ie:

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 4 & 0 \\ 2 & 7 & 3 & 9 & 0 \\ 1 & 5 & 3 & 1 & 0 \\ 1 & 2 & 0 & 8 & 0 \end{array}\right).$$

It is no coincidence that this matrix is almost identical to the original matrix. We when reduce this system by Gauss Jordan Elimination to get

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

. We can clearly see that there is a non-trivial relation between the vectors in the column space. we see that $c_1 - 2c_3 = 0$, $c_2 + c_3 = 0$, and $c_4 = 0$. This means that c_3 is a free variable because c_1, and c_2 both depend on it. This means that if we disregard c_3, all other variables are independent. This means that the vectors in the first, second, and fourth column are all linearly independent. ie. vectors $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 7 \\ 5 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ 9 \\ 1 \\ 8 \end{pmatrix}$ are independent. In shorter words, to find the image of a matrix, reduce it to RREF, and the columns with leading 1's correspond to the columns of the original matrix which span the image.

We also know that there is a non-trivial kernel of the matrix. We know this because the *the dimension of the image + the dimension of the kernel must equal the dimension of the domain of the transformation.* In this case, the dimension of the image is 3, the dimension of the domain is 4, so there must be an element in the kernel. So what is it?

Look at the relation $c_1 - 2c_3 = 0$, and $c_2 + c_3 = 0$. we rephrase that by saying $c_1 = 2c_3$, and $c_2 = -c_3$. We construct the basis of the kernel with these relations. We know that for every c_3, c_1 must be half that, and c_2 must be its negation. This means the vector $\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ suffices as the basis of the kernel.