Linear Algebra Assignment 11. Answer key.

7.4. (14) $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Yes. It is Diagonalizable.

$f(\lambda) = \det(A - \lambda I_n) = \det\begin{bmatrix} 3-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda)(1-\lambda) = 0$

$\Rightarrow \lambda = 1, 2, 3$.

For $\lambda_1 = 1$ $\quad E_{\lambda_1} = \ker\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Solving $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$, we get

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t$ $\quad$ So $E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$

For $\lambda_2 = 2$. $\quad E_{\lambda_2} = \ker\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ solving $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$, we get

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} t$ $\quad$ So $E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$

For $\lambda_3 = 3$ $\quad E_{\lambda_3} = \ker\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ solving $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$, we get

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$. $\quad$ So $E_{\lambda_3} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

As $\dim E_{\lambda_1} + \dim E_{\lambda_2} + \dim E_{\lambda_3} = 3$, A is diagonalizable. and

$S = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(20) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Yes, It is Diagonalizable.

$f(\lambda) = \det(A - \lambda I_n) = \det\begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\lambda(\lambda-2) = 0 \Rightarrow \lambda = 0, 1, 2$.

For $\lambda_1 = 0$, $E_{\lambda_1} = \ker\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$

For $\lambda_2 = \cancel{0}1$, $E_{\lambda_2} = \ker \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

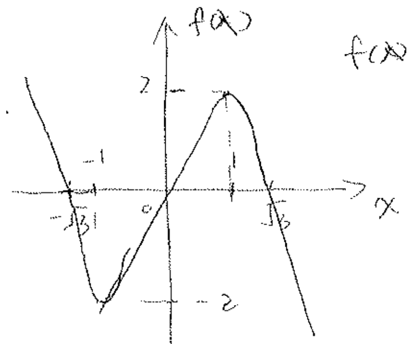
For $\lambda_3 = 2$, $E_{\lambda_3} = \ker \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

So $\dim E_{\lambda_1} + \dim E_{\lambda_2} + \dim E_{\lambda_3} = 3$, Hence A is diagonalizable.

and $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

30 $\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$.

$f(\lambda) = \det \begin{bmatrix} -\lambda & 0 & a \\ 1 & -\lambda & 3 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda(\lambda^2 - 3) + a$.



$f(x) = -\lambda(\lambda^2 - 3)$

(I) So if $-2 < a < 2$, $f(\lambda) = 0$ has 3 distinct solutions. $\lambda_1, \lambda_2, \lambda_3$ (the 3 intersections with x-axis).

As each E_{λ_i} has dimension at least 1,

$\dim E_{\lambda_1} + \dim E_{\lambda_2} + \dim E_{\lambda_3} \geq 3$.

But $\dim E_{\lambda_1} + \dim E_{\lambda_2} + \dim E_{\lambda_3} \leq 3$ since they are linearly independent subspaces of $\mathbb{R}^3$. So $\dim E_{\lambda_1} + \dim E_{\lambda_2} + \dim E_{\lambda_3} = 3$. Hence $\begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ is diagonalizable.

(II). If $a = 2$. $f(\lambda) = (2-\lambda)(\lambda+1)^2$. So $\lambda_1 = 2$ with multiplicity 1
$\lambda_2 = -1$ with multiplicity 2.

Hence $\dim E_{\lambda_1} = 1$

When $\lambda_2 = -1$, we have $E_{\lambda_2} = \ker \begin{bmatrix} +1 & 0 & 2 \\ 1 & +1 & 3 \\ 0 & 1 & +1 \end{bmatrix} = \text{span}\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$.

So $\dim E_{\lambda_2} = 1$. Hence $\dim E_{\lambda_1} + \dim E_{\lambda_2} = 2 \neq 3$. Hence for $a = 2$ it is not diagonalizable.

(III) If $a=-2$. $f(\lambda)=-\lambda^3+3\lambda-2 = \cancel{(\lambda-1)^2(-2-\lambda)}\ (\lambda-1)^2(-2-\lambda)$.

Hence $\lambda_1=-2$ with multiplicity 1, and $\lambda_2=1$ with multiplicity 2.

When $\lambda_2=1$, $E_{\lambda_2}=\ker\begin{bmatrix}-1&0&-2\\1&-1&3\\0&1&-1\end{bmatrix}=\text{span}\left\{\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\}$

~~Since~~ As $\lambda_1=-2$ has multiplicity 1. $\dim E_{\lambda_1}=1$.

So $\dim E_{\lambda_1}+\dim E_{\lambda_2}=2\neq 3$. Hence when $a=-2$, it is not diagonalizable.

~~So~~ In conclusion, $\begin{bmatrix}0&0&a\\1&0&3\\0&1&0\end{bmatrix}$ is diagonalizable if $-2<a<2$.

(37). Yes. As A, B are 2×2 matrices. the characteristic polynomial is $f_A(\lambda)=\lambda^2-\text{tr}A\,\lambda+\det A=\lambda^2-7\lambda+7.=f_B(\lambda)$. as $\text{tr}A=\text{tr}B=\det A=\det B=7$

So $\lambda_1=\frac{7+\sqrt{21}}{2}$ $\lambda_2=\frac{7-\sqrt{21}}{2}$.

As $\lambda_1\neq\lambda_2$, and $\dim E_{\lambda_1}=\dim E_{\lambda_2}=1$ So $\dim E_{\lambda_1}+\dim E_{\lambda_2}=2$

Hence A is similar to $\begin{bmatrix}\frac{7+\sqrt{21}}{2}&0\\0&\frac{7-\sqrt{21}}{2}\end{bmatrix}$, and B is similar to

$\begin{bmatrix}\frac{7+\sqrt{21}}{2}&0\\0&\frac{7-\sqrt{21}}{2}\end{bmatrix}$. Hence A is similar to B.

(40). If $T(f)=\lambda f$ for some number λ and non-trivial C^∞ functions $f(t)$),

we have $5f'-3f=\lambda f \Rightarrow f'=\frac{(3+\lambda)}{5}f$.

Hence $\frac{f'}{f}=\frac{3+\lambda}{5}$ integrate on both sides we get $\log|f|=\frac{3+\lambda}{5}\cdot t+C$.

Hence $f = Ae^{\frac{3+\lambda}{5}t}$

So for any real number λ, we have a nontrivial C^∞ function $f = e^{\frac{3+\lambda}{5}t}$

to be its "~~eige~~ eigenvector".

(42). Let $B = \{ \underset{e_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{e_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{e_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \}$ be the standard basis for $\mathbb{R}^{2\times 2}$. Then

$$L(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{So } [L(e_1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L(e_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{So } [L(e_2)]_B = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$L(e_3) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{So } [L(e_3)]_B = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$L(e_4) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{So } [L(e_3)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Hence } [L]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$f(\lambda) = \det([L]_B - \lambda I_4) = \det \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & -1 & 0 \\ 0 & -1 & 1-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} = -\lambda \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$= \lambda^2 \det \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} = \lambda^2(\lambda^2 - 2\lambda) = \lambda^3(\lambda-2)$$

Hence $\lambda_1 = 0$, multiplicity 3, $\lambda_2 = 2$. multiplicity 1. So $\dim E_{\lambda_2} = 1$

When $\lambda_1 = 0$, $E_{\lambda_1} = \ker \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

~~So~~ As $\dim E_{\lambda_1} + \dim E_{\lambda_2} = 3 + 1 = 4$, L is diagonalizable.

(50) Consider the standard basis $B = \{1, x, x^2\}$ for P_2.

Then As $T(f(x)) = f(x-3)$, we have

$$[T]_B = \begin{bmatrix} -2 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = -2 \cdot \text{ multiplicity} = 1 \Rightarrow \dim E_{\lambda_1} = 1$$

$$\lambda_2 = 1 \quad \text{multiplicity} = 2.$$

$$E_{\lambda_2} = \ker \begin{bmatrix} -3 & -3 & 9 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

So $\dim E_{\lambda_1} + \dim E_{\lambda_2} = 2 \neq 3$. So T is not diagonalizable.

(54) No. ~~If~~ A is similar to B, A^2 is similar to B^2.

However. $A^2 = \vec{0}$ $\quad B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. They all have

eigenvalue 0 ~~eigenvectors~~ of, multiplicity 4. but $\overset{E_0}{\overset{\shortparallel}{\ker B^2}} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

Hence ~~ker~~ $\dim E_0 = 3 \neq 4$. So B^2 is not diagonalizable.

While A^2 is a diagonal matrix.

8.1. (1) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \lambda_2 = 2$. So.

$$E_{\lambda_1} = \ker \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}.$$

$$E_{\lambda_2} = \ker \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}.$$

As $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ already an orthonormal basis. we have

~~$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a orthogonal matrix~~ $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is a orthonormal eigenbasis

(6) $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

$$f(\lambda) = \det \begin{pmatrix} -\lambda & 2 & 2 \\ 2 & 1-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{pmatrix} = \lambda(3-\lambda)(3+\lambda) = 0 \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 3 \\ \lambda_3 = -3 \end{matrix}$$

$$E_{\lambda_1} = \ker \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} = \left\{ \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \right\} \right\} \Rightarrow v_1 = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$E_{\lambda_2} = \ker \begin{pmatrix} -3 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -4 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\} \Rightarrow v_2 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$E_{\lambda_3} = \ker \begin{pmatrix} 3 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\} \Rightarrow v_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

So $S = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

(11) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$f(\lambda) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = \lambda(1-\lambda)(\lambda-2) = 0 \Rightarrow \lambda_1 = 0 \ \lambda_2 = 1 \ \lambda_3 = 2$$

(7).(a)

So $E_{\lambda_1} = \ker\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\} \Rightarrow v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$E_{\lambda_2} = \ker\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$E_{\lambda_3} = \ker\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\} \Rightarrow v_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Hence $S = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(12): $A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, a) $S_{\mathcal{M}}[L] = A(A^T A)^{-1} A^T$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

$f(\lambda) = \det\begin{bmatrix} \frac{1}{5}-\lambda & 0 & \frac{2}{5} \\ 0 & -\lambda & 0 \\ \frac{2}{5} & 0 & \frac{4}{5}-\lambda \end{bmatrix} = -\lambda^2(\lambda-1) = 0 \Rightarrow \lambda_1 = 0$ multiplicity 2

$\lambda_2 = 1$ multiplicity 1

$E_{\lambda_1} = \ker\begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{bmatrix} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$

Apply G-S process to $\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}-2\\0\\1\end{pmatrix}$, we get

$$v_1 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad v_2 = \begin{pmatrix}-\frac{2}{\sqrt5}\\0\\\frac{1}{\sqrt5}\end{pmatrix}$$

$$E_{\lambda_2} = \ker\begin{bmatrix}-\frac45 & 0 & \frac25\\ 0 & -1 & 0\\ \frac25 & 0 & -\frac15\end{bmatrix} = \text{span}\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}\right\}. \quad \text{so } v_3 = \begin{pmatrix}\frac{1}{\sqrt5}\\0\\\frac{2}{\sqrt5}\end{pmatrix}$$

So ~~an orth~~ $B = \left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-\frac{2}{\sqrt5}\\0\\\frac{1}{\sqrt5}\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt5}\\0\\\frac{2}{\sqrt5}\end{pmatrix}\right\}$ is an orthonormal eigenbasis

b) $B = S^{-1}MS = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$

c): $A = M = \begin{bmatrix}\frac15 & 0 & \frac25\\ 0&0&0\\ \frac25 & 0 & \frac45\end{bmatrix}$

(13) No. If $A = I_3$, it is not a reflection about any subspace of $\mathbb{R}^3$

(14) (a): $S = \begin{bmatrix}-1/\sqrt2 & -1/\sqrt6 & 1/\sqrt3\\ 1/\sqrt2 & -1/\sqrt6 & 1/\sqrt3\\ 0 & 2/\sqrt6 & 1/\sqrt3\end{bmatrix}$. $B = S^{-1}\begin{bmatrix}2&2&2\\2&2&2\\2&2&2\end{bmatrix}S = \begin{bmatrix}0&0&0\\0&0&0\\0&0&6\end{bmatrix}$

(b). ~~So~~ same as in (a), $S^{-1}\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}S = S^{-1}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}S + S^{-1}\begin{bmatrix}-3&0&0\\0&-3&0\\0&0&-3\end{bmatrix}S$

$$= \begin{bmatrix}-3&0&0\\0&-3&0\\0&0&0\end{bmatrix}$$

(c) S same as in (a)

$$S^{-1}\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} S = \frac{1}{2}\left[S^{-1}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}S + S^{-1}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}S\right]$$

$$= \frac{1}{2}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} \cancel{0}\,-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)$$

$$= \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(16) a) ~~ker~~ $\ker A = \ker\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

So $\lambda = 0$ is a eigenvalue of A with multiplicity 4.

$$\cancel{f(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1 & 1-\lambda \end{bmatrix} =}$$

As symmetrix matrices is always similar to a diagonal matrices, A is similar to $D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d \end{pmatrix}$ for some number d. As A and D have the same trace, we know that $d = 5$. So $\lambda_2 = 5$ of multiplicity 1 is the other eigenvalue, and

$$E_{\lambda_2} = \ker\begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

So the geometric multiplicity of $\lambda_1 = 0$ is 4., and for $\lambda_2 = 5$ is 1

b). $B = A + 2I_5$. , let S be the orthogonal matrix such that

$$S^{-1}AS = D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

Then $S^{-1}BS = S^{-1}(A + 2I_5)S$

$= S^{-1}AS + 2(S^{-1}I_5S)$

$= D + 2I_5$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

So B has 2 eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 7$. $\lambda_1 = 2$ has multiplicity 4,

and $\lambda_2 = 7$ has multiplicity 1.

c). $\det B = \det(S^{-1}BS) = 2 \times 2 \times 2 \times 2 \times 7 = 112$.

(21). ~~Only 4~~ 6. Because ~~each~~ each of the 3 eigenspaces has dimension 1, so they have unique orthonormal basis u_1, u_2, u_3 respectively. considering the order of u_1, u_2, u_3, we have 6 different such S.