Math 304

Name: ____________________________

Midterm Exam 1

Spring 2014

Instructions: Write your name above. **No books, calculators, notes or tables are allowed. You may assume results proven in class. There is no credit for an unjustified answer or inaccurate references.**

Prob.	**Possible points**	**Score**
1a	25	
1b	25	
2a	25	
2b	25	
3	25	
4a	5	
4b	20	
5	25	
6	25	
TOTAL	200	

Solutions

Consider the sequence of numbers A_n defined by:

$$A_{n+2} = 5A_{n+1} - 6A_n, \quad A_0 = 0, \quad A_1 = 1.$$

1a). Show by induction that A_n is given by the formula:

$$A_n = 3^n - 2^n.$$

Basis step: check for $n=0$ and $n=1$

$3^0 - 2^0 = 0 = A_0$; $3^1 - 2^1 = 1 = A_1$

Induction step: Assume true for $n \leq k$.

Consider

$$\begin{aligned} A_{k+1} &= 5A_k - 6A_{k-1} \\ &= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \\ &= (5-2)3^k - (5-3)2^k \\ &= 3^{k+1} - 2^{k+1} \end{aligned}$$

$\Rightarrow$ true for $n = k+1$.

Remark: The Basis step needs to verify both $n=0$ and $n=1$ because this recursion is a 2-step recursion so for example the $n=1$ case does not follow if we know $n=0$ to be true.

1b). Express the generating function for the sequence A_n as a ratio of two polynomials.

$$f_A(x) = \sum_{n\geq 0} A_n x^n$$

$$= \sum_{n\geq 0} (3^n - 2^n) x^n$$

$$= \sum_{n\geq 0} 3^n x^n - \sum_{n\geq 0} 2^n x^n$$

$$= \frac{1}{1-3x} - \frac{1}{1-2x} \quad \text{(by geom series)}$$

$$= \frac{x}{(1-3x)(1-2x)}$$

2a. Find the greatest common divisor of the numbers 252 and 198 using the Euclidean algorithm.

$252 = 1 \cdot 198 + 54$

$198 = 3 \cdot 54 + 36$

$54 = 1 \cdot 36 + 18$

$36 = 2 \cdot 18$

$\Rightarrow (252, 198) = 18$

2b. Nitu has several samples of two types of boxes that weigh 252 grams and 198 grams respectively. He would like to verify the weight of a new box of weight 360 grams using a weighing scale. Can he do so? Justify your answer.

Since there are integers A, B so that

$$A \cdot 252 + B\,198 = 18$$

we see that

$$20A \cdot 252 + 20B \cdot 198 = 360$$

if either A or B is negative we may take the expression over to the right to obtain a positive integral linear equation.

Hence the scale will balance with those # of samples of the two boxes to verify the weight of the new box.

3. Find the **smallest positive integer** x that is a simultaneous solution of the equations:

$2x \equiv 1 \mod 3$
$3x \equiv 2 \mod 5$
$5x \equiv 3 \mod 2$

$M = 30$

- $2x \equiv 1 \bmod 3 \Rightarrow x \equiv 2 \bmod 3$
- $3x \equiv 2 \bmod 5 \Rightarrow x \equiv 4 \bmod 5$
- $5x \equiv 3 \bmod 2 \Rightarrow x \equiv 1 \bmod 2$

ie simultaneous solution is

$$x_0 = 2 \cdot \frac{30}{3} + 4 \cdot \frac{30}{5} + 1 \cdot \frac{30}{2}$$
$$= 20 + 24 + 15$$
$$= 59$$

But $59 \equiv 29 \bmod 30$

Hence $x_0 = 29$ is the smallest positive integral solution

4a). State the binomial expansion of the expression $(x+y)^n$.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

4b). Give values to x and y in the binomial expansion to show that:

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} \cdots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} \cdots$$

set $x = 1$, $y = -1$ to get

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots$$

$$\Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

5. Show that for any positive integer n, we have:

$$n^3 + 3n^2 + 2n \equiv 0 \mod 6.$$

$n^3 + 3n^2 + 2n$

$= n(n+1)(n+2)$

is a product of three consecutive integers. By class notes, we know that it is divisible by $3! = 6$.

6. Find all incongruent solutions mod 32 of the equation:

$$x^{16} - 1 \equiv 0 \mod 32.$$

Note that $(n, 32) = 1$ is true exactly if $(n, 2) = 1$ ie n is odd.

also note that $\phi(32) = \#$ all odd #'s < 32
$= 16$.

ie $x^{16} \equiv 1 \mod 32$ for any odd x.

if x is even, then clearly x^{16} has 0 remainder on dividing by 32 and so is <u>not</u> a solution.

Therefore $x^{16} - 1 \equiv 0 \mod 32$ has solutions given by all odd nos less than 32

ie $\{1, 3, 5, 7, 9, \ldots\ldots, 31\}$.