1a). Consider the function $f(x) = \sqrt{x^3 + 1}$. Find the domain of $f(x)$.

Need $x^3 + 1 \geq 0$

$x^3 \geq -1$

$x \geq -1$

$\boxed{\text{Domain: } [-1, \infty)}$

1b). Find the inverse of the function $f(x)$ from part a).

$f(g(x)) = x$

$\sqrt{g(x)^3+1} = x$

$g(x)^3+1 = x^2$

$g(x)^3 = x^2-1$

$\boxed{g(x) = (x^2-1)^{1/3}}$

1c). Consider the function $g(x) = \sin(2x)$. Find the smallest positive number a so that $g(x+a) = g(x)$. This number is know as the period of $g(x)$.

$$\sin(2(x+a)) = \sin(2x)$$

$$\sin(2x+2a) = \sin(2x)$$

$$2a = 2\pi$$

$$\boxed{a = \pi}$$

2a). Consider the sequence:

$$a_n = \frac{\sin(n)}{n+1}.$$

Guess (without proof) what the limit L of this sequence is. Then find an integer N so that for any $n > N$, we have

$$|a_n - L| < \frac{1}{100}.$$

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guess: $L = 0$ ← 8 points

Want $\left|\frac{\sin(n)}{n+1}\right| < \frac{1}{100}$

Since $|\sin(n)| \leq 1$,

$\left|\frac{\sin(n)}{n+1}\right| \leq \frac{1}{n+1}$

so we ~~[illegible]~~ want $\frac{1}{n+1} < 100$

$n+1 > 100$

$n > 99$

So $N = 99$ (or anything larger)

≈ 12 points

2b). Consider the recursion:

$$a_{n+1} = \sqrt{(2a_n)}, \quad a_0 = 1.$$

Assuming that a_n converges, find its limit. Justify your answer.

Take $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n}$

Let $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$

so $a = \sqrt{2a}$

$a^2 = 2a$

$a^2 - 2a = 0$

$a(a-2) = 0$

$a = 0, 2$ ← 15 points

Since $a_0 = 1$ and $a_{n+1} = \sqrt{2a_n}$, we see that $a_n \geq 1$ for all n.

so $\boxed{a = 2}$

15 points

3a). If measured today, the soil in a farm is found to be contaminated with 8 ppb (parts per billion) of radioisotope I. Six years ago, the same soil was found to have 32 ppb of I. Find the half-life of I.

$W(0) = 32$

$W(6) = 8$

$W(t) = W_0 e^{-\lambda t} = 32 e^{-\lambda t}$

$8 = W(6) = 32 e^{-6\lambda}$

$\frac{1}{4} = e^{-6\lambda}$

$\ln(4) = 6\lambda$

$\lambda = \frac{\ln(4)}{6}$

10 points

$W(t) = 32 e^{-\frac{\ln(4) t}{6}}$

5 points

Solve for t:

$W(t) = \frac{1}{2} W_0 = \frac{1}{2} 32 = 16$

$32 e^{-\frac{\ln(4) t}{6}} = 16$

$e^{-\frac{\ln(4) t}{6}} = \frac{1}{2}$

$\frac{\ln(4) t}{6} = \ln(2)$

$t = \frac{6 \ln 2}{\ln 4}$

10 points

$\frac{\ln 2}{\ln 4} = \log_4 2 = \frac{1}{2}$

−5 points for not simplifying

so $\boxed{t = \frac{1}{2} \cdot 6 = 3 \text{ years}}$

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3b). Soil is considered safe to farm on if the amount of radioisotope I is less than 1 ppb. After how many years (starting today) would the soil be considered safe.

~~Want to find t such~~

Now $W_0 = 8$ ("today")

-5 points if you had $W_0 = 32$

$W(t) = 8e^{-\lambda t} = 8e^{-\frac{\ln 4}{6}t}$

15 points

Find t so that

$1 > W(t) = 8e^{-\frac{\ln 4}{6}t}$

10 points

$\frac{1}{8} > e^{-\frac{\ln 4}{6}t}$

$\ln\left(\frac{1}{8}\right) > -\frac{\ln 4}{6}t$

$\ln 8 < \frac{\ln 4}{6}t$

$\boxed{t > 6\frac{\ln 8}{\ln 4} = 6 \cdot \log_4 8 = 6 \cdot \frac{3}{2} = 9 \text{ years}}$

4a). The number of predators in a forest are given by the function: $F(t) = t^3 + 1$, where t represents time in years. The number of prey in the same forest is given by the function $G(t) = t^2 + 8$. Show that between two and three years from now, there will be a time when the number of predators equal the number of prey.

Want $F(t) = G(t)$

$t^3 + 1 = t^2 + 8$

Want $t^3 - t^2 - 7 = 0$

10 points

Set $\varphi(t) = t^3 - t^2 - 7$.

$\varphi(t)$ is continuous on $[2, 3]$,

and $\varphi(2) = -3$ and $\varphi(3) = 11$

so $\varphi(2) < 0 < \varphi(3)$

so there exists $c \in (2, 3)$

so that $\varphi(c) = 0$.

by the IVT

10 points

(+5 points for picture)

4b). Consider the equation $e^x = \sin(x)$. Show that it has a negative solution.

Let $f(x) = e^x - \sin x$

only 3 points if it is by chance, i.e. not setting $f(x) = e^x - \sin x$

20 points

$f(0) = 1$

$f\left(-\frac{3\pi}{2}\right) = e^{1/(3\pi/2)} - \sin\left(-3\pi/2\right)$

$= e^{\frac{1}{3\pi/2}} \cancel{\quad} - 1 < 0$

10 points

Since $f(x)$ is continuous on $\left[-\frac{3\pi}{2}, 0\right]$

(it is a difference of continuous functions)

by the IVT there exists a $c \in \left(-\frac{3\pi}{2}, 0\right)$

so that $f(c) = 0$.

(Note: $\frac{-3\pi}{2}$ could have been any $-\frac{(4n+3)\pi}{2}$)

5a). Calculate

$$\lim_{x\to\infty} \frac{8x^4+3x^3+12x+1}{2x^4+19x^2+2}.$$

$$\lim_{x\to\infty} \frac{8x^4+3x^3+12x+1}{2x^4+19x^2+2}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^4}\left(8x^4+3x^3+12x+1\right)}{\frac{1}{x^4}\left(2x^4+19x^2+2\right)}$$

$$= \lim_{x\to\infty} \frac{8+\frac{3}{x}+\frac{12}{x^3}+\frac{1}{x^4}}{2+\frac{19}{x^2}+\frac{2}{x^4}}$$

$$= \frac{8+0}{2+0} = \frac{8}{2} = 4$$

limit laws

5b). Calculate

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2}.$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x}$$

$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \cdot \lim_{x\to 0} \frac{1}{1+\cos x} = 1 \cdot \frac{1}{1+1} = \frac{1}{2}$$

5c). Define a function

$$f(x) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+1}-1} & \text{if } x \neq 0, \\ c & \text{if } x = 0. \end{cases}$$

Find a value for c that makes $f(x)$ continuous at $x = 0$.

Want $\lim_{x \to 0} f(x) = f(0)$

$$\lim_{x \to 0} \frac{x^2}{\sqrt{x^2+1}-1} = c$$

$$\lim_{x \to 0} \frac{x^2(\sqrt{x^2+1}+1)}{(\sqrt{x^2+1}-1)(\sqrt{x^2+1}+1)} = c$$

$$\lim_{x \to 0} \frac{\cancel{x^2}(\sqrt{x^2+1}+1)}{\cancel{x^2}} = c$$

$$\lim_{x \to 0} (\sqrt{x^2+1}+1) = c$$

$$\text{"} \quad \cancel{\lim_{x\to 0}} \; 2$$

$$\boxed{c = 2}$$

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