Homework 8 Solutions

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(1)
Solution: If the length of the sides of the right angle is x and y respectively, the area is given by

$$A = \frac{1}{2}xy.$$

As the hypotenuse is 10 cm long, by Pythagorean Theorem, $x^2 + y^2 = 10^2$, hence $y = \sqrt{100 - x^2}$. This means, the area is depend only on x, where $x \in [0, 10]$,

$$A(x) = \frac{1}{2}x\sqrt{100 - x^2}.$$

And the largest area is the absolute maximum of $A(x)$ on $[0, 10]$.

$$A'(x) = \frac{1}{2}(\sqrt{100 - x^2} + x\frac{1}{2}100 - x^{2-\frac{1}{2}} \cdot (-2x)) = \frac{50 - x^2}{\sqrt{100 - x^2}}.$$

So the critical point is $x = 5\sqrt{2}$.
We can check that $A(0) = 0$,$A(10) = 0$,$A(5\sqrt{2}) = 25$. So the largest area is 25.

(2)
Solution: If the bottom surface of the cylinder has radius r, and the cylinder has hight h, the material used for the cylinder is

$$A = \pi r^2 + 2\pi rh.$$

As the right circular cylinder has volume $100cm^3$, open on the top and closed on the bottom, we have, from the volume formula,

$$100 = \pi r^2 h.$$

So $h = \frac{100}{\pi r^2}$, and the amount of material A only depends on r. So we can have,

$$A(r) = \pi r^2 + 2\pi r\frac{100}{\pi r^2} = \pi r^2 + \frac{200}{r}$$

. Where $r \in (0, +\infty)$.
It is obvious that $\lim_{r\to+\infty} = +\infty$, and $\lim_{r\to 0^+} = +\infty$. Hence the least of $A(r)$ is a local minimum.
Set

$$A'(r) = 2\pi r - \frac{200}{r^2} = 0,$$

We have the only critical is $r = \sqrt[3]{\frac{100}{\pi}}$.
As

$$A''(\sqrt[3]{\frac{100}{\pi}}) = 2\pi + \frac{400}{(\sqrt[3]{\frac{100}{\pi}})^3} = 6\pi > 0,$$

we know that $r = \sqrt[3]{\frac{100}{\pi}}$ is a local minimum. Hence the least amount of material is $A(\sqrt[3]{\frac{100}{\pi}}) = 300\sqrt[3]{\frac{\pi}{100}}(cm^2)$.

(3) **Solution:** Let d be the distance between the point on $y = \frac{1}{x}$ and $(0, 0)$. Then, as $x > 0$,

$$d=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}=\sqrt{x^2+(\frac{1}{x})^2}.$$

So the closest point corresponding to the smallest d, which is the local minimum of d.

$$d'=\frac{1}{2}(x^2+(\frac{1}{x})^2)^{-\frac{1}{2}}(2x-\frac{2}{x^3}).$$

So setting $d'=0$, we get the critical point $x=1$, since $x>0$.

$$d''=\frac{1}{2}[-\frac{1}{2}(x^2+\frac{1}{x^2})^{-\frac{3}{2}}(2x-\frac{2}{x^3})^2+(x^2+\frac{1}{x^2})^{-\frac{1}{2}}(x+\frac{6}{x^4})].$$

So when $x=1$, $d''=\frac{7}{2\sqrt{2}}>0$. Hence $x=1$ is the local minimum we are looking for. And the closest point is $(1,1)$.

(4)
(a)**Solution:**

$$\lim_{x\to 1}\frac{x^4-1}{x^5-1}=\lim_{x\to 1}\frac{4x^3}{5x^4}=\lim_{x\to 1}\frac{4}{5x}=\frac{4}{5}$$

(b)**Solution:**

$$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{2\cdot\frac{1}{2\sqrt{x}}}\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0$$

(c)**Solution:**

$$\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0$$

(d)**Solution:**

$$\lim_{x\to 0}\frac{e^x-1}{\sin x}=\lim_{x\to 0}\frac{e^x}{\cos x}=\frac{1}{1}=1$$