HW 6 Solution

October 22, 2012

1. This problem depends on the formula

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Furthermore, because both f^{-1} and g^{-1} exist, we must have f, g are one-to-one and onto.
(a) We first note that since $f(1) = 2$, so that $f^{-1}(2) = 1$. Hence, $a'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = -\frac{1}{2}$.
(b) By the chain rule, $b'(x) = (f^{-1})'(\sqrt{x}) \cdot (\sqrt{x})' = \frac{1}{f'(f^{-1}(\sqrt{x}))} \cdot \frac{1}{2\sqrt{x}}$. Plugging in $x = 4$, we get

$$\frac{1}{f'(f^{-1}(2))} \cdot \frac{1}{4} = -\frac{1}{2} \cdot \frac{1}{4} = -\frac{1}{8}$$

(c) By the chain rule, $c'(x) = f'(g^{-1}(x)) \cdot (g^{-1}(x))' = f'(g^{-1}(x)) \cdot \frac{1}{g'(g^{-1}(x))}$. Plugging in $x = 2$, we have

$$f'(g^{-1}(2)) \cdot \frac{1}{g'(g^{-1}(2))} = f'(2) \cdot \frac{1}{g'(2)} = -1 \cdot \frac{1}{1} = -1$$

(d) $\frac{d}{dx}ln(f(x)) = \frac{f'(x)}{f(x)}$. Plugging in $x = 2$, we have $d(2) = -1$.
2. (a) We just need to find the tangent line $L(x)$ of $f(x)$ at $x = 2$, which is

$$L(x) = \sqrt[3]{2} + \frac{1}{3\sqrt[3]{4}}(x - 2)$$

(b) Plugging in $x = 8$ in $L(x)$, we have

$$\sqrt[3]{8} \approx \sqrt[3]{2} + \frac{2}{\sqrt[3]{4}}$$

3. (a) Apply chain rule, we have

$$a'(x) = (\frac{x}{x+1})' \cdot \frac{1}{\frac{x}{x+1}} = \frac{1}{x(x+1)}$$

(b) $b'(x) = -2x\ \sin x^2\ \ln(x+2) + \frac{\cos x^2}{x+2}$
(c) You need to know the fundamental derivative $(tan^{-1})'(x) = \frac{1}{1+x^2}$. By chain rule,

$$c'(x) = e^{\tan^{-1} x} \frac{1}{1+x^2}$$

(d) You need to know $(\sec x)' = \sec x\ \tan x$.

$$d'(x) = 2\sec^2(x \ln x)\tan(x \ln x) \cdot (\ln x + 1)$$

(e) Consider $F(x) = \ln e(x) = 2\tan(x)\ln x$.

$$F'(x) = \frac{e'(x)}{e(x)} = 2\sec^2 x \ln x + \frac{2\tan x}{x}$$

. Hence,

$$e'(x) = (2\sec^2 x \ln x + \frac{2\tan x}{x}) \cdot e(x)$$

4. We know that the decay of an isotope obeys the function $f(t) = Ae^{-\lambda t}$, where A is the initial amount of the isotope.
(a) We need to determine the constants A and λ. Since the half life of the isotope is 150 years, we have $0.5A = Ae^{-150\lambda}$. Solving this equation, we obtain $\lambda = 0.02$. We are also giving the initial amount of the isotope is 2000 grams, so that

$$f(t) = 2000e^{-0.02t}$$

(b) $f'(t) = -40e^{-0.02t}$